This course started with the one of Frank Tecker (CERN-BE) in 2010 (I took over from him in 2011), who inherited it from Roberto Corsini (CERN-BE), who gave this course in the previous years, based on the transparencies written by Louis Rinolfi (CERN-BE) who held the course at JUAS from 1994 to 2002 (see CERN/PS 2000-008 (LP)):


Material from Joel LeDuff’s Course at the CERN Accelerator School held at Jyvaskyla, Finland the 7-18 September 1992 (CERN 94-01) has been used as well:


I attended the course given by Louis Rinolfi in 1996 and was his assistant in 2000 and 2001 (and the assistant of Michel Martini for his course on transverse beam dynamics)

This course and related exercises / exams (as well as other courses) can be found in my web page: http://emetral.web.cern.ch/emetral/

Assistant: Benoit Salvant (CERN BE Department)
PURPOSE OF THIS COURSE

Discuss the oscillations of the particles in the longitudinal plane of synchrotrons, called **SYNCHROTRON OSCILLATIONS** (similarly to the betatron oscillations in the transverse planes), and derive the basic equations.

Example of the LHC p beam in the injector chain:

\[ 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \] (energy gain by an e- accelerated by a potential difference of 1 Volt)
PURPOSE OF THIS COURSE
PURPOSE OF THIS COURSE

IN REAL SPACE

IN PHASE SPACE

Horizontal

Vertical

Courtesy of A.W. Chao

\[ \delta = \frac{\Delta p}{p} \]

One particle

Circular design orbit

Single-particle trajectory

In the middle of the vacuum chamber

Longitudinal, bunched beam, below transition

Longitudinal, unbunched beam, below transition

Longitudinal, bunched beam, above transition

Longitudinal, unbunched beam, above transition
PURPOSE OF THIS COURSE

Some movies (in phase space) to have a better idea of what we will work on during this course and what you will be able to understand and do after this course...
“MATCHED” AND “MISMATCHED” BUNCH
MATCHED BUNCH

Surface = Longitudinal EMITTANCE of the bunch
= \varepsilon_L

Surface = Longitudinal ACCEPTANCE of the RF bucket
MISMATCHED BUNCH

Leads to an increase of the longitudinal emittance and/or particle losses
MISMatched Bunch

Energy profile [a.u.]
MISMATCHED BUNCH

Energy profile [a.u.]
SOME “RF GYMNASTICS”
BUNCH ROTATION
(to shorten bunches before extraction)
BUNCH (DOUBLE) SPLITTING

Energy profile [a.u.]
BUNCH MERGING
(reverse process)
BUNCH TRIPLE SPLITTING

Energy profile [a.u.]
**JUAS - TIMETABLE 2018 - WEEK 3**

<table>
<thead>
<tr>
<th>Schedule 2018</th>
<th>Monday Jan 22nd</th>
<th>Tuesday Jan 23rd</th>
<th>Wednesday Jan 24th</th>
<th>Thursday Jan 25th</th>
<th>Friday Jan 26th</th>
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<tbody>
<tr>
<td>09:00</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Linear Imperfections lecture</td>
</tr>
<tr>
<td></td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>H. Bartosik</td>
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<tr>
<td>10:00 - 10:15</td>
<td>Coffee Break</td>
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<td>Coffee Break</td>
<td>Coffee Break</td>
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<tr>
<td>11:15</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Linear Imperfections lecture</td>
</tr>
<tr>
<td></td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>H. Bartosik</td>
</tr>
<tr>
<td>12:15</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Longitudinal Dynamics lecture</td>
<td>Non-linear effects lecture</td>
</tr>
<tr>
<td></td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>E. Métral/B. Salvant</td>
<td>H. Bartosik</td>
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<tr>
<td>13:00 - 14:00</td>
<td>Working Lunch</td>
<td>Break</td>
<td>Break</td>
<td>Break</td>
<td>Break</td>
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<tr>
<td>14:00</td>
<td>Linear Imperfections lecture</td>
<td>Linear Imperfections lecture</td>
<td>Linacs lecture</td>
<td>Linacs tutorial</td>
<td>Non-linear effects lecture</td>
</tr>
<tr>
<td></td>
<td>H. Bartosik</td>
<td>H. Bartosik</td>
<td>J-B. Lallemand</td>
<td>J-B. Lallemand</td>
<td>H. Bartosik</td>
</tr>
<tr>
<td>15:00</td>
<td>Linear Imperfections lecture</td>
<td>Linear Imperfections lecture</td>
<td>Linacs lecture</td>
<td>Linacs tutorial</td>
<td>Non-linear effects lecture</td>
</tr>
<tr>
<td></td>
<td>H. Bartosik</td>
<td>H. Bartosik</td>
<td>J-B. Lallemand</td>
<td>J-B. Lallemand</td>
<td>H. Bartosik</td>
</tr>
<tr>
<td>16:00 - 16:15</td>
<td>Coffee Break</td>
<td>Coffee Break</td>
<td>Coffee Break</td>
<td>Coffee Break</td>
<td>Coffee Break</td>
</tr>
<tr>
<td>16:15</td>
<td>The neutrino physics programme</td>
<td>Free-Electron Lasers Seminar</td>
<td>Linacs lecture</td>
<td>Linacs tutorial</td>
<td>Non-linear effects lecture</td>
</tr>
<tr>
<td></td>
<td>Alain Blondel, CERN &amp; U. of Geneva</td>
<td>E. Prat</td>
<td>J-B. Lallemand</td>
<td>J-B. Lallemand</td>
<td>H. Bartosik</td>
</tr>
<tr>
<td>17:15</td>
<td>+ Examination on WE 07/02/2018 (09:00 to 10:30)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**LESSON I**

Fields & forces
- Acceleration by time-varying electric field
- Relativistic equations

**LESSON II**

Particle acceleration => Synchrotrons
- Transit time factor
- Main RF parameters
- Momentum compaction
- Transition energy

**LESSON III**

Equations related to synchrotrons
- Synchronous particle
- Synchrotron oscillations
- Principle of phase stability

**LESSON IV**

RF acceleration for synchronous particle
- RF acceleration for non-synchronous particle
- Small amplitude oscillations
- Large amplitude oscillations - the RF bucket
- Synchrotron frequency and tune
- Tracking
- Nonadiabatic theory needed “close” to transition
- Double RF systems

**LESSON V**

Measurement of the longitudinal bunch profile and Tomography
- The pyHEADTAIL simulation code (by Benoit Salvant)
# Units of physical quantities

