$$E_r(\boldsymbol{r}) = \frac{\lambda(z)}{2\pi\ \varepsilon_o}\frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda(z)\beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$

$$F_\perp(r) = e(E_r - \beta c\ B_\theta) = \frac{e}{\gamma^2}E_r = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\ \varepsilon_o}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\ \varepsilon_o a^2} = -\frac{\rho_x^2\lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

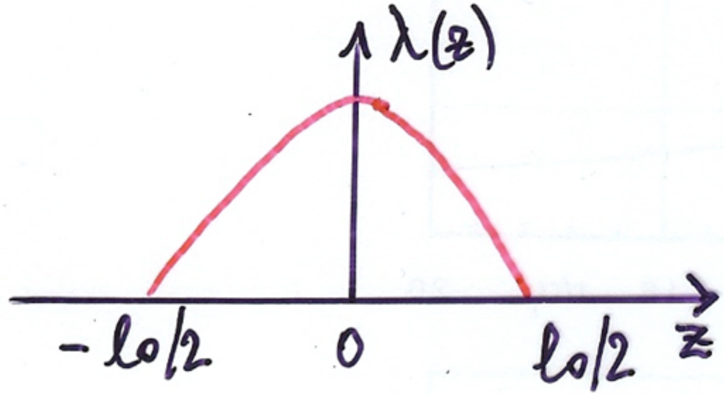
$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2}\ (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

a) Parabolic bunch (q_0 = Ne)

$$\lambda(z) = \frac{3Ne}{2l_o}\left[1-\left(\frac{2z}{l_o}\right)^2\right]$$



$$\Delta Q_{\max}(\text{at } z=0) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3 a^2 Q_{x0}}\frac{3N}{2l_0}$$

$$\Delta Q_{\min}\left(\text{at } z=\pm\frac{l_0}{2}\right) = 0$$

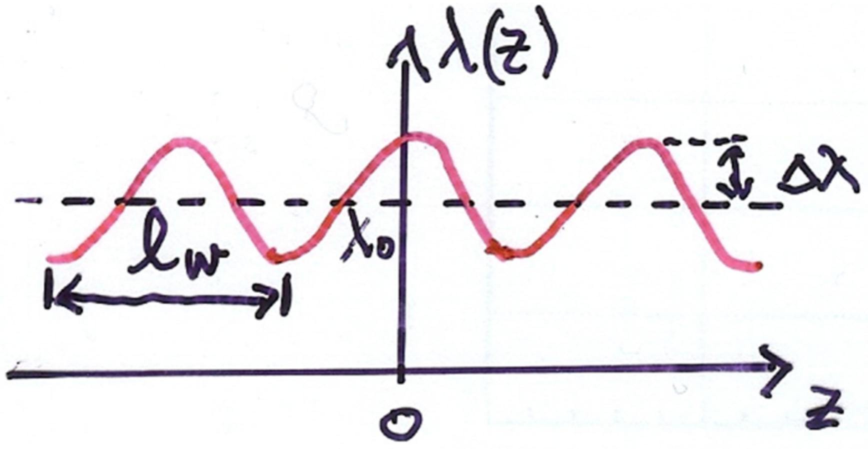
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

b) Sinusoidal modulation (λ_0 =Ne/l_0)

$$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \;\; ; \;\; k_z = 2\pi / \lambda_w$$



$$\Delta Q_{\max}\left(\text{at } k_z z = 2n\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 + \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$\Delta Q_{\min}\left(\text{at } k_z z = (2n+1)\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 - \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

