INTERNATIONAL JOURNAL OF
MODERN PHYSICS A

Volume 23, Number 2
January 20, 2008

# On Quarks and Flavor Symmetry

H. Høgaasen and P. Sorba

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

International Journal of Modern Physics A
Vol. 23, No. 2 (2008) 189–210


World Scientific
www.worldscientific.com

# ON QUARKS AND FLAVOR SYMMETRY

H. HØGAASEN

*Department of Physics, University of Oslo,*
*Box 1048, NO-0316 Oslo, Norway*
*hallstein.hogasen@fys.uio.no*

P. SORBA

*LAPTH,* Université de Savoie, CNRS,*
*9 Chemin de Bellevue, B.P. 110, F-74941 Annecy-le-Vieux Cedex, France*
*sorba@lapp.in2p3.fr*

Received 29 November 2007

Hadronic spectroscopy can be introduced to students and developed rather far without requiring SU($N$) flavor symmetry. In such a "minimalist" presentation, we are naturally led to comment and clarify the concept of the "generalized" Pauli principle.

*Keywords*: Quarks; flavor symmetry.

## 1. Introduction

Internal symmetry groups have a glorious place in the history of physics. One of the highlights was the discovery by Fermi that low energy pion–nucleon scattering is dominated by a single resonance of spin 3/2. By treating the nucleons as an isospin doublet, the pions as an isospin triplet and the resonance $\Delta$ as a state with isospin 3/2, Fermi and his collaborators proved the dynamical fact that, to a good precision, isospin is conserved in strong interactions.

Another highlight was Gell-Mann's prediction of the spin-3/2 $\Omega^-$ at the Geneva conference in 1962. The subsequent discovery of this state at Brookhaven in 1964 made everyone confident that the *flavor-group* SU(3) was as relevant for strong interactions as the isospin group.

There is however a difference between the two cases we have mentioned: Fermi's discovery was a discovery in the dynamics of particles while Gell-Mann's prediction is usually presented as coming from the assumption that baryons should fall into specific representations of SU(3)-flavor, namely the eight-dimensional one for the lightest spin-1/2 baryons, the ten-dimensional one for the lightest spin-3/2 baryons.

*Laboratoire d'Annecy-le-Vieux de Physique Théorique, UMR 5108.

Experimental results from reactions involving the assignment of mesons and baryons to irreducible representations of SU(3) came later, and showed that the concept of "broken flavor symmetry" was useful.

Subsequently the flavor symmetry groups SU(4) and SU(5) have been invoked to classify the multiplicities of states when charm and bottom quantum numbers are added.

But let us stress that the assignment of a multiplet of particles to a group representation is in itself an empty statement. Its usefulness depends on the group to be an (almost) symmetry group. When Fermi's team discovered that the isospin group — historically, this group was first supposed to be $R(3)$ before becoming SU(2) — it was by studying reactions. If a pion–nucleon state is a linear combination of isospin $I = 1/2$ and $I = 3/2$, only two independent amplitudes $A_{1/2}$ and $A_{3/2}$ will describe all $\pi N \to \pi N$ reactions:[a] this turned out to be correct.

Experiments must decide to which extent flavor symmetry is a useful concept. From the multiplication table of SU(3):

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$$

it follows that the 64 reactions we have in "quasielastic" $2 \leftrightarrow 2$ reactions between octet states are given by only six independent amplitudes. From the same table we see, as the representation 10 occurs only once, that in the coupling of a decuplet to two octets there is only one amplitude. This is similar to the decay of $\Delta \to N\pi$ in the isospin symmetric — SU(2) — case. So, if one has baryons made from $u$, $d$ and $s$ quarks, the assumption of flavor symmetry relates the coupling constant in $\Delta^{++} \to P\pi^+$ to the coupling constants in $\Sigma^{*0} \to \Lambda\pi^0$, $\Sigma^{*0} \to \Sigma^+\pi^-$ and $\Xi^{*0} \to \Xi^0\pi^0$. Experimentally they come out correctly within 10%.

1964 was the year Gell-Mann and Zweig invented quarks. Right from the start Zweig was a fervent believer in the objective existence of quarks (that he called aces) as the fundamental constituents of hadrons. With three flavors of aces (quarks) mesons came in $q\bar{q}$ nonets. There was no need to invoke a "magic mixing" between a flavor octet and a singlet; to explain that one of the vector mesons decayed into $K\bar{K}$ pairs was simply the reflection of the fact that it was composed of $s\bar{s}$. If some "magic" is there, it is rather in the structure of the spin-zero mesons.

In the years before quarks were generally accepted as physical beings in the physics community, flavor symmetry arguments were (and often still are!) regarded as more high-brow than arguments simply based on quarks. But today hardly anybody denies that quarks exist. It is easy to criticize the (nonrelativistic) quark model where baryons are made from three valence quarks and mesons from a quark–antiquark pair, but it cannot be denied that it gives a set of rules to classify hadron states, and also to calculate many of their static properties which are quite illuminating.

[a] As $A_{3/2}$ dominates at low energy, the analysis of data were quite simple.

However, as we will see in the following, we do not need any flavor SU($N$) group either to classify hadrons or to estimate many of their physical properties. Hereafter, the fundamental symmetry groups we will use are the group of rotations in the three-dimensional space represented by the SU(2) spin group (together with the O(3) orbital angular momentum one in the case of excited states), and the color SU(3) group.

Indeed, the SU(3) symmetry group of fundamental importance is not the group acting over flavors, but the group acting over color space. It is a group that is gauged and thereby introduces gluon fields while defining QCD. The (yet unproven) dogma that free particles are colorless, i.e. transform as a singlet under the color SU(3) group, implies that the most economical configurations are those involving three quarks (baryon) or a quark–antiquark pair (meson). This comes from the decomposition of (color) SU(3) representations:

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10\,, \qquad 3 \otimes 3 = 6 \oplus \bar{3}\,, \quad 3 \otimes \bar{3} = 1 \oplus 8\,,$$

where the representations 6 and $\bar{3}$ respectively involve symmetric and antisymmetric combinations of the two color indices.

The second essential ingredient we need is the Pauli principle. Of course it is of importance only for baryons,[b] and it is actually this class of hadrons which deserves to be considered for our purpose. Indeed, at least with regard to their classification, $q\bar{q}$ mesons with a given spin and parity are simply gathered in multiplets of dimension $N^2$, $N$ being the number of different quark flavors.

Note that such three-quark states are color-antisymmetric under the exchange of two quarks. Then, the Pauli principle tells us that a state involving two identical quarks, i.e. quarks with identical flavors, must be symmetric in the other quantum numbers.

We will start by showing in Secs. 2 and 3 that the multiplets of low-lying as well as excited baryonic resonances can be directly constructed with the above hypotheses and tools at hand. In Sec. 4, expressions for $S$-wave three-quark states only satisfying permutational symmetry imposed by the Pauli principle are given and used to compute baryon mass splittings through (color) magnetic interactions between the constituent quarks; their magnetic moments are also obtained.

However the basis constructed in Sec. 2 does not appear suitable for computations of physical quantities involving flavor changing forces, such as, for instance, semileptonic decays. A convenient way to circumvent this difficulty is to impose complete symmetry under permutations of the three constituents within the baryon wave function rather than the *partial* symmetry dictated by the Pauli principle: this prescription appeals to the so-called "generalized" Pauli principle, which, as is well known, does not introduce any extra physical assumption. The corresponding developments are presented in Sec. 5.

[b]We do not consider mixing with multiquark states. For mesons we need a generalization of the Pauli principle which follows from local field theory through the CPT operation; but this point will not be discussed hereafter where we limit to (usual) baryons.

We conclude by commenting on the direct connection between irreducible representations of the unitary groups and their properties under the permutation group, which explains, in our opinion, their perfect adaptability to the classification of baryons.

In order not of overload the text and keep clear the main idea of this paper, we have decided to treat in appendices the following points. Appendix A is devoted to a general and explicit construction of the (three-state particle) Jacobi coordinates. In our knowledge, such coordinates have already been explicated in the case of the harmonic oscillator with only one coupling constant, but not with a different coupling associated to each couple of constituents.