<table>
<thead>
<tr>
<th>Quantity</th>
<th>unit</th>
<th>SI unit</th>
<th>SI derived unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitance</td>
<td>F (farad)</td>
<td>m⁻² kg⁻¹s⁴A²</td>
<td>C/V</td>
</tr>
<tr>
<td>Electric charge</td>
<td>C (coulomb)</td>
<td>As</td>
<td></td>
</tr>
<tr>
<td>Electric potential</td>
<td>V (volt)</td>
<td>m² kg s⁻³A⁻¹</td>
<td>W/A</td>
</tr>
<tr>
<td>Energy</td>
<td>J (joule)</td>
<td>m² kg s⁻²</td>
<td>Nm</td>
</tr>
<tr>
<td>Force</td>
<td>N (newton)</td>
<td>m kg s⁻²</td>
<td>N</td>
</tr>
<tr>
<td>Frequency</td>
<td>Hz (hertz)</td>
<td>s⁻¹</td>
<td></td>
</tr>
<tr>
<td>Inductance</td>
<td>H (henry)</td>
<td>m² kg s⁻²A⁻²</td>
<td>Wb/A</td>
</tr>
<tr>
<td>Magnetic flux</td>
<td>Wb (weber)</td>
<td>m² kg s⁻²A⁻¹</td>
<td>Vs</td>
</tr>
<tr>
<td>Magnetic flux density</td>
<td>T (tesla)</td>
<td>kg s⁻²A⁻¹</td>
<td>Wb/m²</td>
</tr>
<tr>
<td>Power</td>
<td>W (watt)</td>
<td>m² kg s⁻³</td>
<td>J/s</td>
</tr>
<tr>
<td>Pressure</td>
<td>Pa (pascal)</td>
<td>m⁻¹ kg s⁻²</td>
<td>N/m²</td>
</tr>
<tr>
<td>Resistance</td>
<td>Ω (ohm)</td>
<td>m² kg s⁻³A⁻²</td>
<td>V/A</td>
</tr>
</tbody>
</table>
### Fundamental physical constants

<table>
<thead>
<tr>
<th>Physical constant</th>
<th>symbol</th>
<th>value</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avogadro’s number</td>
<td>$N_A$</td>
<td>$6.0221367 \times 10^{23}$</td>
<td>/mol</td>
</tr>
<tr>
<td>atomic mass unit ($\frac{1}{12} m(C^{12})$)</td>
<td>$m_u$ or $u$</td>
<td>$1.6605402 \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td>Boltzmann’s constant</td>
<td>$k$</td>
<td>$1.380658 \times 10^{-23}$</td>
<td>J/K</td>
</tr>
<tr>
<td>Bohr magneton</td>
<td>$\mu_B = e\hbar/2m_e$</td>
<td>$9.2740154 \times 10^{-24}$</td>
<td>J/T</td>
</tr>
<tr>
<td>Bohr radius</td>
<td>$a_0 = \frac{e^2}{4\pi\epsilon_0}\hbar^2/m_e \epsilon_0 c^2$</td>
<td>$0.529177249 \times 10^{-10}$</td>
<td>m</td>
</tr>
<tr>
<td>classical radius of electron</td>
<td>$r_e = e^2/4\pi\epsilon_0 m_e c^2$</td>
<td>$2.81794092 \times 10^{-15}$</td>
<td>m</td>
</tr>
<tr>
<td>classical radius of proton</td>
<td>$r_p = e^2/4\pi\epsilon_0 m_p c^2$</td>
<td>$1.5346986 \times 10^{-18}$</td>
<td>m</td>
</tr>
<tr>
<td>elementary charge</td>
<td>$e$</td>
<td>$1.60217733 \times 10^{-19}$</td>
<td>C</td>
</tr>
<tr>
<td>fine structure constant</td>
<td>$\alpha = e^2/2\epsilon_0 \hbar c$</td>
<td>$1/137.0359895$</td>
<td></td>
</tr>
<tr>
<td>$m_u c^2$</td>
<td></td>
<td>$931.49432$ MeV</td>
<td></td>
</tr>
<tr>
<td>mass of electron</td>
<td>$m_e$</td>
<td>$9.1093897 \times 10^{-31}$</td>
<td>kg</td>
</tr>
<tr>
<td>$m_e c^2$</td>
<td></td>
<td>$0.51099906$ MeV</td>
<td></td>
</tr>
<tr>
<td>mass of proton</td>
<td>$m_p$</td>
<td>$1.6726313 \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td>$m_p c^2$</td>
<td></td>
<td>$938.27231$ MeV</td>
<td></td>
</tr>
<tr>
<td>mass of neutron</td>
<td>$m_n$</td>
<td>$1.6749286 \times 10^{-27}$</td>
<td>kg</td>
</tr>
<tr>
<td>$m_n c^2$</td>
<td></td>
<td>$939.56563$ MeV</td>
<td></td>
</tr>
<tr>
<td>molar gas constant</td>
<td>$R = N_A k$</td>
<td>$8.314510$ J/mol K</td>
<td></td>
</tr>
<tr>
<td>neutron magnetic moment</td>
<td>$\mu_n$</td>
<td>$-0.96623707 \times 10^{-26}$</td>
<td>J/T</td>
</tr>
<tr>
<td>nuclear magneton</td>
<td>$\mu_p = e\hbar/2m_u$</td>
<td>$5.0507866 \times 10^{-27}$</td>
<td>J/T</td>
</tr>
<tr>
<td>Planck’s constant</td>
<td>$\hbar$</td>
<td>$6.626075 \times 10^{-34}$</td>
<td>J s</td>
</tr>
<tr>
<td>permeability of vacuum</td>
<td>$\mu_0$</td>
<td>$4\pi \times 10^{-7}$ N/A²</td>
<td></td>
</tr>
<tr>
<td>permittivity of vacuum</td>
<td>$\epsilon_0$</td>
<td>$8.854187817 \times 10^{-12}$</td>
<td>F/m</td>
</tr>
<tr>
<td>proton magnetic moment</td>
<td>$\mu_p$</td>
<td>$1.41060761 \times 10^{-26}$</td>
<td>J/T</td>
</tr>
<tr>
<td>proton $g$ factor</td>
<td>$g_p = \mu_p/\mu_N$</td>
<td>$2.792847386$</td>
<td></td>
</tr>
<tr>
<td>speed of light (exact)</td>
<td>$c$</td>
<td>$299792458$ m/s</td>
<td></td>
</tr>
<tr>
<td>vacuum impedance</td>
<td>$Z_0 = 1/\epsilon_0 c = \mu_0 c$</td>
<td>$376.7303$ Ω</td>
<td></td>
</tr>
</tbody>
</table>
LESSON I

Fields & forces

Acceleration by time-varying electric field

Relativistic equations
Fields and force

Equation of motion for a particle of charge $q$

$$\vec{F} = \frac{d\vec{p}}{dt} = q \left( \vec{E} + \vec{v} \times \vec{B} \right)$$

$\vec{p}$: Momentum
$\vec{v}$: Velocity
$\vec{E}$: Electric field
$\vec{B}$: Magnetic field
**Constant electric field**

1. Direction of the force always parallel to the field
2. Trajectory can be modified, velocity also $\Rightarrow$ momentum and energy can be modified

$$\frac{d\vec{p}}{dt} = -e \vec{E}$$

This force can be used to accelerate and decelerate particles.
Constant magnetic field

\[ \frac{d\vec{p}}{dt} = \vec{F} = -e (\vec{v} \times \vec{B}) \]

