JUAS 2013 exam (K. Wille)

The following constant will be used in the exercises

velocity of light	$c = 2.998 \cdot 10^{8}$ m/s
reduced Planck constant	$\eta = h/2\pi = 1.05 \cdot 10^{-34}$ Kg m^{2}/ s
vacuum dielectric constant	$\varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m
electron rest energy	0.511 MeV
proton rest energy	938 MeV
classical electron radius	$r_e = 1/(4\pi\varepsilon_0)\, e^2/(mc^2) = 2.81 \cdot 10^{-15}$ m

Ex. 1:
a) give a short explanation why charged electrons emit almost no radiation during longitudinal acceleration, for instance in a LINAC;

b) what is the reason to use electron beams as a source of synchrotron radiation instead of other particles as muons and protons?

Ex. 2:
A proton with energy $E_p = 10$ TeV moves through the magnetic field of a neutron star with strength $B = 10^{8}$ T. We assume that at the position of the proton the field is homogenous.

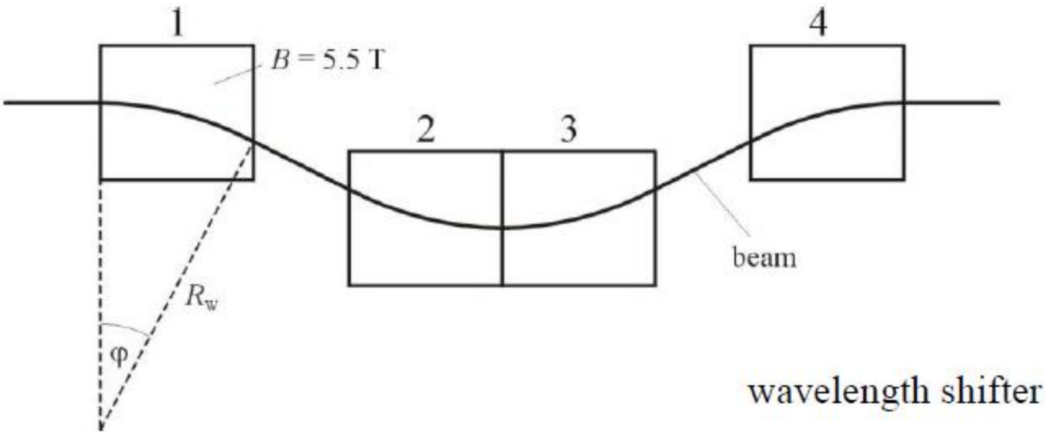
a) Calculate the diameter of the proton trajectory and the revolution frequency
b) How large is the power emitted by synchrotron radiation?
c) How much energy does the proton lose per revolution?

Ex. 3:
From the bending magnet of an electron storage ring synchrotron radiation should be emitted with a critical energy of $E_c = 1.2$ keV. The bending radius of the magnets is $R = 5.5$ m.

a) What is the required energy of the electron beam?
b) A maximum beam current of Imax = 250 mA has to be stored in the machine. How much RF power is at least required? We assume that 50% of the power is transferred to the beam and 50% lost in the cavities.

In order to get higher critical photon energy in one insertion a “wavelength shifter” is installed. It consists of 4 identical short bending magnets. The homogeneous field in the dipoles amounts to $B = 5.5$ T. Each magnet bends the beam by an angle $\varphi = 10$ degrees (see Fig. below)

c) Calculate the critical energy of the radiation emitted by the wavelength shifter. By what amount the power of the RF system has to be increased to compensate the additional energy loss produced by the wavelength shifter



Ex. 4: low emittance lattices

An undulator has total length L = 5.3 m and the period λ_u = 50 mm. The pole tip field is B_0 = 1.2 T. The gap height can be varied between the limits of g = 20 mm and g = 60 mm.

a) The undulator is installed in a storage ring operating with an electron beam energy E_b = 2.9 GeV. What wavelength range is covered by the emitted coherent radiation of the first harmonics? What is the relative width of the radiation spectrum? The radiation is measured at the radiation axis ($\theta = 0$).

b) The same undulator is adjusted at a fixed gap of g = 40 mm. What beam energy E_b is needed to get coherent infrared radiation with a wavelength of λ = 5 μm?

