Transverse Beam Dynamics

JUAS 2018 - tutorial 4 (solutions)

1 Exercise: Double-Bend Achromat (DBA) lattice

A Double-Bend Achromat (DBA) can be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is:

$$M_{\text{DBA}} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}}$$

Note that this magnet configuration does not produce vertical focusing, therefore it will not be enough to create a stable lattice. A full DBA typically comprises additional quadrupole doublets before and after the bending section. For sake of simplicity, we will neglect them.

1. Use the thin-lens approximation for quadrupoles and small-angle approximation for bends to find the dispersion in the middle of the quadrupole. Write the focal length in terms of the drift and bend parameters.

   Answer. Let us consider the 3×3 transfer matrices of each element of the lattice (using the thin lens approximation and small angle approximation for the bending magnets) for the beam coordinates x, $x'$and $\Delta p/p$:

$$M_{\text{bend}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\text{drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{1/2\text{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Assuming the initial dispersion vector $(\eta_0, \eta'_{0,} 1) = (0, 0, 1)$ and propagating it to the centre of the quadrupole:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Here we take into account that η'=0 at the centre of a quadrupole. After matrix multiplication we obtain:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & L\theta \\ -\frac{1}{2f} & 1 - \frac{L}{2f} & \theta\left(1 - \frac{L}{2f}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Therefore, one obtains:

$$\eta_c = L\theta$$

$$1 - \frac{L}{2f} = 0 \Rightarrow L = 2f$$

2. Show that the dispersion vanishes after the bend.

   Answer. Propagate the dispersion vector from the centre of the quadrupole to the end of the lattice:

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{2f} & L & 0 \\ -\frac{1}{2f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix}.$$

Taking into account $L = 2f$, we obtain:

$$\eta_{end} = (2f - L)\theta = 0,$$

$$\eta'_{end} = \theta - \frac{1}{2f}\eta_c = \theta - \frac{1}{2f}(2f\theta) = 0$$

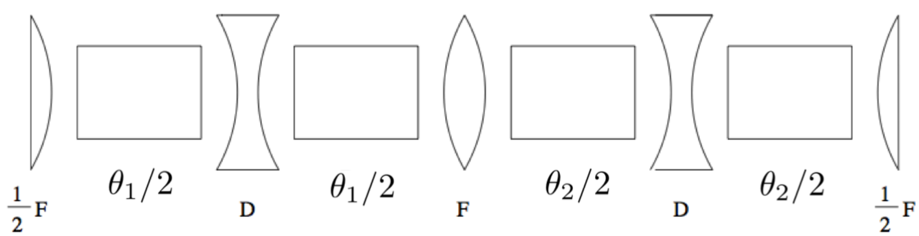
3. Compute the parameters L, f for a 10-meter long DBA which bends the beam by an angle of 1 radians. What is the dispersion in the centre? Given a particle with 1% energy deviation, compute the displacement at the centre of the cell.

   Answer.

$$L = 5 \text{ m}$$
$$f = 2.5 \text{ m}$$
$$D = L \cdot \theta = 5 \text{ m}$$
$$x = \delta D = 0.01 * 5 \text{ m} = 5 \text{ cm}$$

2 Exercise: Dispersion suppressor

In several straight sections of the accelerator, like the ones hosting RF cavities, extraction systems or other devices such as detectors, it is preferable to have no dispersion $\eta(s) = \eta'(s) = 0$. For example, in big colliders, such as the LHC, where small spot sizes are required at the interaction points, the dispersion is reduced to zero at the detector positions. The most common dispersion suppressors consists of two FODO cells of equal length L and equal quadrupole strengths. Bending magnets are placed in the space between the quadrupoles with a different bending field in each FODO. Figure below shows a typical dispersion suppressor.



1. Considering two FODO cells with different total bend angles, $\theta_1 \neq \theta_2$, calculate the relation between the angles θ_1 and θ_2 which must be satisfied to cancel the dispersion at the end of the lattice.

