Exercise 1:

Show that the impedance of an RLC parallel circuit is that of a resonant mode and relate R, L and C to Q, R_s and ω_r

$$Z_R = R \qquad Z_C = \frac{i}{\omega C} \qquad Z_L = -i\omega L$$

$$\frac{1}{Z} = \frac{1}{R} - i\omega C + i\frac{1}{\omega L} =$$

$$\frac{\omega L - i\omega^2 LCR + iR}{R\omega L} = \frac{1}{R}\left(1 + i\left(\frac{R}{\omega L} - \omega CR\right)\right) =$$

$$\frac{1}{Z} = \frac{1}{R}\left(1 + iR\sqrt{\frac{C}{L}}\left(\frac{1}{\omega\sqrt{CL}} - \omega\sqrt{CL}\right)\right)$$

$$Z(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

$$\frac{1}{Z(\omega)} = \frac{1}{R_s}\left(1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)\right)$$

$$\left(R_s = R \qquad \omega_r = \frac{1}{\sqrt{LC}} \qquad Q = R\sqrt{\frac{C}{L}}\right)$$

Exercise 2:

Calculate the amplitude of the resonator wake field given $R_s = 1\ k\Omega$, $\omega_r = 5$ *GHz*, $Q = 10^4$ $(\omega_r R_s / Q = 5 * 10^8\ V/C)$

Calculate the ratio $|Z(\omega_r)| / |Z(2\omega_r)|$ *for* $Q = 1, 10^3, 10^5$ $|1+i\ 3Q/2|$

$Q=1 \rightarrow 1.8$

$Q=10^3 \rightarrow 1.5 \times 10^3$

$Q=10^5 \rightarrow 1.5 \times 10^5$

Exercise 3: Beam Break Up

Consider a beam in a linac at 1 GeV without acceleration. Obtain the growth of the oscillation amplitude after 3 km if:

N = 5e10, $w_{\perp}$(-1 mm) = 63 V/(pC m), L_w = 3.5 cm, k_y = 0.06 1/m

$$\left(\frac{\Delta \hat{y}_2}{\hat{y}_2}\right)_{max} = \frac{cNew_{\perp}(z)L_L}{4\omega_y (E_o / e) L_w} = 180$$

To preserve the beam emittance, it is necessary to have

$$\left(\frac{\Delta \hat{y}_2}{\hat{y}_2}\right)_{max} \hat{y}_2 = 180 \times \hat{y}_2 << \text{ transverse beam size}$$

This means that the beam must be injected onto the linac axis with an accuracy better than a fraction of a per cent of the beam size, which is difficult to achieve.

Exercise 4: Beam Break Up (2)

Consider the same beam of the previous exercise being now accelerated from 1 GeV with a gradient g =16.7 MeV/m. Obtain the growth of the oscillation amplitude

$$E_f = E_0 + gL_L \approx gL_L = 50\,\text{GeV}$$

$$\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_f/e)L_w}\ln\frac{E_f}{E_0} = 14$$

Acceleration is helpful to reduce the instability

Exercise 5: Evaluate the energy lost per unit length by a charge due to the longitudinal wake field of the space charge and compare it with the longitudinal space charge force in r=0

$$\frac{w_{//}}{L} = \frac{dw_{//}(z)}{ds} = \frac{1}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{\partial}{\partial z}\delta(z)$$

$$\frac{dU(z)}{ds} = -e\int_{-\infty}^{\infty}\frac{dw_{//}(z'-z)}{ds}\lambda(z')dz' =$$

$$-\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\int_{-\infty}^{\infty}\frac{\partial}{\partial z'}\delta(z'-z)\lambda(z')dz' =$$

$$= -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{\partial \lambda(z)}{\partial z}$$

Exercise 6: Evaluate the energy spread (U_max-U_min) of a Gaussian bunch of RMS length σ due to the longitudinal wake field of the space charge in a structure of length L

$$\lambda(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} \qquad \frac{\partial\lambda(z)}{\partial z} = -\frac{z}{\sqrt{2\pi}\sigma^3} e^{-\frac{z^2}{2\sigma^2}}$$

$$U(z) = -\frac{eL}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{\partial\ \lambda(z)}{\partial\ z} = \frac{eL}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{z}{\sqrt{2\pi}\sigma^3} e^{-\frac{z^2}{2\sigma^2}}$$

$$\frac{\partial U}{\partial z} = 0 \Rightarrow z = \pm\sigma$$

$$U_{\max} - U_{\min} = 2U_{\max} = \frac{2eL}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2}}$$

Exercise 7: Evaluate the energy lost by a charge inside a uniform beam of length l_0 due to the longitudinal wake field of a pill box cavity of length g at high frequency ω>>c/b (diffraction model), with b the pipe radius

$$w(z) = \frac{Z_0 c\sqrt{2g}}{2\pi^2 b}\frac{1}{z^{1/2}}$$

$$U(z) = -e\int_{-\infty}^{\infty} w_{//}\left(z'-z\right)\lambda(z')\,dz' =$$

$$= -\frac{eqZ_0 c\sqrt{2g}}{l_0 2\pi^2 b}\int_{z}^{l/2}\frac{dz'}{(z'-z)^{1/2}} = -\frac{2eqZ_0 c\sqrt{g(l_0-2z)}}{l_0 2\pi^2 b}$$