1. Direction always perpendicular to the velocity
2. Trajectory can be modified, but not the velocity

\[ e \nu B = \frac{m \nu^2}{\rho} \]

This force cannot modify the energy

magnetic rigidity: \[ B \rho = \frac{p}{e} \]

angular frequency: \[ \omega = 2\pi f = \frac{e}{m} B \]
**Important relationship:**

\[ B \, \rho = \frac{p}{e} \quad \rightarrow \quad \rho = \frac{p}{e \, B} \]

**Practical units:**

\[ B \, \rho \, [\text{Tm}] \approx \frac{p \, [\text{GeV/c}]}{0.3} \]

**Application: spectrometer**
Comparison of magnetic and electric forces

\[ |\vec{B}| = 1 \text{T} \]

\[ |\vec{E}| = 10 \text{ MV/m} \]

\[
\frac{F_{MAGN}}{F_{ELEC}} = \frac{e v B}{e E} = \beta c \frac{B}{E} \approx 3 \times 10^8 \frac{1}{10^7} \beta = 30 \beta
\]
**Acceleration by time-varying electric field**

- Let $V_{RF}$ be the amplitude of the RF voltage across the gap $g$
- The particle crosses the gap at a distance $r$
- The energy gain is:

$$\Delta E = e \int_{g/2}^{g/2} \vec{E}(s, r, t) \, ds$$

In the cavity gap, the electric field is supposed to be:

$$E(s, r, t) = E_1(s, r) \cdot E_2(t)$$

In general, $E_2(t)$ is a sinusoidal time variation with angular frequency $\omega_{RF}$

$$E_2(t) = E_0 \sin \Phi(t) \quad \text{where} \quad \Phi(t) = \int_{t_0}^{t} \omega_{RF} \, dt + \Phi_0$$
Convention

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time $t=0$ chosen such that:

1. $E_2(t) = E_0 \sin(\omega_{RF} t)$

2. $E_2(t) = E_0 \cos(\omega_{RF} t)$
Relativistic Equations

\[ E = m c^2 \]

- **Normalized velocity**
  \[ \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \]

- **Total energy**
  \[ \gamma = \frac{E}{E_0} = \frac{m}{m_0} \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \]

- **Momentum**
  \[ p = m v = \beta \frac{E}{c} = \beta \gamma m_0 c \]

- **Energy**
  \[ E = E_{\text{kin}} + E_0 \]

- **Rest energy**
  \[ E_0 = m_0 c^2 \]

- **Total** (kinetic + rest)
  \[ p^2 c^2 = E^2 - E_0^2 \]

- **Mass**
  \[ \gamma = 1 + \frac{E_{\text{kin}}}{E_0} \]

- **Energy per unit mass**
  \[ p \text{[GeV/c]} \approx 0.3 \quad B[T] \quad \rho[m] \]
normalized velocity

\[ \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \]

total energy

rest energy

\[ \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} \]

protons

electrons

protons

electrons
First derivatives

\[ d\beta = \beta^{-1} \gamma^{-3} \, d\gamma \]
\[ d(cp) = E_0 \gamma^3 \, d\beta \]
\[ d\gamma = \beta(1 - \beta^2)^{3/2} \, d\beta \]

Logarithmic derivatives

\[ \frac{d\beta}{\beta} = (\beta \gamma)^{-2} \frac{d\gamma}{\gamma} \]
\[ \frac{dp}{p} = \frac{\gamma^2}{\gamma^2 - 1} \frac{dE}{E} = \frac{\gamma}{\gamma + 1} \frac{dE_{kin}}{E_{kin}} \]
\[ \frac{d\gamma}{\gamma} = (\gamma^2 - 1) \frac{d\beta}{\beta} \]
LESSON II

Particle acceleration => Synchrotrons

Transit time factor

Main RF parameters

Momentum compaction

Transition energy
1. $\omega_{\text{RF}}$ and $\omega$ increase with energy

2. To keep particles on the closed orbit, $B$ should increase with time
In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc..

Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection.

The bending radius $\rho$ does not coincide to the machine radius $R = L/2\pi$. 

**Synchrotron**
Examples of different proton and electron synchrotrons at CERN
Parameters for circular accelerators

The basic principles, for the common circular accelerators, are based on the two relations:

1. The Lorentz equation: the orbit radius can be expressed as:

   \[ R = \frac{\gamma v m_0}{eB} \]

2. The synchronicity condition: The revolution frequency can be expressed as:

   \[ f = \frac{eB}{2\pi \gamma m_0} \]

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarized in the table below:

<table>
<thead>
<tr>
<th>Machine</th>
<th>Energy ((\gamma))</th>
<th>Velocity</th>
<th>Field</th>
<th>Orbit</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyclotron</td>
<td>~ 1</td>
<td>var.</td>
<td>const.</td>
<td>~ v</td>
<td>const.</td>
</tr>
<tr>
<td>Synchrocyclotron</td>
<td>var.</td>
<td>var.</td>
<td>B(r)</td>
<td>~ p</td>
<td>B(r)/(\gamma(t))</td>
</tr>
<tr>
<td>Proton/Ion synchrotron</td>
<td>var.</td>
<td>var.</td>
<td>~ p</td>
<td>R</td>
<td>~ v</td>
</tr>
<tr>
<td>Electron synchrotron</td>
<td>var.</td>
<td>const.</td>
<td>~ p</td>
<td>R</td>
<td>const.</td>
</tr>
</tbody>
</table>
**Transit time factor**

RF acceleration in a gap $g$

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

**Simplified model**

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

$$E_2(t) = \sin(\omega_{RF} t + \phi_0)$$

At $t = 0, s = 0$ and $v \neq 0$, parallel to the electric field

Energy gain:

$$\Delta E = e \int_{-g/2}^{g/2} E(s,r,t) \, ds$$

$$\Delta E = e V_{RF} T_a \sin \phi_0$$

$T_a$ is called transit time factor

$\cdot T_a < 1$

$\cdot T_a \to 1$ if $g \to 0$

where

$$T_a = \frac{\sin \frac{\omega_{RF} g}{2v}}{\frac{\omega_{RF} g}{2v}}$$
Transit time factor II

In the general case, the transit time factor is given by:

\[
T_a = \frac{\int_{-\infty}^{+\infty} E_1(s, r) \cos\left(\omega_{RF} \frac{s}{v}\right) ds}{\int_{-\infty}^{+\infty} E_1(s, r) ds}
\]

It is the ratio of the peak energy gained by a particle with velocity \(v\) to the peak energy gained by a particle with infinite velocity.
I. Voltage, phase, frequency

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration

\[ E(s,t) = E_1(s) \cdot E_2(t) \]
\[ E_2(t) = E_0 \sin \left[ \int_{0}^{t} \omega_{RF} \, dt + \phi_0 \right] \]
\[ \omega_{RF} = 2\pi f_{RF} \]
\[ \Delta E = e V_{RF} T_a \sin \phi_0 \]

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase

\[ T_{rev} = h T_{RF} \quad \Rightarrow \quad f_{RF} = h f_{rev} \]

II. Harmonic number

- \( f_{rev} \) = revolution frequency
- \( f_{RF} \) = frequency of the RF
- \( h \) = harmonic number

 harmonic number in different machines:

\[
\begin{array}{ccccc}
\text{AA} & \text{EPA} & \text{PS} & \text{SPS} \\
1 & 8 & 20 & 4620 \\
\end{array}
\]
**Dispersion**

Reference = design = nominal trajectory = closed orbit (circular machine)

\[
x(s) = D_x(s) \frac{\Delta p}{p}
\]
**Momentum compaction factor in a transport system**

In a particle transport system, a nominal trajectory is defined for the nominal momentum $p$.