Exercise 1 solution:

See slides – just qualitative answer is ok

a)

From the total radiated power expressed in the relativistic case

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

If v || a we have

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6(\dot{\vec{\beta}})^2$$

We have (W is the energy of the particle)

$$\frac{dW}{dt} = Fv = eEv \qquad \rightarrow \qquad mc^2\frac{d\gamma}{dt} = ec|\bar{E}|\beta \qquad \rightarrow$$

$$mc\frac{d\gamma}{dt} = e|\bar{E}|\beta$$

$$\frac{dp}{dt} = eE$$

putting the two together to eliminate the electric field, we have

$$\frac{dp}{dt} = e|\bar{E}| = \frac{mc}{\beta}\frac{d\gamma}{dt}$$

From relativistic kinematics we have

$$\frac{d\gamma}{dt} = \gamma^3\beta\frac{d\beta}{dt}$$

hence

$$\frac{dp}{dt} = \frac{mc}{\beta}\frac{d\gamma}{dt} = mc\gamma^3\frac{d\beta}{dt}$$

We finally have

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6|\dot{\vec{\beta}}|^2 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{dp}{dt}\right|^2 \qquad \text{(A.1)}$$

In terms of the electric field we have

$$\frac{dp}{dt} = mc\beta\frac{d\gamma}{dt} + mc\gamma\frac{d\beta}{dt} = eE$$

and using

$$mc\frac{d\gamma}{dt} = e|\bar{E}|\beta$$

we have

$$eE\beta^2 + mc\gamma\frac{d\beta}{dt} = eE \qquad \rightarrow \qquad mc\gamma\frac{d\beta}{dt} = eE(1-\beta^2) \qquad \rightarrow \qquad \frac{d\beta}{dt} = \frac{eE}{mc\gamma^3}$$

and therefore

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6 |\dot{\bar{\beta}}|^2 = \frac{e^4}{6\pi\varepsilon_0 m^2 c^3}\gamma^3 |\bar{E}|^2$$

If v ⊥ a we have

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left[|\dot{\bar{\beta}}|^2 - |\bar{\beta}|^2 |\dot{\bar{\beta}}|^2\right] = \frac{e^2 |\dot{\bar{\beta}}|^2}{6\pi\varepsilon_0 c}\gamma^6\left[1 - |\bar{\beta}|^2\right] = \frac{e^2 |\dot{\bar{\beta}}|^2}{6\pi\varepsilon_0 c}\gamma^4$$

and since

$$\frac{d\bar{p}}{dt} = m\gamma\frac{d\bar{v}}{dt} = m\gamma c\dot{\bar{\beta}}$$

we have

$$P = \frac{e^2 |\dot{\bar{\beta}}|^2}{6\pi\varepsilon_0 c}\gamma^4 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\bar{p}}{dt}\right|^2 \gamma^2 \qquad \text{(A.2)}$$

b)
Since

$$P = \frac{e^2 |\dot{\bar{\beta}}|^2}{6\pi\varepsilon_0 c}\gamma^4 = \frac{e^2 |\dot{\bar{\beta}}|^2}{6\pi\varepsilon_0 c}\frac{E^4}{E_0^4}$$

the total radiated power is inversely proportion to m^4. Electrons are ~2000 times lighter than protons.

Exercise 2 solution:

a)
we know the energy of the proton and the magnetic field
we can compute the radius

Proton relativistic factor

$\gamma = 10658$

radius of cyclotron motion

$R = m_p\gamma c/eB = 0.34$ mm $\rightarrow D = 2R$

OK checked with K. Wille

We know the radius of curvature
we can compute the revolution frequency

Revolution frequency

$f = c/2\pi R = 143$ GHz

OK checked with K. Wille

b)
We know the energy of the protons and the radius of curvature
We can compute the power loss

power radiated by protons

$$P = \frac{e^2 c}{6\pi\varepsilon_0 (m_p c^2)^4} \frac{E^4}{\rho^2} = 5.35 \text{ kW}$$