Then we give a proof, in App. B, of the equivalence between the two Hilbert space descriptions introduced in Secs. 4 and 5 respectively, which involve states satisfying partial permutation symmetry (simple Pauli principle) for the former, and total permutation symmetry (generalized Pauli principle) for the later. The Fock space emerges rather naturally from this construction, and we show in App. C how it can be used to represent baryon states. Finally, we give in App. D the explicit wave functions of the eight $S = 1/2$ low-lying baryons in the three bases successively introduced: the permutational "partial" symmetric basis (cf. Sec. 4), the totally symmetric basis (cf. Sec. 5) and the Fock space basis, this last one combining the advantages of the other two representations: simplicity of the expressions and complete symmetry.

As this paper can be partly regarded as a set of lecture notes, there is an almost empty reference list. We apologize for this but we realize that to do justice to all the people who developed the subject, we would need a list longer than the paper.

## 2. The Ground State of Three Quarks

This is the subject of most introductory courses on elementary particles. So, we will treat this first. Although the following is known, it is apparently not well known, as we realized by looking up at lectures notes on the web. Among many popular monographs, we found only one[1] which shares the approach presented below.

We now forget about color, remembering that each pair of quarks are antisymmetric in color so that the baryon states must be symmetric in the other degrees of freedom. This was for a long time called the Dalitz symmetric quark model. Note at this point that it was realized[2] that if quarks were fermions such that there existed an hidden quantum number — which later turned out to be the color — justifying the symmetric quark model, then the multiplicity of the ground states would follow independently of any flavor symmetry group.

In the ground state where no angular momentum is involved, the only degree of freedom of each quark is the spin. As a pair of identical quarks must be symmetric under interchange it must be symmetric in spin: the spin of the pair is 1.

Integrating out the spatial degrees of freedom, we are left with a Hamiltonian over the flavor–spin space of the quarks.[c] Now let us count the ground states.

Suppose that we have $N$ flavors $(u, d, s, c, \ldots)$ and we choose three of these to make baryons.

First, for three identical flavors: there are $N$ ways of choosing these baryons made up of three identical quarks. Each pair has $S = 1$ so all have total spin $3/2$.

For two identical flavors: there are $N(N-1)$ ways of choosing two identical and one different from the two first chosen. The identical quarks are coupled to spin 1, and the total spin is $S = 3/2$ or $S = 1/2$.

For all three quarks of different flavor: there are $N(N-1)(N-2)/6$ ways of choosing three different flavors. The Pauli principle imposes no restriction here, so the total spin can be $3/2$ or $1/2$, the multiplicity of $S = 1/2$ is two.

So the number $N(3/2)$ of flavor states with total spin $3/2$ is

$$\begin{aligned} N(3/2) &= N + N(N-1) + N(N-1)(N-2)/6 \\ &= N(N+1)(N+2)/6 \end{aligned}$$

and the number of flavor states with spin $1/2$ is

$$\begin{aligned} N(1/2) &= N(N-1) + 2N(N-1)(N-2)/6 \\ &= N(N+1)(N-1)/3\,. \end{aligned}$$

We immediately see that we have found the same multiplicities as is commonly inferred from the dimensions of the representations of the flavor symmetry group SU($N$).

For $N = 3$ we have an "eightfold way," more precisely an octet of $S = 1/2$ states and a decuplet of $S = 3/2$ states.

But we also realize that these multiplicities have nothing to do with the existence of an internal symmetry group, they have their sole origin in the Pauli principle. Not only $u$, $d$, $s$ quarks, but also $u$, $c$, $b$, or any triplet of flavor quarks will provide with an octet and a decuplet.

Now, let us count the total number of such quantum mechanical states. Since there are $2S + 1$ states in a spin $S$ representation, one gets

$$\begin{aligned} N(\text{total}) &= 4N(N+1)(N+2)/6 + 2N(N+1)(N-1)/3 \\ &= 2N(N+1)(2N+1)/3 \end{aligned}$$

and that is exactly the dimension of the SU($2N$) completely symmetric representation arising from the tensorial product of three $2N$-dimensional fundamental SU($2N$) representations. Taking as an example $N = 3$, we indeed get 56 states, that is the dimension of the corresponding symmetric representation of the usually called "flavor-spin" SU(6) group: we will comment later on this point, in direct connection with the "generalized" Pauli principle.

[c]The following arguments are not restricted to nonrelativistic quantum mechanics.

## 3. Excited States

Let us now turn our attention to the construction of $P$-wave baryon states, to which we will restrict our presentation in order not to overload this paper. These negative parity baryons are associated to an $L = 1$ orbital momentum. This unit of O(3) orbital angular momentum stands usually in one of the two quark relative coordinates, and the most common model which is well adapted to represent such an effect is that of the harmonic oscillator. The reason for treating confining forces between quarks using harmonic oscillator potentials is simply that the center-of-mass motion can be separated out using Jacobi coordinates. Note that this choice has no importance for our purpose, namely the *number* of exited states.

We will use as a Hamiltonian

$$H_{\text{h.o.}} = \sum_i \left( m_i + \frac{p_i^2}{2m_i} \right) + \sum_{i<j} \left( \frac{k}{2} \right) \cdot (\mathbf{r}_i - \mathbf{r}_j)^2 \,, \tag{1}$$

where $\mathbf{r}_i$, $i = 1, 2, 3$, denote the respective positions of the three quarks. Although we do not start with a more general Hamiltonian $H_{\text{gen}}$, we can obtain the general solution for the spectrum by first using $H_{\text{h.o.}}$ to get states of a harmonic oscillator basis and then perturb them with the perturbation $H_{\text{gen}} - H_{\text{h.o.}}$. The energy of the levels will change with respect to the ones of the harmonic oscillator, but their number will be the same.

In terms of the Jacobi coordinates $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$ and the center-of-mass coordinates $\mathbf{R}_{\text{cm}}$, the Hamiltonian separates into $H_{\text{h.o.}} = \sum_i m_i + H_{\text{cm}} + H_\rho + H_\lambda$ with

$$\begin{gathered} H_{\text{cm}} = \frac{P^2}{2M} \,, \quad M = \sum_i m_i \,, \quad \mathbf{P} = \sum_i \mathbf{p}_i \,, \\ H_\rho = \frac{p_\rho^2}{2m_+} + \frac{3k}{2} \cdot (\boldsymbol{\rho})^2 \,, \quad H_\lambda = \frac{p_\lambda^2}{2m_\lambda} + \frac{3k}{2} \cdot (\boldsymbol{\lambda})^2 \,. \end{gathered} \tag{2}$$

The explicit expressions of $m_+$ and $m_-$ are given in App. A, where a detailed construction of Jacobi coordinates is developed, and also generalized to the case where we have a different coupling $k_{ij}$ for each quark pair $q_i q_j$ in the potential term.

For notational simplicity, we consider here the equal mass case, i.e. $m_i = m$ to which correspond: $m_+ = m_- = m$ and the relative coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ reduce to

$$\boldsymbol{\rho} = \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{\sqrt{2}} \,, \qquad \boldsymbol{\lambda} = \frac{(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3)}{\sqrt{6}} \,.$$

So, let us separately study the different configurations.