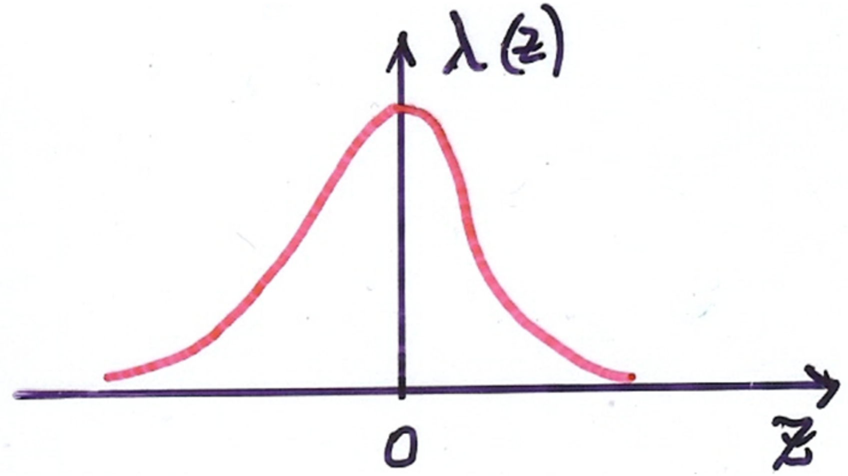
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = -\frac{2\rho_x^2 r_{e,p}\Delta\lambda}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

c) Gaussian bunch (q_0 = Ne)

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$



$$\Delta Q_{\max}\left(\text{at } z = 0\right) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3 a^2 Q_{x0}}\frac{N}{\sqrt{2\pi}\sigma_z}$$

$$\Delta Q_{\min}\left(\text{at } z \rightarrow \pm\infty\right) = 0$$

$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

bi − Gaussian

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r,z) = \frac{\lambda(z)}{2\pi\sigma_r^2}\exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

$$\int \vec{E}\cdot\hat{n}\; dS = \frac{q(r)}{\varepsilon_0}$$

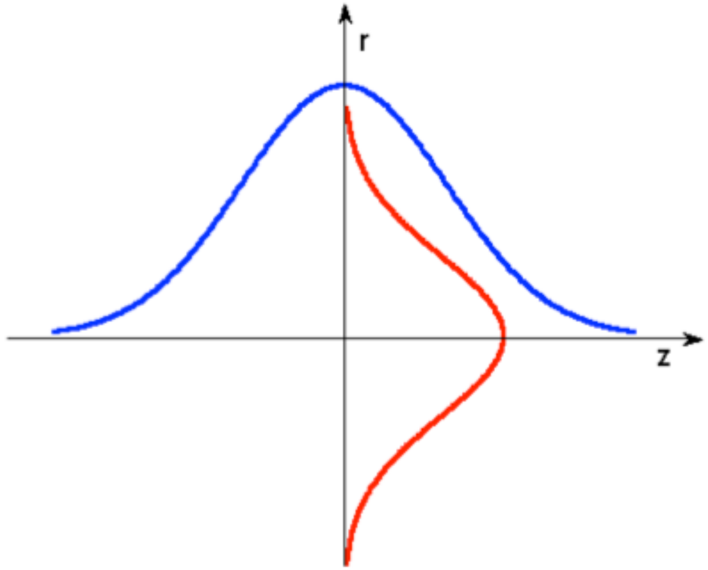
$$\left(\gamma >> 1 \qquad E_z \approx 0\right)$$

