Transverse Beam Dynamics

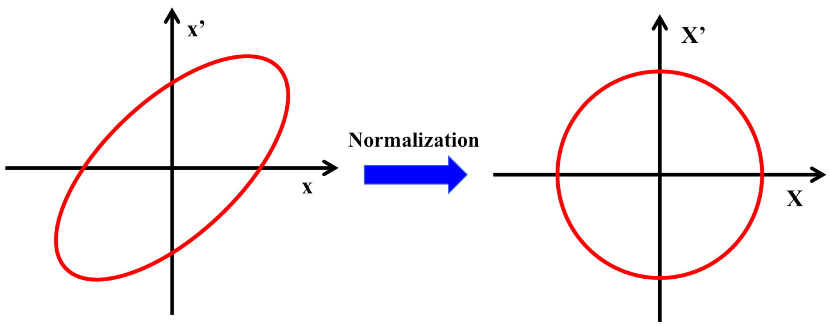
JUAS 2018 - tutorial 2

1 Exercise: Normalised phase space

Let us consider the following phase space vector: (x, x'). The transformation to a *normalised phase space* (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalisation process of the phase space is illustrated in the figure below:



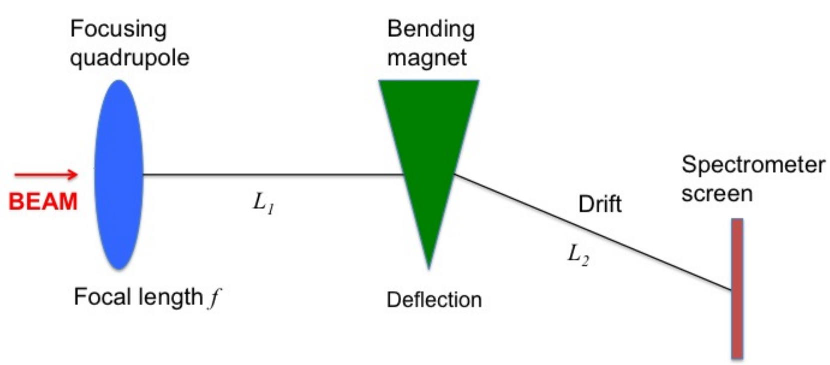
If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x - (1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalised phase space.

2 Exercise: The spectrometer line of CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring. A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring. The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length f, a drift space of length L_1, a bending magnet of deflection angle θ in the horizontal plane, and a drift space of length L_2. We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



1. If the effective length of the dipole is $l_B = 0.43$ m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?

2. Starting from the general horizontal 3×3 transfer matrix of a sector dipole of deflection angle θ, show that the transfer matrix of a dipole in the thin-lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Which approximations are done?

Hint: A sector dipole has the following 3×3 transfer matrix:

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

3. In the thin-lens approximation, derive the horizontal extended 3×3 transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1+L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

4. Assuming $D = D' = 0$ at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of D' at the end of the spectrometer line for the angle of 35 degrees.

5. What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

6. Derive the betatron function β_2 at the end of the spectrometer line in terms of L_1, L_2, f and β_1, assuming $\alpha_1 = 0$.

Hint 1. Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

An alternative way to transport the Twiss parameters is through the σ matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

This matrix multiplied by the emittance ϵ gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

If σ_1 is the matrix at the entrance of the transfer line, the matrix σ_2 at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1 M^T$$

where M is the 2×2 transfer matrix of the line extracted from the extended 3×3 transfer matrix (see question 3), and M^T the transpose matrix of M.

Hint 2. For the calculations, write M as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and replace the values of the matrix elements only at the end.

7. Given the numerical values $L_1 = 1$ m, $L_2 = 2$ m, $\beta_1 = 10$ m, $\alpha_1 = 0$, compute the value of the focal length f such that the betatron function at the end of the spectrometer line is minimum.

8. For an off-momentum particle, compute the deviation from the design orbit? Why is it important to minimise the β function in the spectrometer?

3 Exercise: Basics of lattice design

Design a FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?