

# Transverse Beam Dynamics

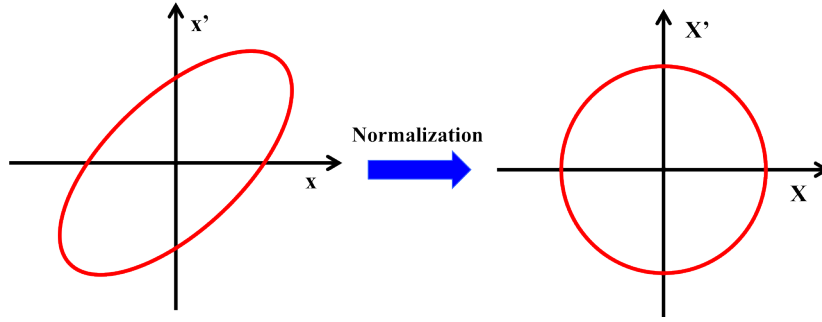
JUAS 2018 - tutorial 2 (solutions)

## 1 Exercise: Normalised phase space

Let us consider the following phase space vector:  $(x, x')$ . The transformation to a *normalised phase space*  $(X, X')$  is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalisation process of the phase space is illustrated in the figure below:



If we know that the transfer matrix between two points 1 and 2 (with phase advance  $\phi_x$  between them) in the phase space  $(x, x')$  is given by:

$$M_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}} (\cos \phi_x + \alpha_{x1} \sin \phi_x) & \sqrt{\beta_{x1} \beta_{x2}} \sin \phi_x \\ \frac{(\alpha_{x1} - \alpha_{x2}) \cos \phi_x - (1 + \alpha_{x1} \alpha_{x2}) \sin \phi_x}{\sqrt{\beta_{x2} \beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}} (\cos \phi_x - \alpha_{x2} \sin \phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalised phase space.

**Answer.** If one writes

$$M_{1 \rightarrow 2} = U_2^{-1} \cdot R \cdot U_1$$

with  $U_1$  the transformation into normalised coordinates for the Twiss parameters at 1, and  $U_2$  its inverse for the Twiss parameters at 2: i.e.,

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}; \quad U_2^{-1} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}$$

It can be shown that the matrix  $M_{12}$  can be written as:

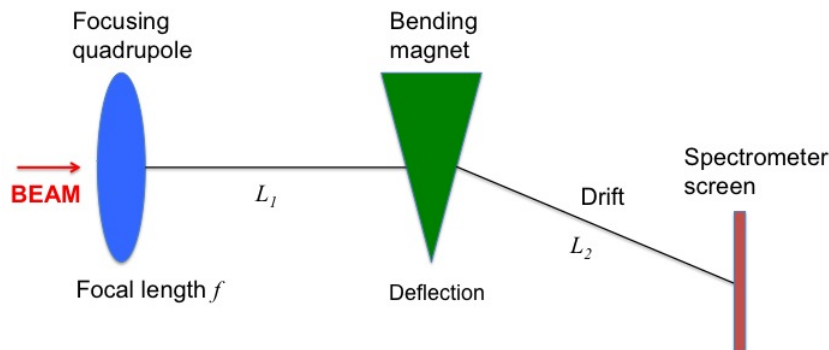
$$M_{12} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos \Delta\phi & \sin \Delta\phi \\ -\sin \Delta\phi & \cos \Delta\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos \Delta\phi & \sin \Delta\phi \\ -\sin \Delta\phi & \cos \Delta\phi \end{pmatrix}.$$

## 2 Exercise: The spectrometer line of CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring. A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring. The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length  $f$ , a drift space of length  $L_1$ , a bending magnet of deflection angle  $\theta$  in the horizontal plane, and a drift space of length  $L_2$ . We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



1. If the effective length of the dipole is  $l_B = 0.43$  m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?

**Answer.** One has  $\theta = \frac{l}{\rho}$  and  $B\rho = 3.356 p$ :  $B = \frac{3.356 p \theta}{l} = 1.66$  T.

2. Starting from the general horizontal  $3 \times 3$  transfer matrix of a sector dipole of deflection angle  $\theta$ , show that the transfer matrix of a dipole in the thin-lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Which approximations are done?

**Hint:** A sector dipole has the following  $3 \times 3$  transfer matrix:

$$M_{dipole} = \begin{pmatrix} \cos \theta & \rho \sin \theta & \rho(1 - \cos \theta) \\ -\frac{\sin \theta}{\rho} & \cos \theta & \sin \theta \\ 0 & 0 & 1 \end{pmatrix}$$

**Answer.** We need to compute the limit for  $l \rightarrow 0$  while keeping  $\theta = \frac{l}{\rho} = \text{const}$ . Remember that, if  $\theta$  is a small angle,  $\cos \theta \approx 1$ ,  $\sin \theta \approx \theta$ . Besides the trivial elements, such as  $m_{11}$ ,  $m_{22}$ , and  $m_{23}$ , the others read:

$$m_{12} : \quad \lim_{l \rightarrow 0} \rho \sin \theta = \lim_{l \rightarrow 0} \frac{\sin \theta}{\frac{1}{\rho}} = \lim_{l \rightarrow 0} l \cdot \underbrace{\frac{\sin \frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{13} : \quad \lim_{l \rightarrow 0} \rho(1 - \cos \theta) = \lim_{l \rightarrow 0} \rho \left(1 - \cos \frac{l}{\rho}\right) = \lim_{l \rightarrow 0} l \cdot \underbrace{\frac{1 - \cos \frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{21} : \quad \lim_{l \rightarrow 0} -\frac{\sin \theta}{\rho} = \lim_{\rho \rightarrow \infty} -\frac{\sin \theta}{\rho} = 0$$

therefore, in thin-lens approximation the matrix of a dipole magnet, is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