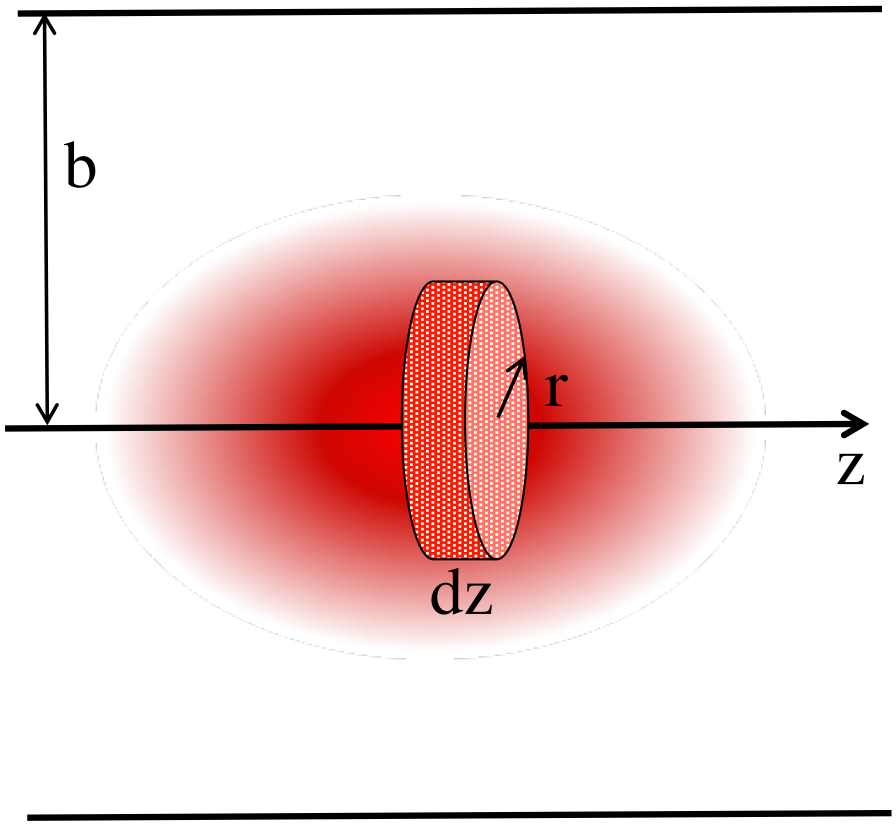
$$E_r(r)2\pi r dz = \frac{dz}{\varepsilon_0}\int_0^r \rho(r',z)2\pi r'\,dr'$$

$$E_r(r) = \frac{\lambda(z)}{2\pi\varepsilon_0\sigma_r^2 r}\int_0^r \exp\left(\frac{-r'^2}{2\sigma_r^2}\right)r'dr'$$

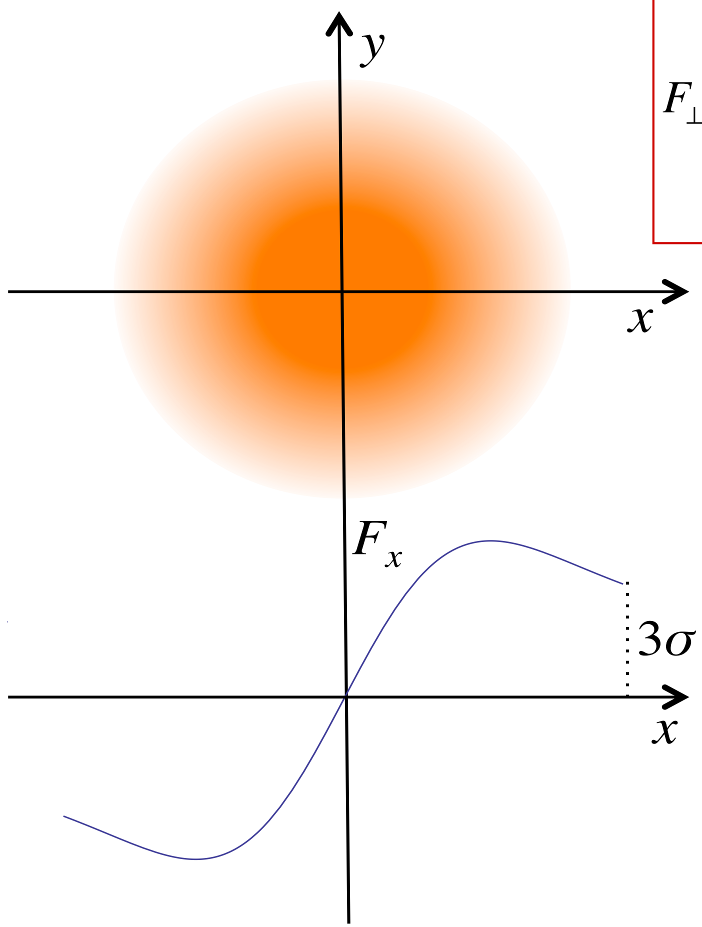
$$E_r(r) = \frac{\lambda(z)}{2\pi\varepsilon_0 r}\left[-\exp\left(\frac{-r'^2}{2\sigma_r^2}\right)\right]_0^r = \frac{\lambda(z)}{2\pi\varepsilon_0}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$