   Hint:

   For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation we have the following 3×3 matrix:

$$M_{\text{FODO } j} = \begin{pmatrix} 1-\frac{L^2}{8f^2} & L\left(1+\frac{L}{4f}\right) & \frac{L}{2}\left(1+\frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1-\frac{L}{4f}\right) & 1-\frac{L^2}{8f^2} & \left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & \frac{L}{2}\left(1+\frac{L}{8f}\right)\theta_j \\ -\frac{\sin\mu}{\beta} & \cos\mu & \left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell).

The following condition must be satisfied:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = M_{\text{suppressor}} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

where η_0 is the initial dispersion (at the middle of the first focusing quadrupole). It can be demonstrated that for a FODO lattice the dispersion has its maximum at the middle of the focusing quadrupole:

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta \tag{2}$$

with $\theta = \theta_1 + \theta_2$ the total bend in the suppressor.

Answer.

Performing the corresponding matrix multiplication yields

$$M_{\text{suppressor}} = \begin{pmatrix} \cos 2\mu & \beta \sin 2\mu & D_x \\ -\frac{\sin 2\mu}{\beta} & \cos 2\mu & D_x' \\ 0 & 0 & 1 \end{pmatrix}$$

where:

$$\begin{aligned}
\cos 2\mu &= 1 - \frac{L^2}{2f^2} + \frac{L^4}{34f^4} \\
\beta \sin 2\mu &= 2L\left(1 - \frac{L^2}{8f^2}\right)\left(1 + \frac{L}{4f}\right) \\
\frac{\sin 2\mu}{\beta} &= \frac{L}{2f^2}\left(1 - \frac{L^2}{8f^2}\right)\left(1 - \frac{L}{4f}\right) \\
D_x &= \cos\mu \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \beta \sin\mu \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_2 \\
D_x' &= -\frac{\sin\mu}{\beta}\frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \cos\mu\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \left(1 - \frac{L}{8f^2} - \frac{L^2}{32f^2}\right)\theta_2
\end{aligned} \tag{3}$$

Taking into account:

$$\cos\mu = 1 - \tfrac{L^2}{8f^2};\ \beta \sin\mu = L + \tfrac{L^2}{4f} \text{ and } \tfrac{\sin\mu}{\beta} = \tfrac{1}{4f^2}\left(1 - \tfrac{L}{4f}\right)$$

the elements D_x and D_x'may also be written as

$$\begin{aligned}
D_x &= \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \\
D_x' &= \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]
\end{aligned} \tag{4}$$

From the condition Eq. (1) we have

$$\begin{aligned}
\eta_0 \cos 2\mu + D_x &= 0 \\
-\eta_0 \frac{\sin 2\mu}{\beta} + D_x' &= 0
\end{aligned} \tag{5}$$

Substituting Eq. (2) in Eq. (5) one obtains:

$$\left(3-\frac{L^2}{4f^2}\right)\theta_1+\theta_2=\left(4-\frac{L^2}{4f^2}-\frac{8f^2}{L^2}\right)\theta$$
$$\left(1-\frac{L^2}{4f^2}\right)\theta_1+\theta_2=\left(2-\frac{L^2}{4f^2}\right)\theta$$

In terms of phase advance μ this can be written as:

$$\begin{aligned}\theta_1&=\left(1-\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta\\ \theta_2&=\frac{1}{4\sin^2\frac{\mu}{2}}\theta\end{aligned} \tag{6}$$

where $\theta_1+\theta_2=\theta$.

2. Obtain the relation between the angles for the cases of phase-advance per cell $\mu=\pi/3$ and $\pi/2$

Answer.

- For $\mu=\pi/3 \longrightarrow 4\sin^2\frac{\mu}{2}=1$ and therefore (using Eq. (6)) $\theta_1=0$ and $\theta_2=\theta$. This corresponds to a dispersion suppressor with missing magnets.
- For $\mu=\pi/2 \longrightarrow 4\sin^2\frac{\mu}{2}=2$ and therefore $\theta_1=\theta_2=\theta/2$.