OK checked with K. Wille

c)
we know the power loss and the revolution period
we can compute the energy loss per turn

energy loss per turn

$$E = PT = P\frac{2\pi\rho}{c} = \frac{e^2}{3\varepsilon_0 (m_p c^2)^4} \frac{E^4}{\rho} = 233.34 \text{ GeV}$$

OK checked with K. Wille

Exercise 3 solution:

a)
we know the critical energy and the bending radius
we can compute the beam energy

from the critical energy

$$\varepsilon_c = \frac{3}{2}\frac{\eta c}{\rho}\gamma^3 \qquad \rightarrow \qquad \gamma^3 = \frac{2\rho\varepsilon_c}{3\eta c}$$

beam energy required

$E = 1.43$ GeV

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b)
we know the beam energy and the bending radius
we can compute the energy loss per turn

energy loss per turn

$$U_0 = \frac{e^2}{3\varepsilon_0 (mc^2)^4} \frac{E^4}{\rho} = \frac{e^2}{3\varepsilon_0 \rho}\gamma^4$$

$$U_0(\text{keV}) = 88.46 \frac{E[\text{GeV}]^4}{\rho(\text{m})} = 88.46 \frac{1.43^4}{5.5} = 67.25 \text{ keV}$$

Total power loss lost by the beam for a stored 250 mA current

$P = I_b \, \Delta E = 0.25 \text{ (A)} * 67.25 \text{ (keV)} = 16.8 \text{ kW}$

Only 50% is given to the beam hence the total RF power is 16.8*2 = 33.6 kW

--- alternatively use slide 42 directly **SLIDE I.42**

$$P(kW) = 88.46 \frac{E[GeV]^4}{\rho(m)} I(A) = 88.46 \frac{1.43^4}{5.5} = 67.25\,(keV) \cdot 0.25\,(A) = 16.8\,kW$$

P lost by a beam with current I, given the energy lost by one electron U_0 is given by

P = total energy lost / revolution time
P = total number of electrons * energy lost by one electron / revolution time
P = total charge / e * energy lost by one electron / revolution time
P = average current * revolution time / e * energy lost by one electron /revolution time
P = average current / e * energy lost by one electron

$$P(kW) = U_0(keV) I(A) = 88.46 \frac{1.43^4}{5.5} 0.25 = 67.25\,(keV) \cdot 0.25\,(A) = 16.8\,kW$$

OK checked with K. Wille

c)
we know the magnetic field in the dipole and the energy of the electrons (calculated before in a))
we can compute the radius in the dipole
we can compute the critical energy

With a magnetic field of B = 5.5T the bending radius is

$$R = \frac{p}{e}\frac{1}{B} = \frac{pc}{ec}\frac{1}{B} = \frac{E\beta}{ec}\frac{1}{B} = \frac{1.4e9(eV)}{3e8 * 5.5(T)} = 0.87\ m$$

OK checked with K. Wille

The critical energy is

$$\varepsilon_c = \frac{3}{2}\frac{\eta c}{\rho}\gamma^3$$

Or in slide 48 **SLIDE 48**

$$\varepsilon_c(keV) =$$
$$= 2.218 \frac{E(GeV)^3}{\rho(m)} = 0.665\,E(GeV)^2 B(T) =$$
$$= 2.218 \frac{1.43^3}{0.87} = 0.665 * 1.43^2 * 5.5 = 7.46\,keV$$

OK checked with K. Wille

Additional RF power required:
We know the power radiated P
We know the time of flight in the four bending of the wavelength shifter T
We know the total energy radiated per electron U_0 = PT
using
$$P(kW) = U_0(keV) I(A)$$
We get the additional RF power required (times 2 again to cater for the RF cavity losses)

Alternatively, from the energy U_0 lost per electron in the wavelength shifter
We compute the total number N of electrons N = IT_{rev}/e
We compute the total energy loss by N electrons $E_0 = NU_0 = IT_{rev}U_0/e$
We divied by the revolution time to get the power lost (per second in W) $P_0 = IU_0/e$