For a particle with a momentum $p + \Delta p$ the trajectory length can be different from the length $L$ of the nominal trajectory.

The momentum compaction factor is defined by the ratio:

\[
\alpha_p = \frac{dL/L}{dp/p}
\]

Therefore, for small momentum deviation, to first order it is:

\[
\frac{\Delta L}{L} = \alpha_p \frac{\Delta p}{p}
\]
By definition of dispersion $D_x$

$ds = \rho \, d\theta$

$ds_1 = (\rho + x) \, d\theta$

$$\frac{ds_1 - ds}{ds} = \frac{(\rho + x) \, d\theta - \rho \, d\theta}{\rho \, d\theta} = \frac{x}{\rho} = \frac{D_x \, dp}{\rho \, p}$$

To first order, only the bending magnets contribute to a change of the trajectory length ($r = \infty$ in the straight sections)


**Momentum compaction in a ring**

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum \( p \).

For a particle with a momentum deviation \( \Delta p \) produces an orbit length variation \( \Delta C \) with:

\[
\frac{\Delta C}{C} = \alpha_p \frac{\Delta p}{p} \quad \text{and} \quad C = 2\pi R
\]

The momentum compaction factor is defined by the ratio:

\[
\alpha_p = \frac{dC}{dp} \frac{1}{C} = \frac{dR}{dp} \frac{1}{R} \quad \text{and} \quad \alpha_p = \frac{1}{C} \int_C \frac{D_x(s)}{\rho(s)} \, ds
\]

N.B.: in most circular machines, \( \alpha_p \) is positive \( \Rightarrow \) higher momentum means longer circumference.
Momentum compaction as a function of energy

\[
E = \frac{pc}{\beta} \quad \rightarrow \quad \frac{dE}{E} = \beta^2 \frac{dp}{p}
\]

\[
\alpha_p = \beta^2 \frac{E}{R} \frac{dR}{dE}
\]
Momentum compaction as a function of magnetic field

Definition of average magnetic field

\[
< B > = \frac{1}{2\pi R} \int B_f \, ds = \frac{1}{2\pi R} \left( \int_{\text{straights}} B_f \, ds + \int_{\text{magnets}} B_f \, ds \right)
\]

\[
< B > = \frac{B_f \rho}{R}
\]

\[
B_f \rho = \frac{p}{e}
\]

\[
< B > R = \frac{p}{e}
\]

\[
\frac{d}{d\rho} \frac{d < B >}{< B >} + \frac{d}{dR} \frac{d < B >}{R} = \frac{d}{dp} < B >
\]

For \( B_f = \text{const} \),

\[
\alpha_p = 1 - \frac{d < B >}{< B >} \left/ \frac{d}{dp} < B > \right.
\]
Transition energy

Proton (ion) circular machine with $\alpha_p$ positive

1. Momentum larger than the nominal ($p + \Delta p$) $\Rightarrow$ longer orbit ($C + \Delta C$)

2. Momentum larger than the nominal ($p + \Delta p$) $\Rightarrow$ higher velocity ($v + \Delta v$)

What happens to the revolution frequency $f = \frac{v}{C}$?

- At low energy, $v$ increases faster than $C$ with momentum
- At high energy $v \approx c$ and remains almost constant

There is an energy for which the velocity variation is compensated by the trajectory variation $\Rightarrow$ transition energy

Below transition: higher energy $\Rightarrow$ higher revolution frequency
Above transition: higher energy $\Rightarrow$ lower revolution frequency
Transition energy - quantitative approach

We define a parameter $\eta$ (revolution frequency spread per unit of momentum spread), called slip or slippage factor:

$$\eta = \frac{\frac{df}{df}}{\frac{dp}{p}} = \frac{d\omega/\omega}{dp/p}$$

$$f = \frac{v}{C} \quad \Rightarrow \quad \frac{df}{f} = \frac{d\beta/\beta}{\beta} - \frac{dC/C}{C}$$

from \( p = \frac{m_0 c \beta}{\sqrt{1 - \beta^2}} \)

$$\frac{d\beta}{\beta} = \frac{1}{\gamma^2} \frac{dp}{p}$$

definition of momentum compaction factor:

$$\frac{dC}{C} = \alpha_p \frac{dp}{p}$$

$$\frac{df}{f} = \left( \frac{1}{\gamma^2} - \alpha_p \right) \frac{dp}{p}$$
Transition energy - quantitative approach

\[ \eta = \frac{1}{\gamma^2} - \alpha_p \]

The transition energy is the energy that corresponds to \( \eta = 0 \)
(\( \alpha_p \) is fixed, and \( \gamma \) variable)

\[ \gamma_{tr} = \sqrt{\frac{1}{\alpha_p}} \]

The parameter \( \eta \) can also be written as

\[ \eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2} \]

• At low energy \( \eta > 0 \)
• At high energy \( \eta < 0 \)

N.B.: for electrons, \( \gamma \gg \gamma_{tr} \Rightarrow \eta < 0 \)
for linacs \( \alpha_p = 0 \Rightarrow \eta > 0 \)
LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability
Equations related to synchrotrons

\[ \frac{dp}{p} = \gamma_{tr}^2 \frac{dR}{R} + \frac{dB}{B} \]

\[ \frac{dp}{p} = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R} \]

\[ \frac{dB}{B} = \gamma_{tr}^2 \frac{df}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{dp}{p} \]

\[ \frac{dB}{B} = \gamma^2 \frac{df}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \frac{dR}{R} \]

- \( p \) [MeV/c] \quad \text{momentum}
- \( R \) [m] \quad \text{orbit radius}
- \( B \) [T] \quad \text{magnetic field}
- \( f \) [Hz] \quad \text{rev. frequency}
- \( \gamma_{tr} \) \quad \text{transition energy}
I - Constant radius

Beam maintained on the same orbit when energy varies

\[
\frac{dp}{p} = \frac{dB}{B}
\]

\[
\frac{dp}{p} = \gamma^2 \frac{df}{f}
\]

If \( p \) increases
- \( B \) increases
- \( f \) increases
II - Constant energy

\[ V_{RF} = 0 \]