(i) *Baryons made of three flavor identical quarks qqq.*
Consider the oscillator relative to $\boldsymbol{\rho}$: it is antisymmetric in the exchange (1)–(2). The doublet made with the first two quarks $qq$, that is of quarks in positions (1) and (2), has automatically spin $S = 0$, in order for the total spin and orbital momentum part to be symmetric (Pauli principle). It follows that in this $\boldsymbol{\rho}$ configuration, the total spin of the baryon is $S = 1/2$, and the spin/orbit

part of the wave function reads, for $Sz = +1/2$ and up to a normalization factor:

$$(\mathbf{r}_1 - \mathbf{r}_2)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \text{sym}\,.$$

Replacing now $\boldsymbol{\rho}$ by $\boldsymbol{\lambda}$, which is symmetric in the first two quarks, one easily deduces that the corresponding $qq$ doublet has a (symmetric) spin $S = 1$. However in the product $S = 1$ by $S = 1/2$, the totally spin symmetric $S = 3/2$ part cannot provide, when combined with the (not completely symmetric) $\boldsymbol{\lambda}$, a completely symmetric wave function. It follows that the only possibility for the resulting baryon is to have spin 1/2. Moreover, one notes that the spin/orbit part of the wave function reads

$$(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3)(2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow) + \text{sym}$$

which is exactly the same, up to a scale factor, as the one obtained just above for the $\boldsymbol{\rho}$ case.

Therefore, for $qqq$ configurations, the only solution is given by the $\boldsymbol{\rho}$ oscillator with two quarks of spin 1 *or* by the $\boldsymbol{\lambda}$ oscillator with two quarks of spin 0, and total spin of $S = 1/2$.

(ii) *Baryons made of two (and only two) identical flavor quarks $qqq'$.*
Now the Pauli principle imposes permutation symmetry only in the first two quarks $qq$. Note that, as explicated in App. A, the antisymmetry (resp. symmetry) of $\boldsymbol{\rho}$ (resp. $\boldsymbol{\lambda}$) persists when $m_1 = m_2 = m$ and $m_3 = m'$ with $m$ and $m'$ different. Then one gets:

- with the $\boldsymbol{\rho}$ oscillator: the $qq$ spin part must be 0, leading for the ($qqq'$) baryon to $S = 1/2$, with in the wave function a $qq$ spin/orbit part proportional to

$$(\mathbf{r}_1 - \mathbf{r}_2)(\uparrow\downarrow - \downarrow\uparrow) + \text{sym}$$

- with the $\boldsymbol{\lambda}$ oscillator: the $qq$ spin part is now 1, allowing the total spin to be $S = 3/2$ and $S = 1/2$.

(iii) *Baryons made of three different flavor quarks $qq'q''$.*
Now there is no restriction imposed by the Pauli principle. It follows that with the $\boldsymbol{\rho}$ as well as with the $\boldsymbol{\lambda}$ oscillator, one can get one baryon with total spin $S = 3/2$ and two baryons of total spin $S = 1/2$ for each triplet of flavors.

As a conclusion, let us count the number of states of three different flavors:

- With $S = 3/2$: there are six states of $qqq'$-type with the $\boldsymbol{\lambda}$ oscillator, one state $qq'q''$ with each of the $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ oscillator, that is a total of eight states.
- With $S = 1/2$: there are three states of $qqq$-type made from the $\boldsymbol{\rho}$ oscillator (and partial $qq$ spin $S = 0$) *or* from the $\boldsymbol{\lambda}$ oscillator (and partial $qq$ spin $S = 1$). There are also six states of $qqq'$-type made from the $\boldsymbol{\rho}$ oscillator (and partial $qq$ spin $S = 0$), six states of $qqq'$-type made from the $\boldsymbol{\lambda}$ oscillator (and partial $qq$ spin $S = 1$). Finally, there are four states of $qq'q''$-type respectively made with the $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ oscillator and of $qq$ partial spin $S = 0$ and 1.

Then, with three flavors, one obtains a total of 70 states, with eight states of $S = 3/2$ and 19 states of $S = 1/2$ in perfect accordance with the usual SU(6) approach where the $P$-wave baryons ($L = 1$) are classified in the irreducible 70-dimensional SU(6) representation, itself decomposing with respect to SU(3) flavor and SU(2) spin as

$$70 = (8, 3/2) + (8 + 10 + 1, 1/2)\,.$$

In a similar manner, one can show that, for higher excitations, the number of excited states and their configurations are always what one should infer from the SU(6) approach.

## 4. Flavor Nonchanging Forces

Let us now focus on forces acting on the quarks (gluonic and electromagnetic in particular) that do not change the quark flavor.

We can represent any baryon made of the three quarks $q_1$, $q_2$, $q_3$ as follows:

$$B(q_1, q_2, q_3) = (q_1 q_2)_s \otimes (q_3)_{1/2}\,, \tag{3}$$

where $s$ is the spin of the doublet of the quark pair $q_1 q_2$. One obviously has $S = 1$ or $S = 0$.

If we denote by $q_i^{\uparrow}$ the $i$ quark with spin up, by $q_i^{\downarrow}$ the corresponding spin down state, then with the help of Clebsch–Gordan coefficients, any $S = 3/2$ state with $S_z = 3/2$ writes

$$q_1^{\uparrow} q_2^{\uparrow} q_3^{\uparrow}$$

while it becomes for $S_z = 1/2$:

$$\frac{[q_1^{\uparrow} q_2^{\uparrow} q_3^{\downarrow} + q_1^{\uparrow} q_2^{\downarrow} q_3^{\uparrow} + q_1^{\downarrow} q_2^{\uparrow} q_3^{\uparrow}]}{\sqrt{3}}$$

and so on. As we have seen there are ten such flavor states, if we limit ourselves to three flavors.

For the 8 flavor states with spin 1/2 and $S_z = +1/2$, one must distinguish the case when the first two quarks are identical in flavor from the case when all flavors are different. In the first situation, the spin of $q_1$ and $q_2$ must couple to 1 (we write below: $q_1 = q_2$) and Clebsch–Gordan coefficients give

$$\frac{1}{\sqrt{6}}\,[2 \cdot q_1^{\uparrow} q_1^{\uparrow} q_3^{\downarrow} - (q_1^{\uparrow} q_1^{\downarrow} + q_1^{\downarrow} q_1^{\uparrow}) q_3^{\uparrow}]\,.$$

In the second configuration, the Pauli principle gives no restriction, and we have two states, the spin of the two first quarks being coupled to produce either a spin 1 ($\psi_1$) or 0 ($\psi_2$):

$$\psi_1 = \frac{1}{\sqrt{6}}\,[2 \cdot q_1^{\uparrow} q_2^{\uparrow} q_3^{\downarrow} - (q_1^{\uparrow} q_2^{\downarrow} + q_1^{\downarrow} q_2^{\uparrow}) q_3^{\uparrow}]\,,$$

$$\psi_2 = \frac{[(q_1^{\uparrow} q_2^{\downarrow} - q_1^{\downarrow} q_2^{\uparrow}) q_3^{\uparrow}]}{\sqrt{2}}\,.$$

Clearly it does not matter which ordering we have for the different flavors, but it will be shown in a moment that it is often convenient to order them by placing the lightest quark(s) in front of the heaviest.

We stress again that the *classification* of states is into a spin-3/2 decuplet and a spin-1/2 octet, *no matter* which three flavors we choose. With quarks $d$, $c$ and $b$ we have the same structure as with the lightest states $u$, $d$ and $s$. From this remark it should be clear for students that the classification of states is one thing; the question whether we have a flavor symmetry group SU(3) is completely different.

Now, from the eight states with total spin 1/2 that we have constructed, we can span a Hilbert space and consider how interactions act on states therein.

If there were no spin dependent interquark forces, it is evident that the states made of the same quarks would have the same mass. The interquark Hamiltonian would be diagonal in our Hilbert space.

This is definitely not so. In the beginning it seemed strange that the $\Lambda$ and $\Sigma_0$ had different masses. An early explanation, based on spin–spin interactions, was given by Sakharov and Zeldowich in 1966 with the Hamiltonian

$$H_S = \sum_{i,j} c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,, \tag{4}$$

where all coefficients $c_{ij}$ being equal. In 1975 this spin–spin interaction was shown to be a consequence of QCD by De Rujula, Georgi and Glashow:

$$H_{\mathrm{CM}} = -\sum_{i,j} C_{ij} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,. \tag{5}$$

As all quark pairs are in a $\bar{3}$ state of color, the color part $\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$ factorizes, giving a common factor: 8/3, so that

$$H_{\mathrm{CM}} = \left(\frac{8}{3}\right) \sum_{i,j} C_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,. \tag{6}$$

Here the coefficients $C_{ij}$ are, among other things, dependent on the quark masses and properties of the spatial wave functions of the quarks in the system.