3. In the thin-lens approximation, derive the horizontal extended  $3 \times 3$  transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1 + L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

**Answer.** For the spectrometer line, one has

$$M_{spectro} = M_{Drift2} \cdot M_{Dipole} \cdot M_{Drift1} \cdot M_{Quad}$$

therefore:

$$M_{spectro} = \begin{pmatrix} 1 & L_2 & L_2\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 - \frac{L_1}{f} & L_1 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Assuming  $D = D' = 0$  at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of  $D'$  at the end of the spectrometer line for the angle of 35 degrees.

**Answer.** At the entrance of the line,  $D = 0$  and  $D' = 0$ . If  $M$  is the transfer matrix of a system the dispersion  $D$  at exit is the element  $m_{13}$  of  $M$ , whereas  $D'$  is the element  $m_{23}$ :

$$D = L_2\theta, \\ D' = \theta = 35 \text{ degrees} = 0.61.$$

5. What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

**Answer.** In a periodic lattice the  $\beta$ -functions are periodic and contained in the (periodic) transfer matrix of the lattice. In transfer line one needs to know the initial conditions in order to calculate the  $\beta$ -functions at any point (using the transfer matrix).

6. Derive the betatron function  $\beta_2$  at the end of the spectrometer line in terms of  $L_1$ ,  $L_2$ ,  $f$  and  $\beta_1$ , assuming  $\alpha_1 = 0$ .

**Hint 1.** Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

An alternative way to transport the Twiss parameters is through the  $\sigma$  matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

This matrix multiplied by the emittance  $\epsilon$  gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

If  $\sigma_1$  is the matrix at the entrance of the transfer line, the matrix  $\sigma_2$  at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1M^T$$

where  $M$  is the  $2 \times 2$  transfer matrix of the line extracted from the extended  $3 \times 3$  transfer matrix (see question 3), and  $M^T$  the transpose matrix of  $M$ .

**Hint 2.** For the calculations, write  $M$  as  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  and replace the values of the matrix elements only at the end.

**Answer.** One has  $\sigma_2 = M\sigma_1M^T$ . If  $\alpha_1 = 0$ , then  $\sigma_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$

$$\sigma_2 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \beta_1 m_{11}^2 + m_{12}^2/\beta_1 & \beta_1 m_{11} m_{21} + m_{12} m_{22}/\beta_1 \\ \beta_1 m_{11} m_{21} + m_{12} m_{22}/\beta_1 & \beta_1 m_{21}^2 + m_{22}^2/\beta_1 \end{pmatrix}$$

Therefore:

$$\beta_2 = \beta_1 m_{11}^2 + m_{12}^2/\beta_1$$

Since  $m_{11} = \frac{f-L_1-L_2}{f}$  and  $m_{12} = L_1 + L_2$ , one has:

$$\beta_2 = \beta_1 \left(1 - \frac{L_1 + L_2}{f}\right)^2 + \frac{(L_1 + L_2)^2}{\beta_1}.$$

7. Given the numerical values  $L_1 = 1$  m,  $L_2 = 2$  m,  $\beta_1 = 10$  m,  $\alpha_1 = 0$ , compute the value of the focal length  $f$  such that the betatron function at the end of the spectrometer line is minimum.

**Answer.** If  $L_1 = 1$  m,  $L_2 = 2$  m, and  $\beta_1 = 10$  m, then  $\beta_2 = 0.9 + 10 \left(1 - \frac{3}{f}\right)^2$ . To have  $\beta_2$  minimum one needs  $\left(1 - \frac{3}{f} = 0\right)$ . Therefore,  $f = 3$  m.

8. For an off-momentum particle, compute the deviation from the design orbit? Why is it important to minimise the  $\beta$  function in the spectrometer?

**Answer.** With dispersion, the deviation from the design orbit is  $\Delta x = D \frac{\Delta P}{P_0}$ . Measuring  $\Delta x$  allows to determine  $\Delta P$  and therefore  $P$ , if one has calibrated the spectrometer at  $P_0$ . It is important to minimise  $\beta_2$  (at the screen location) in order to have the best possible resolution for  $\Delta x$ : a smaller  $\beta_2$  will result in a smaller transverse beam size on the screen, which favours an accurate measurement of the momentum.

### 3 Exercise: Basics of lattice design

Design a FODO cell such that it has: phase advance  $\mu = 90$  degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are  $\beta_{max}$ ,  $\beta_{min}$ ,  $D_{max}$ ,  $D_{min}$ ?

**Answer.**

Lattice parameters:  $L = 10$  m,  $\theta = 5$  degrees = 0.087266 rad,  $f = \frac{1}{\sqrt{2}} \frac{L}{2} = 3.535$  m.

Maximum and minimum betatron functions:

$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin \mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin \mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta \left(1 + \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f + \frac{L}{2}\right) \theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta \left(1 - \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f - \frac{L}{2}\right) \theta = 0.28207 \text{ m}$$