Beam debunches

\[
\frac{dp}{p} = 0 = \gamma_{tr}^2 \frac{dR}{R} + \frac{dB}{B}
\]

\[
\frac{dp}{p} = 0 = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R}
\]

If \( B \) increases

\( R \) decreases

\( f \) increases
III - Magnetic flat-top

Beam bunched with constant magnetic field

\[
\frac{dp}{p} = \gamma_{tr}^2 \frac{dR}{R} \quad \frac{dB}{B} = 0 = \gamma_{tr}^2 \frac{df}{f} + \left[ 1 - \left( \frac{\gamma_{tr}}{\gamma} \right)^2 \right] \frac{dp}{p}
\]

\[
\frac{dB}{B} = 0 = \gamma^2 \frac{df}{f} + \left( \gamma^2 - \gamma_{tr}^2 \right) \frac{dR}{R}
\]

If \( p \) increases
\( R \) increases
\( f \) increases
\( \gamma < \gamma_{tr} \)
\( \gamma > \gamma_{tr} \)
IV - Constant frequency

Beam driven by an external oscillator

\[
\frac{dp}{p} = \gamma^2 \frac{dR}{R}
\]

\[
\frac{dB}{B} = \left[ 1 - \left( \frac{\gamma_{tr}}{\gamma} \right)^2 \right] \frac{dp}{p}
\]

\[
\frac{dB}{B} = \left( \gamma^2 - \gamma_{tr}^2 \right) \frac{dR}{R}
\]

If \( p \) increases

- \( R \) increases
- \( B \) decreases

\( \gamma < \gamma_{tr} \)

If \( p \) increases

- \( R \) increases
- \( B \) decreases

\( \gamma > \gamma_{tr} \)
### Four conditions - resume

<table>
<thead>
<tr>
<th>Beam</th>
<th>Parameter</th>
<th>Variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debunched</td>
<td>$\Delta p = 0$</td>
<td>$B \uparrow, R \downarrow, f \uparrow$</td>
</tr>
<tr>
<td>Fixed orbit</td>
<td>$\Delta R = 0$</td>
<td>$B \uparrow, p \uparrow, f \uparrow$</td>
</tr>
<tr>
<td>Magnetic flat-top</td>
<td>$\Delta B = 0$</td>
<td>$p \uparrow, R \uparrow, f \uparrow (\eta &gt; 0)$ $f \downarrow (\eta &lt; 0)$</td>
</tr>
<tr>
<td>External oscillator</td>
<td>$\Delta f = 0$</td>
<td>$B \uparrow, p \downarrow, R \downarrow (\eta &gt; 0)$ $p \uparrow, R \uparrow (\eta &lt; 0)$</td>
</tr>
</tbody>
</table>

- $p$ momentum
- $R$ orbit radius
- $B$ magnetic field
- $f$ frequency
Synchronous particle

Simple case (no accel.): $B = \text{const.}$ \quad $\gamma < \gamma_{tr}$

**Synchronous particle**: particle that sees always the same phase (at each turn) in the RF cavity

In order to keep the resonant condition, the particle must keep a constant energy.

The phase of the synchronous particle must therefore be $\phi_0 = 0$ (circular machines convention).

Let’s see what happens for a particle with the same energy and a different phase (e.g., $\phi_1$)

\[
\omega = \frac{eB}{\gamma m_0} = \frac{\omega_{RF}}{h}
\]

\[
\Delta E = e \hat{V}_{RF} \sin \phi
\]
**Synchrotron oscillations**

\( \phi_1 \)
- The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier - tends toward \( \phi_0 \)

\[ V_{RF} = \omega_{RF} \phi \]

\( \phi_2 \)
- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later - tends toward \( \phi_0 \)
Synchrotron oscillations

Phase space picture

\[ \phi = \omega_{RF} t \]
**Longitudinal phase space**

The particle trajectory in the phase space \((\phi, \Delta p/p, )\) describes its longitudinal motion.

Emittance: phase space area including all the particles.

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam).
Case with acceleration \( B \) increasing \( \gamma < \gamma_{tr} \)

\[
\Delta E = e \hat{V}_{RF} \sin \phi
\]

The phase of the synchronous particle is now \( \phi_s > 0 \) (circular machines convention)

The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius \( R \)

\[
R = \frac{\gamma v m_0}{eB}
\]

The RF frequency is increased as well in order to keep the resonant condition

\[
\omega = \frac{eB}{\gamma m_0} = \frac{\omega_{RF}}{h}
\]
Phase stability

\[ \phi = \omega_{RF} t \]

\[ V_{RF} \]

\[ \phi_s \]
Phase stability

The symmetry of the case with $B = \text{const.}$ is lost

$$\frac{\Delta p}{p}$$

stable region

unstable region

separatrix
LESSON IV

RF acceleration for synchronous particle

RF acceleration for non-synchronous particle

Small amplitude oscillations

Large amplitude oscillations - the RF bucket

Synchrotron frequency and tune

Tracking

Nonadiabatic theory needed “close” to transition

Double RF systems
RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given $\phi_s > 0$

We want to calculate its rate of acceleration, and the related rate of increase of $B, f$.

$$p = e B \rho$$

Want to keep $\rho = \text{const}$

$$\frac{dp}{dt} = e \rho \frac{dB}{dt} = e \rho \dot{B}$$

Over one turn:

$$\Delta p_{\text{turn}} = e \rho \dot{B} T_{\text{rev}} = e \rho \dot{B} \frac{2\pi R}{\beta c}$$

We know that (relativistic equations) :

$$\Delta p = \frac{\Delta E}{\beta c}$$

$$\Delta E_{\text{turn}} = e \rho \dot{B} \ 2\pi R$$
RF acceleration for synchronous particle - phase

\[(\Delta E)_{\text{turn}} = e \rho \dot{B} 2\pi R\]

On the other hand, for the synchronous particle:

\[(\Delta E)_{\text{turn}} = e \hat{V}_{RF} \sin \phi_s\]

Therefore:

1. Knowing \(\phi_s\), one can calculate the increase rate of the magnetic field needed for a given RF voltage:

\[
\dot{B} = \frac{\hat{V}_{RF}}{2\pi \rho R} \sin \phi_s
\]

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

\[
\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \Rightarrow \phi_s = \arcsin \left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}}\right)
\]
RF acceleration for synchronous particle - frequency

\[ \omega_{RF} = h \omega_s = h \frac{e}{m} < B > \]
\[ \left( v = \frac{e}{m} B \rho \right) \]

\[ \omega_{RF} = h \frac{e}{m} \frac{\rho}{R} B \]

From relativistic equations:

\[ \omega_{RF} = \frac{hc}{R} \sqrt{\frac{B^2}{B^2 + (E_0/\epsilon c \rho)^2}} \]

Let

\[ B_0 \equiv \frac{E_0}{\epsilon c \rho} \]

\[ f_{RF} = \frac{hc}{2\pi R} \left( \frac{B}{B_0} \right) \frac{1}{\sqrt{1 + (B/B_0)^2}} \]
Example: PS