A natural scaling assumption for the coefficients $C_{ij} \propto \frac{1}{m_i m_j}$ comes from the analogy with the hyperfine splitting in atoms that originates in the interaction between (electro)magnetic moments. In the physics of quarks the analog interaction is between the quarks *colormagnetic* moments.

Let us now imagine that we have integrated out all the spatial variables for the three quarks. We can then write an effective Hamiltonian over the spin-space of the quark:

$$H = \sum_i m_i + \left(\frac{8}{3}\right) \sum_{i,j} C_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,. \tag{7}$$

Here effective masses $m_i$ incorporate the masses of the quarks as well as their kinetic energy.[d] If we *assume* that each effective mass $m_i$ is (almost) the same in different baryons, then we get mass formulae. For this, we have only to determine the eigenvalues of $H_{\rm CM}$.

The solution of the eigenvalue problem comes easily when one uses the following identity for the Pauli matrices — as can be directly tested out by applying $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ on the (symmetric) spin 1 state and on the (antisymmetric) spin 0 state:

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = -1 + 2P_{i\leftrightarrow j}\,, \tag{8}$$

where $P_{i\leftrightarrow j}$ is the operator that permutes the spin states of the two particles $i$ and $j$. One sees at once that the Hamiltonian is almost diagonal in our Hilbert space.

For states with total spin $S = 3/2$ the eigenvalue of $H_{\rm CM}$ is then

$$\frac{8}{3}\,(C_{12} + C_{13} + C_{23}) \tag{9}$$

while for the spin $S = 1/2$ baryons, it reduces to

$$\frac{8}{3}\,(C_{12} - 4C_{13}) \tag{10}$$

for all states with two identical quarks (they are chosen above as being $q_1$ and $q_2$). The mass of corresponding $S = 1/2$ states reads then

$$2m_1 + m_3 + \frac{8}{3}\,(C_{12} - 4C_{13})\,. \tag{11}$$

The only two states that are mixed are the spin-1/2 states where all three flavors are different. Using the same spin coupling scheme as before, where the spin of the first two quarks is coupled to 1 ($\psi_1$) or 0 ($\psi_2$), and where the total spin is 1/2:

$$\begin{aligned} \psi_1 &= |(q_1q_2)_1\rangle \otimes |(q_3)_{1/2}\rangle\,, \\ \psi_2 &= |(q_1q_2)_0\rangle \otimes |(q_3)_{1/2}\rangle \end{aligned} \tag{12}$$

one easily finds the color–spin Hamiltonian over these two states as

$$H_{\rm CM} = \frac{8}{3}\begin{bmatrix} C_{12} - 2(C_{13} + C_{23}) & -\sqrt{3}(C_{13} - C_{23}) \\ -\sqrt{3}(C_{13} - C_{23}) & -3C_{12} \end{bmatrix}. \tag{13}$$

Now we see that if $C_{13} = C_{23}$ this matrix is also diagonal. If $q_1$ is the $u$-quark and $q_2$ is the $d$-quark, we would expect this to be approximately true. The same remark holds for the effective masses $m_1 \simeq m_2$. Then we note that in this approximation the largest eigenvalue leads to a mass

$$2m_1 + m_3 + \frac{8}{3}(C_{12} - 4C_{13}) \tag{14}$$

[d]If we added on $H_S = -\sum_{i,j} c_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ we would get the most general Hamiltonian we can have for the system of three quarks when the spatial variable are integrated out. So $H$ is more general than *one* gluon exchange only.

and that is the mass of both the $uuq_3$ and $ddq_3$ systems given by expression (11). Therefore we have *three* states in the octet made from quarks $u$, $d$ and $q_3$ ($q_3$ being $s$, $c$ or $b$), with the same mass (as they would have if we had an isospin 1 state and isospin invariance in nature).

The approximate isospin independence that sits in the QCD Lagrangian — due to the smallness of the (current) $u$ and $d$ mass compared to $\Lambda_{\rm QCD}$ — reappears in the masses of the three-quark bound states.

If we make the baryons out of the lightest quarks $u$, $d$ and $s$, we see that the states we called $\psi_1$ and $\psi_2$ are those commonly denoted $\Sigma_0$ and $\Lambda$.

The mixing between $\psi_1$ and $\psi_2$ — in this case called $\Lambda$–$\Sigma_0$ mixing — is induced by an (isospin breaking) inequality $C_{13} \neq C_{23}$.

So this type of mixing is the general one for all $\psi_1$ and $\psi_2$ states, it is quite small for *uds*, *udc*, *udb*, much bigger for states like *ucb*.

In the general case the eigenvectors for $H_{\rm cm}$ are

$$V_+ = \cos\theta \cdot \psi_1 + \sin\theta \cdot \psi_2\,, \qquad V_- = -\sin\theta \cdot \psi_1 + \cos\theta \cdot \psi_2$$

with corresponding eigenvalues for $H_{\rm cm}$:

$$\lambda_\pm = \frac{8}{3}\Big[-(C_{12} + C_{13} + C_{23}) \\ \pm 2\sqrt{C_{12}{}^2 - C_{12}C_{13} - C_{12}C_{23} + C_{13}{}^2 - C_{13}C_{23} + C_{23}{}^2}\,\Big]$$

and mixing angle $\theta$ given by

$$\tan(2\cdot\theta) = \frac{-\sqrt{3}\cdot(C_{13} - C_{23})}{2\cdot C_{12} - (C_{13} + C_{23})}\,. \tag{15}$$

For light ($q = u$, $d$ and $s$) quarks, baryon masses are well reproduced with masses

$$m_q = 360\,, \qquad m_s = 535\ \text{MeV}\,, \tag{16}$$

and strength factors

$$C_{qq} = 18.5\,, \qquad C_{qs} = 12.5\,, \qquad C_{ss} = 9.5\ \text{MeV}\,. \tag{17}$$

If one accepts that effective masses could differ when heavy quarks are present, a possible choice for $S$-wave charmed baryon states could be

$$m_c = 1550\,, \qquad m_q = 450\,, \qquad m_s = 590\ \text{MeV}\,, \tag{18}$$

with associated strength factors

$$\begin{aligned} C_{qq} &= 20\,, \quad C_{qc} = 5\,, \quad C_{qs} = 15\,, \\ C_{ss} &= 10\,, \quad C_{cs} = 4\,, \quad C_{cc} = 4\ \text{MeV}\,. \end{aligned} \tag{19}$$

The states we have constructed are considerably simpler than those one often[e] finds in textbooks and lecture notes. No mention is here of "mixed symmetry octet

[e]But not always!

states" or "generalized Pauli principle." The states are symmetrized only where the Pauli principle demands it, and can be used for all calculations where flavor is not changed.

If one wants to complicate calculations by symmetrizing in all three flavors, one is of course free to do so.[f]

Mathematically there is *indeed* a one to one correspondence between the states that we have used and the states that are symmetrized in all three particles. But to ask a student to compute the magnetic moment of the nucleons by using nine terms in the state, instead of three, is not very kind. That calculation is very simple indeed, using the magnetic moment operator for a baryon:

$$\boldsymbol{\mu} = \sum_i Q_i \cdot \mu_i \cdot \boldsymbol{\sigma}_i \,,$$

where in the definition of $\mu_i$ we have taken out the electric charge factor $Q_i$ of the quark $i$, but not the expected $\frac{1}{m_i \cdot c}$ dependence. All the $\mu_i$'s are therefore positive.