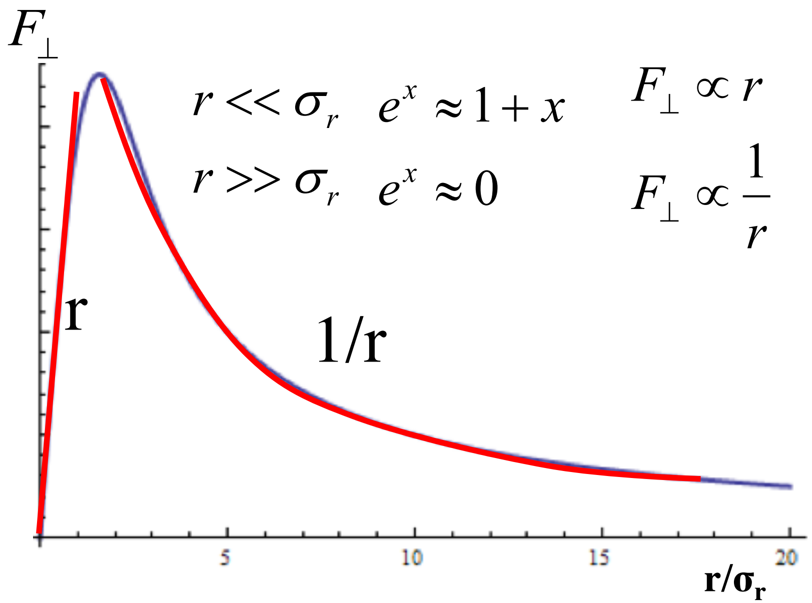
$$F_\perp(r) = \frac{e}{\gamma^2}E_r = \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$





$$F_\perp(r) = \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right] \approx \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\,\frac{r}{2\sigma_r^2} \qquad (r \ll \sigma_r)$$





$(r \ll \sigma_r)$

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\,\frac{e}{\gamma^2}\,\frac{\lambda(z)}{2\pi\,\varepsilon_o}\,\frac{1}{2\sigma_x^2}$$

If the charge distribution is Gaussian but with different σ_x and σ_y (not cylindrical geometry), it is still possible to obtain the transverse electric field. The expression is known as Bassetti-Erskine formula: M. Bassetti and G.A. Erskine, “Closed expression for the electrical field of a two-dimensional Gaussian charge”, CERN-ISR-TH/80-06 (1980).

$$E_x = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]}\,w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]}\,w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

with the complex error function w(z) given by

$$w(z) = e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2}\,d\zeta\right]$$

NB: here Q is the line density.

In the limit $\sigma_x \to \sigma_y$ the above electric field is the one that we have obtained previously

This complicated expression is highly non-linear. It is however possible to obtain a simple expression in the linear approximation which gives

$$E_x \approx \frac{\lambda(z)}{2\pi\varepsilon_0} \frac{x}{\sigma_x(\sigma_x + \sigma_y)}$$

$$E_y \approx \frac{\lambda(z)}{2\pi\varepsilon_0} \frac{y}{\sigma_y(\sigma_x + \sigma_y)}$$

As for the cylindrical symmetry case, there are also magnetic fields associated with the electric fields, so that the transverse force is