At the CERN Proton Synchrotron machine, one has:

\[ R = 100 \text{ m} \]
\[ \dot{B} = 2.4 \text{ T/s} \]

100 dipoles with \( l_{\text{eff}} = 4.398 \text{ m} \). The harmonic number is 20

Calculate:
1. The energy gain per turn
2. The minimum RF voltage needed
3. The RF frequency when \( B = 1.23 \text{ T} \) (at extraction)
RF acceleration for non synchronous particle

Parameter definition (subscript “s” stands for synchronous particle):

\[
\begin{align*}
  f &= f_s + \Delta f & \text{revolution frequency} \\
  \phi &= \phi_s + \Delta \phi & \text{RF phase} \\
  p &= p_s + \Delta p & \text{Momentum} \\
  E &= E_s + \Delta E & \text{Energy} \\
  \theta &= \theta_s + \Delta \theta & \text{Azimuth angle}
\end{align*}
\]

\[
ds = R \, d\theta
\]

\[
\theta(t) = \int_{t_0}^{t} \omega(\tau) \, d\tau
\]
\[ \Delta \theta > 0 \quad \Rightarrow \quad \Delta \phi < 0 \]

Since \[ f_{RF} = h f_{rev} \]

\[ \Delta \phi = -h \Delta \theta \]

Over one turn \( \theta \) varies by \( 2 \pi \)
\( \phi \) varies by \( 2 \pi h \)
1. Angular frequency

\[
\theta(t) = \int_{t_0}^{t} \omega(\tau) \, d\tau
\]

\[
\Delta \omega = \frac{d}{dt} (\Delta \theta)
\]

\[
= -\frac{1}{h} \frac{d}{dt} (\Delta \phi)
\]

\[
= -\frac{1}{h} \frac{d}{dt} (\phi - \phi_s)
\]

\[
\frac{d\phi_s}{dt} = 0 \quad \text{by definition}
\]

\[
\Delta \omega = -\frac{1}{h} \frac{d\phi}{dt}
\]
2. Momentum

\[ \eta = \frac{\frac{d\omega}{dp}}{\omega p} = \frac{\frac{\Delta\omega}{\Delta p}}{p} \]

\[ \Delta p = \frac{p_s}{\omega_s \eta} \Delta\omega = \frac{p_s}{\omega_s \eta} \left( - \frac{1}{h} \frac{d\phi}{dt} \right) \]

\[ \Delta p = - \frac{p_s}{\omega_s \eta h} \frac{d\phi}{dt} \]

3. Energy

\[ \frac{dE}{dp} = \nu \]

\[ \frac{\Delta E}{\Delta p} = \nu = \omega R \]

\[ \Delta E = - \frac{R p_s}{\eta h} \frac{d\phi}{dt} \]
Derivation of equations of motion

Energy gain after the RF cavity

\[ (\Delta E)_{\text{turn}} = e \hat{V}_{RF} \sin \phi \]

\[ (\Delta p)_{\text{turn}} = \frac{e}{\omega R} \hat{V}_{RF} \sin \phi \]

Average increase per time unit

\[ \frac{(\Delta p)_{\text{turn}}}{T_{\text{rev}}} = \frac{e}{2\pi R} \hat{V}_{RF} \sin \phi \]

\[ 2\pi R \dot{p} = e \hat{V}_{RF} \sin \phi \quad \text{valid for any particle!} \]

\[ 2\pi (R \dot{p} - R_s \dot{p}_s) = e \hat{V}_{RF} \left( \sin \phi - \sin \phi_s \right) \]
Derivation of equations of motion

\[ R \dot{p} - R_s \dot{p}_s = \left( R_s + \Delta R \right) \left( \dot{p}_s + \Delta \dot{p} \right) - R_s \dot{p}_s \]

\[ \approx R_s \Delta \dot{p} + \dot{p}_s \Delta R \]

\[ \approx R_s \Delta \dot{p} + \dot{p}_s \left( \frac{dR_s}{dp} \right)_s \Delta p \]

\[ = R_s \Delta \dot{p} + \frac{dR_s}{dp} \frac{dp_s}{dt} \Delta p \]

\[ = R_s \Delta \dot{p} + \dot{R}_s \Delta p \]

\[ = \frac{d}{dt} \left( R_s \Delta p \right) \]

\[ = \frac{d}{dt} \left( \frac{\Delta E}{\omega_s} \right) \]
Derivation of equations of motion

\[ 2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_s} \right) = e \, \hat{V}_{RF} \left( \sin \phi - \sin \phi_s \right) \]

An approximated version of the above is

\[ \frac{d(\Delta p)}{dt} = e \, \hat{V}_{RF} \left( \frac{\sin \phi - \sin \phi_s}{2\pi R_s} \right) \]

Which, together with the previously found equation

\[ \frac{d\phi}{dt} = -\frac{\omega_s \eta h}{p_s} \Delta p \]

Describes the motion of the non-synchronous particle in the longitudinal phase space (\( \Delta p, \phi \))
Equations of motion I

\[\begin{align*}
\frac{d(\Delta p)}{dt} &= A \left(\sin \phi - \sin \phi_s\right) \\
\frac{d\phi}{dt} &= B \Delta p
\end{align*}\]

with

\[A = \frac{e \hat{V}_{RF}}{2\pi R_s}\]

\[B = -\frac{\eta h \beta_s c}{p_s R_s}\]

\(B\) is not the magnetic field (induction) here!
Equations of motion II

1. First approximation - combining the two equations:

\[
\frac{d}{dt} \left( \frac{1}{B} \frac{d \phi}{dt} \right) - A \left( \sin \phi - \sin \phi_s \right) = 0
\]

We assume that \( A \) and \( B \) change very slowly compared to the variable \( \Delta \phi = \phi - \phi_s \)

\[
\frac{d^2 \phi}{dt^2} + \frac{\Omega_{\text{sync}}^2}{\cos \phi_s} \left( \sin \phi - \sin \phi_s \right) = 0
\]

with \( \frac{\Omega_{\text{sync}}^2}{\cos \phi_s} = -AB \)

We can also define:

\[
\Omega_0^2 = \frac{\Omega_{\text{sync}}^2}{\cos \phi_s} = \frac{e \hat{V}_{RF} \eta h c^2}{2\pi R_s^2 E_s}
\]
Small amplitude oscillations

2. Second approximation

\[ \sin \phi = \sin(\phi_s + \Delta \phi) \]
\[ = \sin \phi_s \cos \Delta \phi + \cos \phi_s \sin \Delta \phi \]

\[ \Delta \phi \text{ small} \quad \Rightarrow \quad \sin \phi \approx \sin \phi_s + \cos \phi_s \Delta \phi \]

\[ \frac{d\phi_s}{dt} = 0 \quad \Rightarrow \quad \frac{d^2 \phi}{dt^2} = \frac{d^2}{dt^2} (\phi_s + \Delta \phi) = \frac{d^2 \Delta \phi}{dt^2} \]

by definition

\[ \frac{d^2 \Delta \phi}{dt^2} + \Omega_{sync}^2 \Delta \phi = 0 \]