The expectation value of $\boldsymbol{\mu}$ for a $|B, S = 1/2, S_z = 1/2\rangle$ state, composed of two identical quarks $\alpha$ and a different one $\beta$ gives the value of the magnetic moment of the baryon $B$:

$$\mu_B = \left\langle B, \frac{1}{2}, \frac{1}{2} |\mu_z| B, \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{4}{3} \cdot Q_\alpha \cdot \mu_\alpha - \frac{1}{3} \cdot Q_\beta \cdot \mu_\beta \,.$$

With these conventions one gets for the proton and neutron: $\mu_P = \frac{8}{9}\mu_u + \frac{1}{9}\mu_d$, $\mu_N = -\frac{2}{9}\mu_u - \frac{4}{9}\mu_d$ and so on.

In the case where all three flavors are different, $q_1 = \alpha$, $q_2 = \beta$ and $q_3 = \gamma$, one finds for the two states:

$$\mu_{\psi_1} = \langle \psi_1 | \mu_z | \psi_1 \rangle = \frac{(2 \cdot Q_\alpha \cdot \mu_\alpha + 2 \cdot Q_\beta \cdot \mu_\beta - Q_\gamma \cdot \mu_\gamma)}{3} \,,$$

$$\mu_{\psi_2} = \langle \psi_2 | \mu_z | \psi_2 \rangle = Q_\gamma \cdot \mu_\gamma \,.$$

We include the off-diagonal matrix element for those that want to compute the effect of $\psi_1 \psi_2$ mixing:

$$\langle \psi_2 | \mu_z | \psi_1 \rangle = \langle \psi_1 | \mu_z | \psi_2 \rangle = \frac{(-Q_\alpha \cdot \mu_\alpha + Q_\beta \cdot \mu_\beta)}{\sqrt{3}} \,.$$

Neglecting small mixing, the magnetic moment of $\Lambda$ is given as $\mu_\Lambda = -\frac{1}{3}\mu_s$. A tolerable estimation of all the magnetic moments of the lightest baryons can be obtained by using as input the observed nucleon and $\Lambda$ values, i.e. $\mu_u = \mu_d = \mu_P$, $\mu_s = -3 \cdot \mu_\Lambda$.

The student will remark that $\mu_s < \mu_u$ as expected, but also that not all is well with this kind of calculation. An obvious problem is that experimentally $\mu_\Lambda - \mu_{\Xi^-}$ is

[f]In this case our notation is not optimal. The coefficients $C_{ij}$ should then be labeled by the quark flavors and not by numbers.

positive, whereas this conventional type of calculation gives $\mu_\Lambda - \mu_{\Xi^-} = (\mu_s - \mu_d)/9$ which should be negative!

We now turn to interactions that change flavor, and we shall see that permutation symmetry reestablishes something equivalent to the "generalized Pauli principle" without invoking any "principle," just the consistency of the Hilbert space.

## 5. Flavor Changing Forces and the Emergence of the Generalized Pauli Principle

Suppose now that we want to teach flavor changing weak decays, as it is the case in semileptonic hyperon decays. Let us take the example of an $s$-quark turning into a $u$-quark as in the $\Lambda \to Pe^-\bar{\nu}$ decay.

The state $\Lambda$ is constituted by three quarks of different flavors: $u$, $d$, and $s$, so the Pauli principle does not bring any constraint in this case. This resonance has a total spin $S = 1/2$, and its $(ud)$ part is of spin $S = 0$ when we ignore $\Lambda\Sigma_0$ mixing. We are therefore inclined to write it as

$$\Lambda = \frac{(u^\uparrow d^\downarrow - u^\downarrow d^\uparrow)s^\uparrow}{\sqrt{2}}.$$

In the transformation $s \to u$, the above expression changes into

$$\frac{(u^\uparrow d^\downarrow - u^\downarrow d^\uparrow)u^\uparrow}{\sqrt{2}}$$

and this state is not in the Hilbert space defined in the previous section! It is not a state allowed by the Pauli principle.

The result is a state vector where the first term is symmetric under the interchange of the two $u$-quarks — and that is fine — but the second term is not! As a consequence, our Hilbert space is not appropriate to admit the action of Hamiltonian corresponding to the weak $|\Lambda\rangle$ decays.

Looking at the list of semileptonic decays of low-lying spin-1/2 baryons, one realizes that the same kind of pathology is present as soon as the state $|abc\rangle$ is transformed into the state $|aac\rangle$, or inversely when the state $|aac\rangle$ is transformed into a combination of $|abc\rangle$ states, with $a$, $b$ and $c$ figuring out the quark flavors. But we see that this problem can be easily cured in the above considered $|\Lambda\rangle$ decay by choosing for $|\Lambda\rangle$ a wave function symmetric in $u$ and $s$:

$$\Lambda = \frac{(u^\uparrow d^\downarrow s^\uparrow + s^\uparrow d^\downarrow u^\uparrow - u^\downarrow d^\uparrow s^\uparrow - s^\uparrow d^\uparrow u^\downarrow)}{2}$$

and in the general case by imposing the baryon wave-functions to be completely symmetric in all the three quarks whether they were identical or not. The necessity of having a "complete" Hilbert space is what leads to the "generalized" Pauli principle.

Actually, it can be shown that the Hilbert space description in terms of completely symmetrized states in the three quarks is equivalent to the description in terms of states where the symmetry is imposed only for quarks of the same flavor. In other words, there is an isomorphism between the physics described in terms of two Hilbert spaces, the first one submitted to the usual Pauli principle and the second one submitted to what has been called the "generalized" Pauli principle. The mathematical proof is sketched in App. B.

Let us point out that, instead of the commonly adopted representation for the baryon wave-function (we call it "generalized Pauli symmetric" in App. D), one might prefer the equivalent, simpler and more elegant expressions offered by the Fock space formalism, as developed in App. C and explicated in App. D for the octet of $S = 1/2$ low-lying baryons.

Finally, combining the completely antisymmetric color part of the three-quark wave functions with its completely symmetric complementary part provides, of course, with a wave function which is completely antisymmetric in the three constituent fermions, and that is namely what is prescribed by the generalized Pauli principle.

## 6. Conclusion

We have shown that the classification of baryons, and reasonable estimates of their masses, their magnetic moments, as well as of the corresponding form factors relative to semileptonic decays can be obtained without any explicit reference to the flavor unitary group. All this can be included as well in fairly elementary quantum mechanics courses, as in elementary particle courses.

As widely explicated, the "minimal" or "partial" permutational symmetry imposed on wave functions by the Pauli principle is well adapted as long as flavor changing forces do not operate. When they do, one is naturally led to the choice of wave functions satisfying complete symmetry under permutation of the three constituent quarks, that is, in other words, to the application of the generalized Pauli principle.

At this point it is interesting to make a connection between the above discussion in Sec. 5 and the classification of baryons in irreducible representions of the SU($N$) group, $N$ being the number of different flavors. Actually, one knows that each irreducible representation $\mathcal{R}$ of the SU($N$) group can be represented by a Young tableau with $n$ boxes associated with an irreducible representation of the symmetric group $S_n$ also called permutation group $\mathcal{P}_n$ of $n$ elements. That means that any element of $\mathcal{R}$ satisfies a special $\mathcal{P}_n$ symmetry which is the same for all elements in $\mathcal{R}$. Taking as an example SU(2), the representations above denoted $S = 3/2$ and $S = 1/2$ resulting from the tensorial product of three times the fundamental representation $S = 1/2$ are respectively completely symmetric and "mixed symmetric" under $\mathcal{P}_3$. Now, considering the group SU(3), its irreducible eight-dimensional $\underline{8}$ and ten-dimensional $\underline{10}$ representations obtained by tensorial product of three fundamental representations $\underline{3}$ are respectively completely and mixed symmetric under $\mathcal{P}_3$. Complete symmetry under flavor and spin will therefore be obtained by combining the spin–SU(2) and flavor–SU(3) representations as follows: ($S = 1/2; \underline{8}$) and ($S = 3/2; \underline{10}$). And we can here recognize the decomposition of the completely symmetric $\underline{56}$ representation of the so-called flavor-spin SU(6) group commonly used to classify the $S$-wave baryons (in perfect accordance with results of Sec. 2). That also makes unitary groups so natural candidates for (broken) symmetry groups.