The energy loss is equal to the power radiated multiplied the time of travelling of the electrons in the wavelength shifter. The power radiated is the usual

$$P = \frac{e^2 c}{6\pi\varepsilon_0} \frac{\gamma^4}{\rho^2} = \frac{e^2 c}{6\pi\varepsilon_0 (m_e c^2)^4} \frac{E^4}{\rho^2}$$

SLIDE I.35

The time travelled in each magnet

$\Delta t = \Delta s/c = R\Delta\phi/c$

and

$U_0 = P\Delta t = P\, R\Delta\phi/c = 12.05$ keV time 4 is 48 keV

OK checked with K. Wille

The required additional RF power for the whole set of 4 bending of the wavelength shifter is

$P_{RF} = I_b\, U_0 = 12$ kW

CHECK It can also be done using the total power spent in a bending from the slide

OK checked with K. Wille

Exercise 4 solution:

We know the length of the undulator, the period, the tip field and the gap.

a)
we know the energy of the electron beam and the period
we can calculate the undulator parameter as a function of the gap
we can calculate the wavelength emitted by the undulator

We use

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

and

$$K = \frac{e\lambda_u B}{2\pi mc}$$

and the peak field on axis as a function of the gap

$$B = \frac{B_0}{\cosh\left(\pi \frac{g}{\lambda_u}\right)}$$

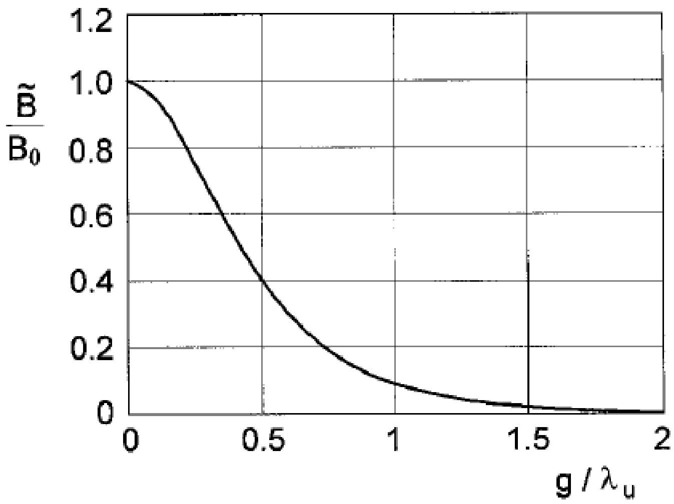

Fig: peak field vs gap

Note that a simpler formula has been used in the lectures in following years (using the pole tip field).

a)
The beam energy is 2.9 GeV (γ = 5675), L_u = 5.3m, λ_u = 50mm, field at tip = 1.2 T. The gap varies from 20 to 60 mm. This means that

the ratio g/λ_u varies from 0.4 to 1.2,
the ratio B/B_0 varies from 0.5 to 0.1,
the peak field varies from 0.6T to 0.12 T

$K = eB\lambda_u/ (2\pi mc) = 93\ B\lambda_u = 4.7\ B \rightarrow$ K varies from 2.35 to 0.47

The fundamental varies as

$\lambda = \lambda_u/2/\gamma^2(1+K^2/2) = 7.8e\text{-}9\ (1+K^2/2) \rightarrow$ λ varies from 29 nm to 8.7 nm

The relative linewidth is $1/N = \lambda_u/L_u$ = 1/106 = 0.01 = 1%

b)
we know the gap
we can compute the K
using the fundamental wavelength we can compute the energy necessary to radiate in the THz

gap fixed at 40 mm. g/λ_u is 0.8, the ratio B/B_0 is 0.15, the peak field is 0.18 T

$K = eB\lambda_u/ (2\pi mc) = 93\ B\lambda_u = 4.7\ B \rightarrow$ K is 0.85

To radiate at 5 um we need

$\gamma = \text{sqrt}(\lambda_u/\lambda/2*(1+K^2/2)) = 81.6$

i.e E = 41.7 MeV