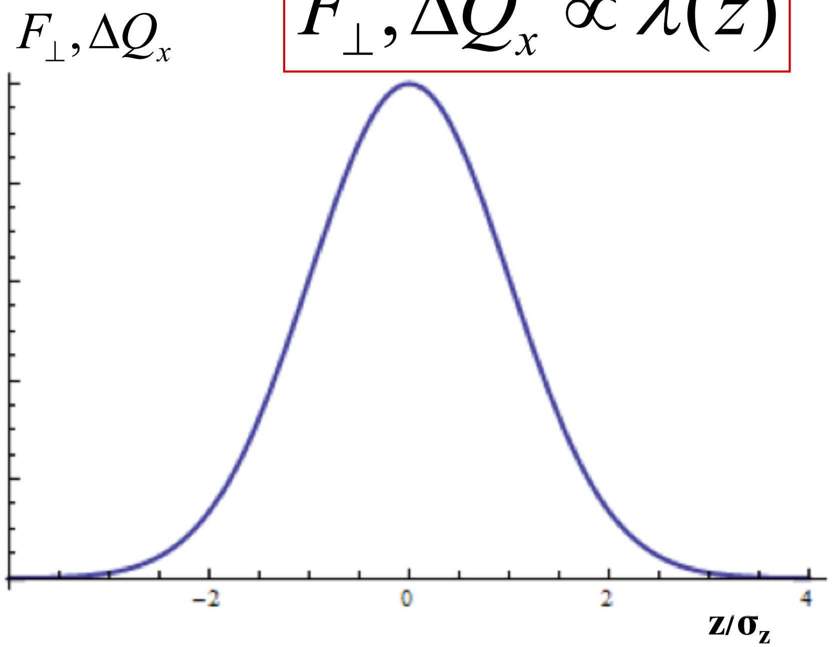
$$F_{x,y} \approx \frac{e}{\gamma^2} E_{x,y}$$

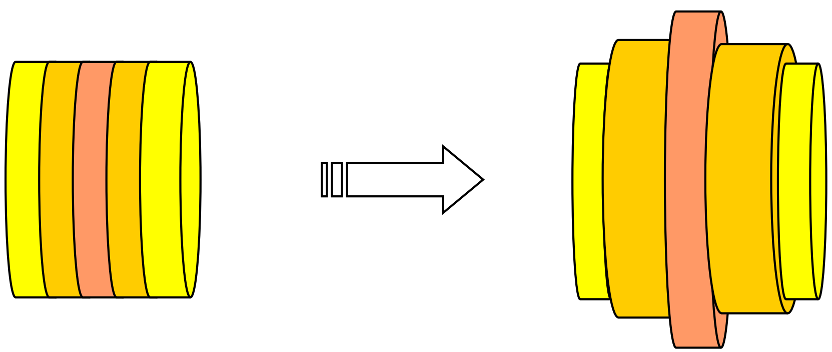
and, as in the previous cases, it is possible to obtain the incoherent tune shift (but remember that we are in the linear approximation).

Exercise 3

Evaluate the dependence of the longitudinal and transverse space charge force with z at fixed r (e.g. << σ_r) for the bi-Gaussian distribution