**Acknowledgments**

P. Sorba is indebted to P. Aurenche and E. Guadagnini for numerous and valuable discussions. We both must specially thank R. Stora for his precious help in some algebraic aspects developed in this paper, particularly in Apps. B and C, as well as for a critical reading of the manuscript.

## Appendix A. Jacobi Coordinates

The spatial wave functions $\psi$ of our problem are the harmonic-oscillator eigenfunctions $\psi_{nLM}(\boldsymbol{\rho}, \boldsymbol{\lambda})$, where $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ are the Jacobi coordinates which separate the Hamiltonian in Eq. (A.1) into two independent three-dimensional oscillators.

Hereafter, we give expressions for Jacobi coordinates that makes the separation of the center-of-mass motion in the case of unequal masses, and even when the interaction potential is different for each pair of particles.

So, starting from the Hamiltonian

$$H_{\text{h.o.}} = \sum_i \left(m_i + \frac{p_i^2}{2m_i}\right) + \sum_{i<j} \left(\frac{k_{ij}}{2}\right) \cdot (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{A.1}$$

with each coupling $k_{ij} = k_{ji}$ explicitly depending of the quark pair $q_i q_j$, one can rewrite the potential part as

$$\sum_{i<j} \left(\frac{k_{ij}}{2}\right) \cdot (\mathbf{r}_i - \mathbf{r}_j)^2 = \frac{1}{2}\bar{Y}\mathcal{M}Y$$

with

$$\bar{Y} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$$

and $\mathbf{y}_i$ defined as $\mathbf{y}_i = \sqrt{m_i}\mathbf{r}_i$, and the entries of the matrix $\mathcal{M}$ satisfying

$$\mathcal{M}_{ii} = \frac{k_{ij} + k_{ik}}{m_i} \qquad (i \neq j \neq k),$$

$$\mathcal{M}_{ij} = \mathcal{M}_{ji} = -\frac{k_{ij}}{(m_i m_j)^{\frac{1}{2}}} \qquad (i \neq j).$$

The diagonalization of $\mathcal{M}$ provides with one eigenvalue $\lambda_0 = 0$ associated to an eigenvector proportional to

$$\mathbf{R}_{\text{cm}} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3), \tag{A.2}$$

where $M = m_1 + m_2 + m_3$, and two nonzero eigenvalues $\lambda_\pm$ respectively associated to the eigenvectors:

$$\boldsymbol{\rho} = h_+\left(m_1^{1/2}\mathbf{r}_1 + m_2^{1/2}f_+\mathbf{r}_2 + m_3^{1/2}g_+\mathbf{r}_3\right) \tag{A.3}$$

and

$$\boldsymbol{\lambda} = h_-\left(m_1^{1/2}\mathbf{r}_1 + m_2^{1/2}f_-\mathbf{r}_2 + m_3^{1/2}g_-\mathbf{r}_3\right) \tag{A.4}$$

with

$$h_\pm = \sqrt{\frac{\lambda_\pm}{(3K)^{\frac{1}{2}}(1 + f_\pm^2 + g_\pm^2)}}\,,$$

$$K = k_{12}k_{13} + k_{21}k_{23} + k_{31}k_{32}$$

and

$$f_\pm = \sqrt{\frac{m_2}{m_1}}\,\frac{[k_{23} + k_{31} - m_3\lambda_\pm][k_{31} + k_{12} - m_1\lambda_\pm] - k_{31}^2}{k_{23}k_{31} + k_{12}[k_{23} + k_{31} - m_3\lambda_\pm]}\,,$$

$$g_\pm = \frac{\sqrt{m_3}}{k_{23}}\left[-\frac{k_{12}}{\sqrt{m_1}} + \frac{k_{23} + k_{12} - m_2\lambda_\pm}{\sqrt{m_2}}f_\pm\right].$$

Finally, the explicit expressions of $\lambda_\pm$ read

$$\lambda_\pm = \frac{1}{2}\left[\sum_{i;i\neq j\neq k\neq i}\frac{k_{ij} + k_{ik}}{m_i} \pm \sqrt{\left(\sum_{i;i\neq j\neq k\neq i}\frac{k_{ij} + k_{ik}}{m_i}\right)^2 - 4K\sum_{j<k}\frac{1}{m_j m_k}}\,\right].$$

In terms of $\mathbf{R}_{\rm cm}$, $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$, the total Hamiltonian acquires the following simple form in which the center-of-mass motion explicitly separates:

$$H_{\rm h.o.} = \sum_i m_i + \frac{p_{R_{\rm cm}}^2}{2M} + \frac{p_\rho^2}{2m_+} + \frac{p_\lambda^2}{2m_-} + \frac{\sqrt{3K}}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2)\,, \tag{A.5}$$

where $m_\pm = \frac{\sqrt{3K}}{\lambda_\pm}$.

It might be useful to consider the special case with only one coupling constant, that is $k_{ij} = k$ for $i, j = 1, 2, 3$. Then, $K$ becomes $= 3k^2$ and, in the Hamiltonian, the potential part reduces to $\frac{3}{2}k(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2)$. In the kinetic part, one gets $m_\pm = \frac{3k}{\lambda_\pm}$ and

$$\lambda_\pm = k\left[\sum_i\frac{1}{m_i} \pm \sqrt{\sum_i\frac{1}{m_i^2} - \sum_{i<j}\frac{1}{m_i m_j}}\,\right]$$

while $f_\pm$ and $g_\pm$ simplify as

$$f_\pm = \sqrt{\frac{m_2}{m_1}}\,\frac{4[1 - m_3(\lambda_\pm/2k)][1 - m_1(\lambda_\pm/2k)] - 1}{3 - 2m_3(\lambda_\pm/2k)}\,,$$

$$g_\pm = \sqrt{m_3}\left[-\frac{1}{\sqrt{m_1}} + 2\frac{1 - m_2(\lambda_\pm/2k)}{\sqrt{m_2}}f_\pm\right].$$

Another relevant case is the one with two identical quarks, that is $m_1 = m_2 = m$ and $m_3 = m'$. Then it is reasonable to have $k_{31} = k_{23} = k'$ and $k_{12} = k$, and the $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ Jacobi vectors simply become

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}\left[\frac{2k+k'}{3k'}\right]^{\frac{1}{4}}(\mathbf{r}_1 - \mathbf{r}_2)\,, \tag{A.6}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{2}}\left[\frac{k'}{3(2k+k')}\right]^{\frac{1}{4}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3)\,. \tag{A.7}$$

Finally, we give the limit case with all three masses equal $m_1 = m_2 = m_3 = m$ and $k_{12} = k_{23} = k_{31} = k$:

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2)\,, \tag{A.8}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3)\,. \tag{A.9}$$

## Appendix B. Generalized Pauli Principle vs Pauli Principle: A Mathematical Proof of Their Equivalence

This section is not directly dedicated to students. Of course, one can try and convince oneself on some examples that there is a one-to-one correspondence between the "mathematical" states and observables submitted to the generalized Pauli principle (or GPP) and those simply satisfying the Pauli principle (or PP). The proof that we propose hereafter has two advantages. First, it is general, and so adaptable to situations other than that of regular baryons (for example multiquark states). Second, it introduces the Fock space formalism in a rather natural way. All that follows is taken from Ref. 3. We start with some definitions:

Let $k$ be a commutative ring, $E$ a $k$-module, and $N$ the set of integers.

**Definition B.1 (Symmetric tensor algebra).** Denoting $T^n(E)$ the set of elements $z = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ with $x_1, \ldots, x_n \in M$, the action of $S_n$ on $T^n(E)$ reads

$$\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}}(2) \otimes \cdots \otimes x_{\sigma^{-1}(n)}\,. \tag{B.1}$$

The elements $z$ such that

$$\sigma \cdot z = z$$

are called symmetric tensors of order $n$. They form a sub-$k$-module of $T^n(E)$, denoted $TS^n(E)$. One sets

$$TS(E) = \bigoplus_{n=0}^{\infty} TS^n(E)$$

on which one can define a symmetric product (we do not give here the rule in order not to overload the text).