Harmonic oscillator!
Stability condition for $\phi_s$

Stability is obtained when the angular frequency of the oscillator, $\Omega^2_{\text{sync}}$, is real positive:

$$\Omega^2_{\text{sync}} = \frac{e \hat{V}_{RF} \eta hc^2}{2\pi R_s^2 E_s} \cos \phi_s \quad \Rightarrow \quad \Omega^2_{\text{sync}} > 0 \iff \eta \cos \phi_s > 0$$

Stable in the region if $\eta > 0$.
Small amplitude oscillations - orbits

For \( \eta \cos \phi_s > 0 \) the motion around the synchronous particle is a stable oscillation:

\[
\begin{align*}
\Delta \phi &= \Delta \phi_{\text{max}} \sin \left( \Omega_{\text{sync}} t + \phi_0 \right) \\
\Delta p &= \Delta p_{\text{max}} \cos \left( \Omega_{\text{sync}} t + \phi_0 \right)
\end{align*}
\]

with \( \Delta p_{\text{max}} = \frac{\Omega_{\text{sync}}}{B} \Delta \phi_{\text{max}} \)
Synchrotron (angular) frequency and synchrotron tune (for small amplitudes)

$$\Omega_{\text{sync}} = \omega_s \sqrt{\frac{e \hat{V}_{RF} h}{2\pi \beta^2 E_s}} \eta \cos \phi_s$$

$$\Omega_{\text{sync}} = 2\pi f_{\text{sync}}$$

$$\omega_s = 2\pi f_s$$

Number of synchrotron oscillations per turn:

$$Q_{\text{sync}} = \frac{\Omega_{\text{sync}}}{\omega_s} = \sqrt{\frac{e \hat{V}_{RF} h}{2\pi \beta^2 E_s}} \eta \cos \phi_s$$

“synchrotron tune”
Large amplitude oscillations

\[ \ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \]

Multiplying by \( \dot{\phi} \) and integrating

\[ \frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = cte \]

Constant of motion

here \( \dot{\phi} = 0 \)

\( \phi = \pi - \phi_s \)

Equation of the separatrix

\[ \frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = -\frac{\Omega_s^2}{\cos \phi_s} \left[ \cos (\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s \right] \]
Phase space separatrix and particle trajectories

- Equation of the bucket separatrix

\[
\frac{\dot{\phi}}{\Omega_s} = \pm \sqrt{\frac{2}{\cos \phi_s}} \left[ \cos \phi + \phi \sin \phi_s - \cos (\pi - \phi_s) - (\pi - \phi_s) \sin \phi_s \right]
\]

- Equation of a particle trajectory

\[
\frac{\dot{\phi}}{\Omega_s} = \pm \sqrt{\frac{2}{\cos \phi_s}} \left[ \cos \phi + \phi \sin \phi_s \right] + Cte
\]
Phase space separatrix and particle trajectories

- (Bucket) separatrices: Below transition
- Above transition

\[
\phi_s = 0° \quad \phi_s = 30° \quad \phi_s = 60° \quad \phi_s = 85°
\]

\[
\phi_s \Rightarrow \pi - \phi_s
\]

\[
\phi_s = 180° \quad \phi_s = 150° \quad \phi_s = 120° \quad \phi_s = 95°
\]
Phase space separatrix and particle trajectories

- Particle trajectories: Below transition

\[ \phi_s = 0^\circ \]

\[ \phi_s = 30^\circ \]
Change of variables if one wants to use \((\Phi, \Delta E)\) or \((\Delta t, \Delta E)\) instead of \((\Phi, d\Phi/dt)\)

\[
\begin{align*}
\Delta \phi &= \phi - \phi_s \\
&= \omega_{RF} \Delta t \\
&= h \omega_s \Delta t \\
\Delta p &= \frac{\Delta E}{\beta_s c} \\
\dot{\phi} &= -\frac{\eta h c}{\beta_s E_s R_s} \Delta E
\end{align*}
\]

=> System of 2 equations to be solved

\[
\frac{d}{dt} (\Delta E) = e \hat{V}_{RF} \frac{\omega_s}{2\pi} \left[ \sin \left( \phi_s + h \omega_s \Delta t \right) - \sin \phi_s \right]
\]

\[
\frac{d}{dt} (\Delta t) = -\frac{\eta}{\beta_s^2 E_s} \Delta E
\]
2 questions

\[ \phi_{\text{min}} = ? \]

\[ \phi_s \]

\[ \phi_{\text{max}} = \pi - \phi_s \]

\[ \Delta E_{\text{sep}} = ? \]

\[ \Phi_{\text{max}} = \pi - \Phi_s \]
- $\Phi_{\min}$ is obtained from the equation of the separatrix when $\dot{\phi} = 0$

$$\Rightarrow \cos \phi + \phi \sin \phi_s - \cos (\pi - \phi_s) - (\pi - \phi_s) \sin \phi_s = 0$$
- $\Delta E_{\text{max}}^{\text{sep}}$ is obtained from the equation of the separatrix when $\phi = \phi_s$

$$\Delta E_{\text{max}}^{\text{sep}} (\phi_s) = \sqrt{\frac{2 \beta_s^2 E_s e \hat{V}_{RF}}{\pi h |\eta|}} \ G(\phi_s)$$

with

$$G(\phi_s) = \sqrt{\frac{2 \cos \phi_s - (\pi - 2 \phi_s) \sin \phi_s}{\sqrt{2}}}$$

\[\phi_s = 0^\circ \quad \phi_s = 30^\circ \quad \phi_s = 60^\circ \quad \phi_s = 85^\circ\]
- nTOF bunch in the CERN PS (near transition)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average machine radius: $R$ [m]</td>
<td>100</td>
</tr>
<tr>
<td>Bending dipole radius: $\rho$ [m]</td>
<td>70</td>
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<tr>
<td>$\dot{B}$ [T/s]</td>
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<tr>
<td>$\hat{V}_{RF}$ [kV]</td>
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<td>$h$</td>
<td>8</td>
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<tr>
<td>$\alpha_p$</td>
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<td>Longitudinal (total) emittance: $\varepsilon_L$ [eVs]</td>
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</tr>
<tr>
<td>Number of protons/bunch: $N_b$ [1E10 p/b]</td>
<td>800</td>
</tr>
<tr>
<td>Norm. rms. transverse emittance: $\varepsilon_{x,y}^*$ [$\mu$m]</td>
<td>5</td>
</tr>
<tr>
<td>Trans. average betatron function: $\beta_{x,y}$ [m]</td>
<td>16</td>
</tr>
<tr>
<td>Beam pipe [cm x cm]</td>
<td>3.5 x 7</td>
</tr>
<tr>
<td>Trans. tunes: $Q_{x,y}$</td>
<td>6.25</td>
</tr>
</tbody>
</table>