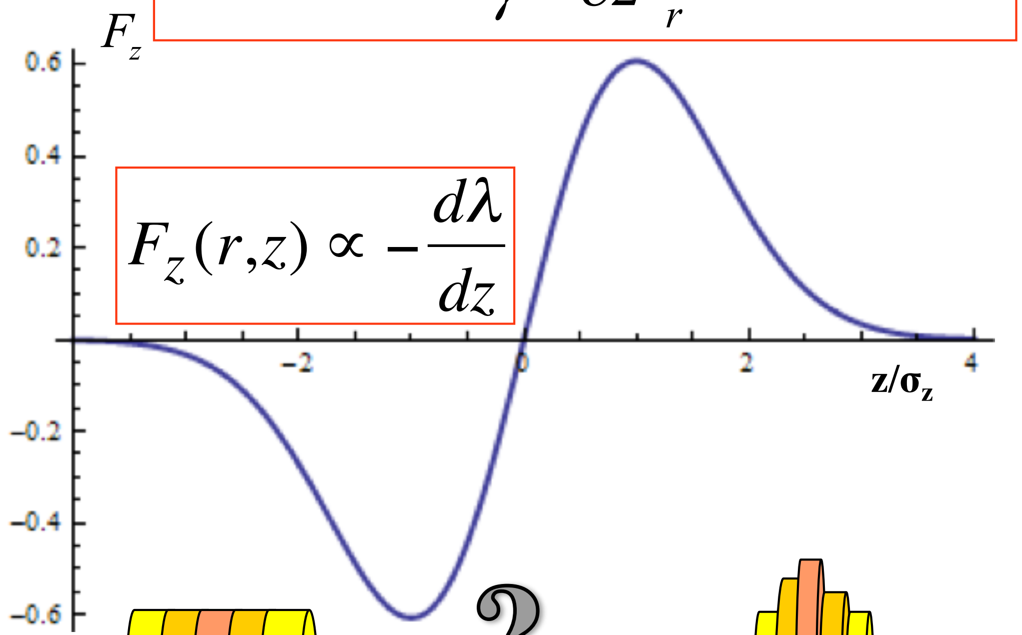
$$F_\perp, \Delta Q_x \propto \lambda(z)$$

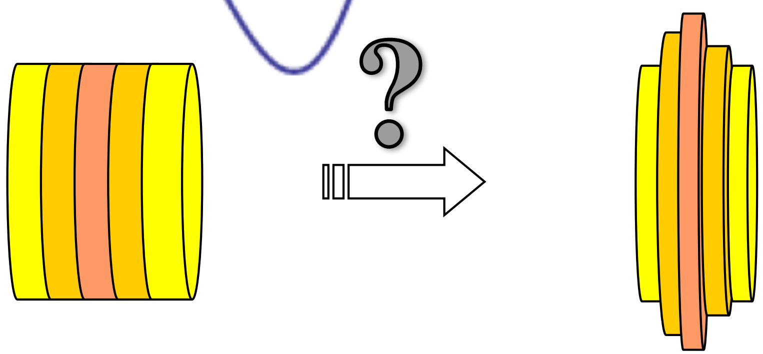




$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'$$

$$F_z(r,z) \propto -\frac{d\lambda}{dz}$$





Exercise 4

Compute the longitudinal space charge force of a transverse uniform cylindrical beam in a circular perfectly conducting beam pipe

$$E_z(r,z) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)\,dr'$$

$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)\,dr'$$

$$E_r(r \le a) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\frac{r}{a^2}$$

$$E_r(r \ge a) = \frac{\lambda(z)}{2\pi\varepsilon_0 r}$$

$$F_z(r,z) = -\frac{e}{2\pi\varepsilon_0\gamma^2}\left[\int_r^a \frac{r'}{a^2}dr' + \int_a^b \frac{1}{r'}dr'\right]\frac{\partial\lambda(z)}{\partial z}$$

$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$

Exercise 5

Compute the longitudinal space charge forces for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian

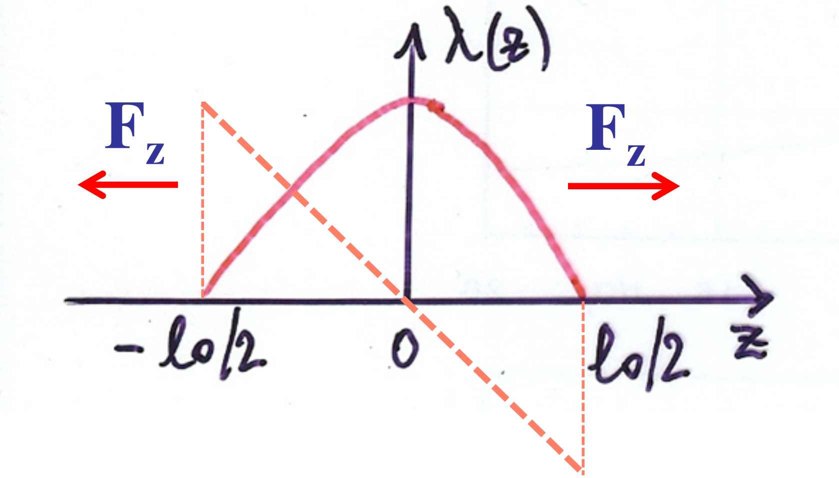
parabolic $$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$$

sinusoidal modulation $$\lambda(z) = \lambda_o + \Delta\lambda \cos\left(k_z z\right) \quad ; \quad k_z = 2\pi / l_w$$

Gaussian $$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

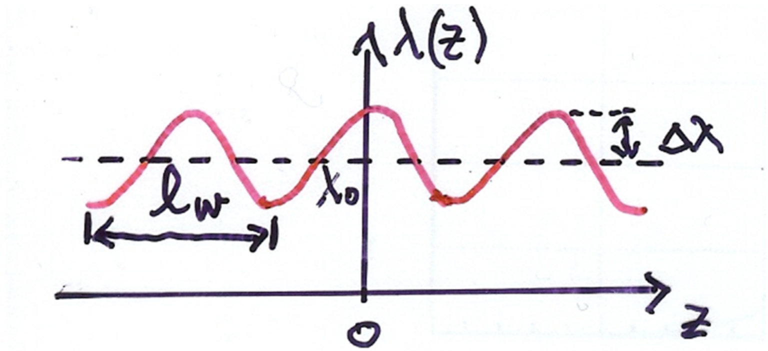
$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'$$

$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a})\frac{\partial\lambda(z)}{\partial z}$$



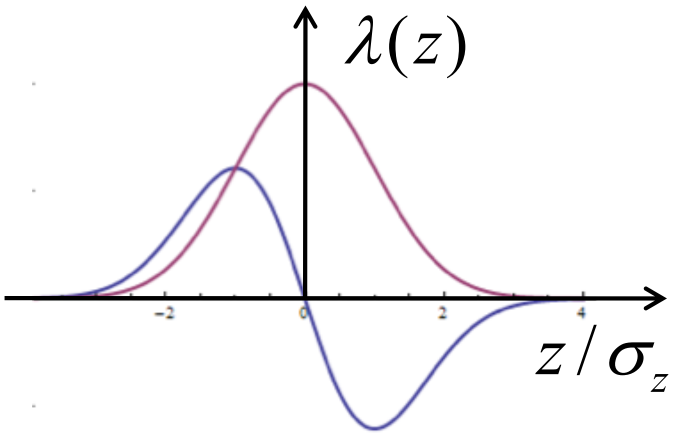
$$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$$

$$\frac{d\lambda(z)}{dz} = -\frac{12Ne}{l_0^3}z$$



$$\lambda(z) = \lambda_o + \Delta\lambda\cos(k_z z) \quad ; \quad k_z = 2\pi / l_w$$

$$\frac{d\lambda(z)}{dz} = -\Delta\lambda k_z \sin(k_z z)$$



$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\frac{d\lambda(z)}{dz} = -\frac{Ne}{\sqrt{2\pi}\sigma_z^3}z\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

Exercise 6

Compute the incoherent betatron tune shift of a uniform proton beam inside two perfectly conducting parallel plates

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)$$

$$F_x(z,x) = \frac{e\lambda_0 x}{\pi \varepsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$\Delta Q_x = -\frac{\rho_x^2 e\lambda_0}{2\pi \varepsilon_0 \beta^2 E_0 Q_{x0}}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$