**Definition B.2 (Gamma algebra).** We call Gamma algebra of $E$ and denote by $\Gamma(E)$ the associative, unifer, commutative, algebra defined by the set of generators $N \times E$ and relations:

$$(0, x) = 1\,, \tag{B.2}$$

$$(p, \lambda x) = \lambda^p (p, x)\,, \tag{B.3}$$

$$(p, x + y) = \sum_{q=0}^{p} (p, x)(p - q, y)\,. \tag{B.4}$$

**Definition B.3 (Exponential type sequence).** We call exponential type sequence of $E$ elements a sequence $a = (a_p)_{p \in N}$ such that

$$a_0 = 1\,, \tag{B.5}$$

$$a_p a_q = \frac{(p+q)!}{p!q!}\, a_{p+q}\,. \tag{B.6}$$

Now, we come to the results we need.

**Proposition B.1.** *$\{(p, x)\}_{p \in N}$ with $x \in E$ is an exponential type sequence.*
*An interesting example of exponential type sequence is given by $f(x)$ with*

$$f(x)_p = \frac{1}{p!}\,(x)^p\,. \tag{B.7}$$

**Proposition B.2.** *There exists one and only one isomorphism $g$ between $\Gamma(E)$ and $TS(E)$:*

$$\Gamma(E) \cong TS(E) \tag{B.8}$$

*such that*

$$(g \circ \gamma_p)(x) = x \otimes x \otimes \cdots \otimes x \tag{B.9}$$

*with $p$ $x$-factors, and where we have denoted by $\gamma_p$, with $p \in N$, the application from $E$ into $\Gamma(E)$ product of the injection $x \to (p, x)$ and of the canonical homomorphism of the free commutative algebra $N \times E$ to $\Gamma(E)$.*

**Proposition B.3.** *Let $E$ and $F$ be two $k$ modules. There exists one and only one isomorphism $\phi$ from $\Gamma(E \times F)$ into $\Gamma(E) \otimes_k \Gamma(F)$:*

$$\Gamma(E \times F) \cong \Gamma(E) \otimes_k \Gamma(F) \tag{B.10}$$

*such that*

$$(\phi \circ \gamma_p)(x, y) = \sum_{q=0}^{p} \gamma_q(x) \otimes \gamma_{p-q}(y)\,. \tag{B.11}$$

The above properties can be reformulated by considering the case of polynomials. Indeed, one directly remarks that $\Gamma(E)$ can be seen as the set of polynomials on the dual $E^*$ of $E$, that is also

$$\Gamma(E) \cong \mathrm{Polyn}(E^*)\,. \tag{B.12}$$

Now, considering $E$ finite-dimensional, that is made of elements $\mathbf{x} = \sum_i x^i e_i \in E$ where $(e_i)_{i\in I}$ is a basis of $E$, with $x_i \in k$ for each $i \in I$, and $I$ being finite, then Proposition B.2 insures that

$$\Gamma(E) \cong \mathrm{Polyn}(E^*) \cong (\mathrm{Polyn}(E))^* \cong TS(E) \tag{B.13}$$

and from Proposition B.3 we get

$$\mathrm{Polyn}(\mathbf{x}, \mathbf{y}) = \mathrm{Polyn}(\mathbf{x}) \otimes \mathrm{Polyn}(\mathbf{y}) \tag{B.14}$$

with $\mathbf{x} \in E$ and $\mathbf{y} \in F$, keeping in mind the relations (B.3), (B.6) and (B.10).

Coming back to our flavored quarks, let us associate to each flavor $f = u, d, \ldots$ the $S = 1/2$ spin representation space $E_f$ generated by $f^\uparrow$, $f^\downarrow$. One can write

$$\Gamma(E_f) \cong \mathrm{Polyn}(f^\uparrow, f^\downarrow) \tag{B.15}$$

and also

$$\Gamma(E_f \times E_{f'}) \cong \Gamma(E_f) \otimes_R \Gamma(E_{f'}) \tag{B.16}$$

our states being defined on the ring of real numbers $R$.

## Appendix C. Fock Space Wave Functions

Actually, the above developed framework is nothing else than Fock space. Let us associate to each just above defined couple $f^\uparrow$, $f^\downarrow$ in $E_f$ the — commuting or "bosonic" — creator operators $a^\dagger_{f\uparrow}$, $a^\dagger_{f\downarrow}$, acting on the $|0\rangle$ vacuum. Again, we can write

$$\Gamma(E_f) \cong \mathrm{Polyn}(a^\dagger_{f\uparrow}, a^\dagger_{f\downarrow}) \tag{C.1}$$

and also

$$\mathrm{Polyn}(a^\dagger_{f\uparrow}, a^\dagger_{f\downarrow}) \otimes \mathrm{Polyn}(a^\dagger_{f'\uparrow}, a^\dagger_{f'\downarrow}) \cong \mathrm{Polyn}(a^\dagger_{f\uparrow}, a^\dagger_{f\downarrow}, a^\dagger_{f'\uparrow}, a^\dagger_{f'\downarrow})\,. \tag{C.2}$$

Due to the invariance under permutations of a monomial in $a^\dagger_f$ operators, one can get, for baryon states, expressions as simple as the ones used in Sec. 4 and denoted in App. D below "simply Pauli symmetric" — so much simpler than those called "generalized Pauli symmetric" in the same appendix — but unambiguously defined, as these last ones, for computations involving flavor changes. More precisely, it is rather immediate to translate a wave function written in our "Pauli symmetric basis" into the corresponding one in the Fock basis. One has just to take care of the respective normalization of states: as an example, the spin $S = 1$, $S_z = 0$ state $(u^\uparrow u^\downarrow + u^\downarrow u^\uparrow)/\sqrt{2}$ will correspond to $a^\dagger_{u\uparrow} a^\dagger_{u\downarrow}$, while $u^\uparrow u^\uparrow$ will be associated to $a^\dagger_{u^\uparrow} a^\dagger_{u^\uparrow}/\sqrt{2}$, using the usual normalized power product $(a^\dagger_f)^n/\sqrt{n!}$. An explicit comparison of the wave functions for $S = 1/2$ low-lying baryons in the different bases can be performed with App. D.

Finally, associating, as usual, to each creator $a^\dagger_f$ its annihilator counter part $a_f$ such that $a_f|0\rangle = 0$ and $(a^\dagger_f)^\dagger = a$, and satisfying the well-known Heisenberg algebra commutation relations:

$$\left[a_i, a^\dagger_j\right] = \delta_{ij} \,, \tag{C.3}$$

$$\left[a_i, a_j\right] = \left[a^\dagger_i, a^\dagger_j\right] = 0 \tag{C.4}$$

with $i$, $j$ representing any $f^\uparrow$ or $f^\downarrow$, simple expressions can be given for the self-adjoint parts of the Hamiltonian representing the flavor changing $f$ into $f'$:

$$H^{\pm}_{f\to f'} = \left(a^\dagger_{f'\uparrow} a_{f\uparrow} \pm a^\dagger_{f'\downarrow} a_{f\downarrow} + \text{h.c.}\right) \tag{C.5}$$

involved in the computations of the vector $f_1$ and axial $g_1$ form factors of baryon semileptonic decays.

It might look surprising to the reader that we use a bosonic Fock space although we are dealing with fermionic quarks. We should not forget that antisymmetry is carried by the color part. In other words, let us say that in the fermionic space of colored quarks, there exists a subspace constituted by three quark states which are color singlets — our baryons — this subspace being isomorphic, as a Hilbert space, to the three particles subspace of a bosonic Fock space.