$\Rightarrow \gamma_t \approx 6.1$
Tracking

- The motion of the particles can be tracked turn by turn using the recurrence relation (between turn \( n \) and turn \( n+1 \))

\[
\Delta E_{n+1} = \Delta E_n + e \hat{V}_{RF} \left[ \sin \phi_n - \sin \phi_s \right]
\]

\[
\phi_{n+1} = \phi_n - \frac{2 \pi h \eta}{\beta_s^2 E_s} \Delta E_{n+1}
\]
Tracking applied to the nTOF bunch at PS injection

\[ n_{\text{max}} = 758 = 1 / Q_s \]

\[ \phi_s = 0 \text{ deg} \]
Tracking applied to the nTOF bunch at PS injection

One can show (but the detailed computation is beyond the scope of this course) that

$$Q_s(\phi) = Q_s(0) \frac{\pi}{2 K_{cei} \left[ \sin^2 \left( \phi/2 \right) \right]}$$

$$K_{cei}(x) = \int_0^{\pi/2} \frac{dy}{\sqrt{1 - x \sin^2 y}}$$

Complete elliptical integral of the first kind

$$\phi_s = 0 \text{ deg}$$
Tracking applied to the nTOF bunch at PS injection

\[ n_{\text{max}} = 782 = 1 / Q_s \]

\[ \phi_s = 20 \text{ deg} \]
Bucket height near transition (with "adiabatic" theory)

- Case of a stationary bucket in the PS with the nTOF bunch from injection (~ 2.4 GeV total energy) till top energy (~ 20 GeV total energy) assuming a constant RF voltage (200 kV)

Goes to $\infty$ at transition (meaning that the bunch length would go to 0) => Nonadiabatic theory needed "close" to transition
Nonadiabatic theory needed “close” to transition

- Reminder: the (general, nonlinear) equations, which have to be solved, using the variables \((\Delta \Phi, \Delta E)\), are

\[
\frac{d \Delta \phi}{dt} = - \frac{h \eta \omega_s}{\beta_s^2 E_s} \Delta E
\]

\[
\frac{d \Delta E}{dt} = e \hat{V}_{RF} \omega_s \frac{\sin(\phi_s + \Delta \phi) - \sin \phi_s}{2\pi}
\]

- Assuming here only small amplitude particles

\[
\frac{d \Delta E}{dt} = e \hat{V}_{RF} \omega_s \frac{\sin(\phi_s + \Delta \phi) - \sin \phi_s}{2\pi} \approx e \hat{V}_{RF} \omega_s \frac{\cos \phi_s \Delta \phi}{2\pi}
\]
Nonadiabatic theory needed “close” to transition

\[
\frac{d}{dt} \left( \beta_s^2 E_s \frac{d \Delta \phi}{dt} \right) - \frac{e \hat{V}_{RF} \omega_s}{2 \pi} \cos \phi_s \Delta \phi = 0
\]

where in general \( \beta_s, E_s, \eta \) and \( \omega_s \) depend on time

- Until now we assumed that these parameters were slowly moving \( \Rightarrow \) Adiabatic theory

- However, close to transition the particle will not be able to catch up with the rapid modification of the bucket shape and a nonadiabatic theory is needed
Nonadiabatic theory needed “close” to transition

- Neglecting the slow time variations of all the parameters except $\frac{\eta}{E_s}$, one has to solve

$$\frac{d}{dt}\left(\frac{E_s}{\eta} \frac{d \Delta \phi}{dt}\right) - \frac{h e \hat{V}_{RF} \omega_s^2 \cos \phi_s}{2 \pi \beta_s^2} \Delta \phi = 0$$

- Assuming then that $\gamma = \gamma_t + \dot{\gamma} t$, with $t = 0$ at transition,

$$- \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \approx 2 \frac{\dot{\gamma} t}{\gamma_t^3} \quad E_s = \gamma E_0 \approx \gamma_t E_0$$

$$\frac{\eta}{E_s} \approx - \frac{2 \dot{\gamma} t}{\gamma_t^4 E_0}$$
Nonadiabatic theory needed “close” to transition

- The (small amplitude) equation which needs to be solved close to transition is

\[
\frac{d}{dt} \left( \frac{T_c^3}{|t|} \frac{d \Delta \phi}{dt} \right) + \Delta \phi = 0
\]

with \( T_c \) a nonadiabatic time defined by (with \( E_0 \) in eV)

\[
T_c = \left( \frac{\beta_s^2 E_0 \gamma_t^4}{4\pi f_s^2 \hat{\gamma} \hbar \hat{V}_{RF} \cos \phi_s} \right)^{1/3}
\]

\( \sim 1.9 \text{ ms for the nTOF bunch in the CERN PS} \)
**Nonadiabatic theory needed “close” to transition**

- This equation can be solved but the detailed computation is beyond the scope of this course => See for instance (for those interested)
Nonadiabatic theory needed “close” to transition

- Numerical (analytical) result for the case of the nTOF bunch in the CERN PS
Show that the motion of the particles can be tracked turn by turn using the recurrence relation (between turn $n$ and turn $n+1$)

$$
\Delta E_{n+1} = \Delta E_n + e \hat{V}_{RF} \left[ \sin \phi_n - \sin \phi_s + \frac{V_{RF2}}{\hat{V}_{RF}} \sin \left[ \phi_{s2} + \frac{h_2}{h} \left( \phi_n - \phi_s \right) \right] - \sin \phi_{s2} \right]
$$

$$
\phi_{n+1} = \phi_n - \frac{2 \pi h \eta}{\beta_s^2 \, E_s} \Delta E_{n+1}
$$
Double RF systems

\[ \frac{h_2}{\hbar} = 2 \quad \phi_s = \phi_{s2} = 0 \]

\[
\frac{V_{RF2}}{\hat{V}_{RF}} = 0
\]

\[
\frac{V_{RF2}}{\hat{V}_{RF}} = -0.5
\]

\[
\frac{V_{RF2}}{\hat{V}_{RF}} = -1
\]

\[
\frac{V_{RF2}}{\hat{V}_{RF}} = +1
\]
LESSON V

Measurement of the longitudinal bunch profile and Tomography

The pyHEADTAIL simulation code (by Benoit Salvant)
Measurement of the longitudinal bunch profile

=> WALL CURRENT MONITOR = Device used to measure the instantaneous value of the beam current

For the vacuum + EM shielding

Induced or wall current

⇒ High-frequency signals do not see the short circuit

Longitudinal bunch profiles

Courtesy
J. Belleman

A Wall Current Monitor
Tomography

TOMOSCOPE (developed by S. Hancock, CERN/BE/RF)

The aim of TOMOGRAPHY is to estimate an unknown distribution (here the 2D longitudinal distribution) using only the information in the bunch profiles.

\[ \text{Surface} = \text{Longitudinal EMITTANCE of the bunch} = \varepsilon_L \text{[eV.s]} \]

\[ \text{Surface} = \text{Longitudinal ACCEPTANCE of the bucket} \]
The pyHEADTAIL simulation code

See Tutorial by Benoit Salvant