## Appendix D. Wave Functions of Spin-1/2 *S*-Wave Baryons

Different expressions for the wave functions of baryons have been used following the kind of computations we had to consider. As long as classification and flavor non-changing quantities are involved, wave functions satisfying the symmetry imposed by the Pauli principle are perfectly adapted: we will call them "partial symmetric" or "simple Pauli symmetric" wave functions. As soon as flavor changing forces are involved, complete permutational invariance of the three quarks is required: we will denote them as "generalized Pauli symmetric" wave functions. In this last case, the use of the Fock space formalism, as seen in App. C, looks rather appealing: their natural denomination will be "Fock" wave functions. Let us once more mention that the color part, and so the antisymmetry nature of the baryon, has been "factored out," that is why one is concerned below only with permutational symmetry aspects, and in particular with bosonic Fock operators.

### *Simple Pauli symmetric wave functions*

$$N = \frac{\left[2d^\uparrow d^\uparrow u^\downarrow - (d^\uparrow d^\downarrow + d^\downarrow d^\uparrow)u^\uparrow\right]}{\sqrt{6}} \,,$$

$$P = \frac{\left[2u^{\uparrow}u^{\uparrow}d^{\downarrow} - (u^{\uparrow}u^{\downarrow} + u^{\downarrow}u^{\uparrow})d^{\uparrow}\right]}{\sqrt{6}},$$

$$\Lambda = \frac{\left[(u^{\uparrow}d^{\downarrow} - u^{\downarrow}d^{\uparrow})s^{\uparrow}\right]}{\sqrt{2}},$$

$$\Sigma^0 = \frac{\left[2u^{\uparrow}d^{\uparrow}s^{\downarrow} - (u^{\uparrow}d^{\downarrow} + u^{\downarrow}d^{\uparrow})s^{\uparrow}\right]}{\sqrt{6}},$$

$$\Sigma^+ = \frac{\left[2u^{\uparrow}d^{\uparrow}s^{\downarrow} - (u^{\uparrow}u^{\downarrow} + u^{\downarrow}u^{\uparrow})s^{\uparrow}\right]}{\sqrt{6}},$$

$$\Sigma^+ = \frac{\left[2d^{\uparrow}d^{\uparrow}s^{\downarrow} - (d^{\uparrow}d^{\downarrow} + d^{\downarrow}d^{\uparrow})s^{\uparrow}\right]}{\sqrt{6}},$$

$$\Xi^- = \frac{\left[2s^{\uparrow}s^{\uparrow}d^{\downarrow} - (s^{\uparrow}s^{\downarrow} + s^{\downarrow}s^{\uparrow})d^{\uparrow}\right]}{\sqrt{6}},$$

$$\Xi^0 = \frac{\left[2s^{\uparrow}s^{\uparrow}u^{\downarrow} - (s^{\uparrow}s^{\downarrow} + s^{\downarrow}s^{\uparrow})u^{\uparrow}\right]}{\sqrt{6}}.$$

***Generalized Pauli symmetric wave functions***

$$N = \frac{\left[2d^{\uparrow}d^{\uparrow}u^{\downarrow} - (d^{\uparrow}d^{\downarrow} + d^{\downarrow}d^{\uparrow})u^{\uparrow} + \text{permutations}\right]}{\sqrt{18}},$$

$$P = \frac{\left[2u^{\uparrow}u^{\uparrow}d^{\downarrow} - (u^{\uparrow}u^{\downarrow} + u^{\downarrow}u^{\uparrow})d^{\uparrow} + \text{permutations}\right]}{\sqrt{18}},$$

$$\Lambda = \frac{\left[(u^{\uparrow}d^{\downarrow} - u^{\downarrow}d^{\uparrow})s^{\uparrow} + \text{permutations}\right]}{\sqrt{12}},$$

$$\Sigma^0 = \frac{\left[2u^{\uparrow}d^{\uparrow}s^{\downarrow} - (u^{\uparrow}d^{\downarrow} + u^{\downarrow}d^{\uparrow})s^{\uparrow} + \text{permutations}\right]}{6},$$

$$\Sigma^+ = \frac{\left[2u^{\uparrow}d^{\uparrow}s^{\downarrow} - (u^{\uparrow}u^{\downarrow} + u^{\downarrow}u^{\uparrow})s^{\uparrow} + \text{permutations}\right]}{\sqrt{18}},$$

$$\Sigma^- = \frac{\left[2d^{\uparrow}d^{\uparrow}s^{\downarrow} - (d^{\uparrow}d^{\downarrow} + d^{\downarrow}d^{\uparrow})s^{\uparrow} + \text{permutations}\right]}{\sqrt{18}},$$

$$\Xi^- = \frac{\left[2s^{\uparrow}s^{\uparrow}d^{\downarrow} - (s^{\uparrow}s^{\downarrow} + s^{\downarrow}s^{\uparrow})d^{\uparrow} + \text{permutations}\right]}{\sqrt{18}},$$

$$\Xi^0 = \frac{\left[2s^{\uparrow}s^{\uparrow}u^{\downarrow} - (s^{\uparrow}s^{\downarrow} + s^{\downarrow}s^{\uparrow})u^{\uparrow} + \text{permutations}\right]}{\sqrt{18}}.$$

***Fock wave functions***

$$N = \frac{1}{\sqrt{3}}\left[a^\dagger_{d\uparrow}a^\dagger_{d\uparrow}a^\dagger_{u\downarrow} - a^\dagger_{d\uparrow}a^\dagger_{d\downarrow}a^\dagger_{u\uparrow}\right]|0\rangle\,,$$

$$P = \frac{1}{\sqrt{3}}\left[a^\dagger_{u\uparrow}a^\dagger_{u\uparrow}a^\dagger_{d\downarrow} - a^\dagger_{u\uparrow}a^\dagger_{u\downarrow}a^\dagger_{d\uparrow}\right]|0\rangle\,,$$

$$\Lambda = \frac{1}{\sqrt{2}}\left[a^\dagger_{u\uparrow}a^\dagger_{d\downarrow}a^\dagger_{s\uparrow} - a^\dagger_{u\downarrow}a^\dagger_{d\uparrow}a^\dagger_{s\uparrow}\right]|0\rangle\,,$$

$$\Sigma^0 = \frac{1}{\sqrt{6}}\left[2a^\dagger_{u\uparrow}a^\dagger_{d\uparrow}a^\dagger_{s\downarrow} - a^\dagger_{u\uparrow}a^\dagger_{d\downarrow}a^\dagger_{s\uparrow} - a^\dagger_{u\downarrow}a^\dagger_{d\uparrow}a^\dagger_{s\uparrow}\right]|0\rangle\,,$$

$$\Sigma^+ = \frac{1}{\sqrt{3}}\left[a^\dagger_{u\uparrow}a^\dagger_{u\uparrow}a^\dagger_{s\downarrow} - a^\dagger_{u\uparrow}a^\dagger_{u\downarrow}a^\dagger_{s\uparrow}\right]|0\rangle\,,$$

$$\Sigma^- = \frac{1}{\sqrt{3}}\left[a^\dagger_{d\uparrow}a^\dagger_{d\uparrow}a^\dagger_{s\downarrow} - a^\dagger_{d\uparrow}a^\dagger_{d\downarrow}a^\dagger_{s\uparrow}\right]|0\rangle\,,$$

$$\Xi^- = \frac{1}{\sqrt{3}}\left[a^\dagger_{s\uparrow}a^\dagger_{s\uparrow}a^\dagger_{d\downarrow} - a^\dagger_{s\uparrow}a^\dagger_{s\downarrow}a^\dagger_{d\uparrow}\right]|0\rangle\,,$$

$$\Xi^0 = \frac{1}{\sqrt{3}}\left[a^\dagger_{s\uparrow}a^\dagger_{s\uparrow}a^\dagger_{u\downarrow} - a^\dagger_{s\uparrow}a^\dagger_{s\downarrow}a^\dagger_{u\uparrow}\right]|0\rangle\,.$$

## References


1. B. R. Martin and G. Shaw, *Particle Physics* (John Wiley and Sons, Chichester, 1997).
2. J. Franklin, *Phys. Rev.* **172**, 1807 (1968).
3. N. Bourbaki, *Eléments de Mathématique, Algébre*, (Masson, Paris, 1981), Chap. 4–7, pp. 87–89.