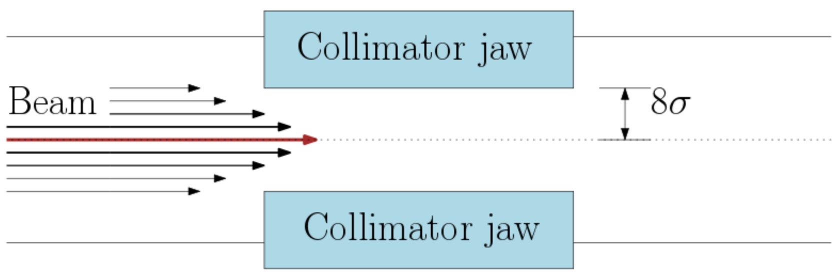
Transverse Beam Dynamics
(solutions)

JUAS 2018 - Exam

1 Exercise: LHC momentum cleaning [3pt]

Particles with large deviations from the nominal momentum impose an important hazard to any accelerator, especially if superconductive. Even the loss of a small fraction of the beam, in a cold region, can induce a magnet quench. In the LHC, in order to intercept losses, movable collimator jaws are installed in the so called "momentum cleaning" section. The collimator jaws are located in a high-dispersion region in order to intercept particles with large energy deviations. Assume that:

- the collimator's cut is at 8σ (where σ is the half-width beam size for on-momentum particles),
- at the collimator location $\beta_x = 200$ m and $D_x = -2$ m, and
- the beam has a normalised emittance of $\epsilon_{nx} = 3.5\,\mu$m.



1. Compute the maximum momentum deviation allowed by this collimator configuration at injection energy ($E_b = 450\,$GeV). [1.5pt]

   Answer.

   The physical aperture, σ, is given by:

   $$\sigma = 8\sigma_\beta = 8\sqrt{\epsilon\beta_x} = 8\sqrt{\epsilon_{nx}\beta_x/\gamma}$$

   where

   $$\gamma = \frac{E}{m_p c^2} = \frac{450 \text{ GeV}}{0.938 \text{ GeV}} = 479.74$$

   therefore

   $$8\sigma_\beta = 8 \cdot \sqrt{3.5 \cdot 10^{-6} \cdot 200/479.74} = 9.66 \text{ mm}$$

   The real beam size is given by the betatron amplitude plus the dispersive contribution of the off-momentum particles.

   $$\sigma^2 = \sigma_\beta^2 + D_x^2 \left(\frac{\Delta p}{p_0}\right)^2$$

   Therefore, the maximum energy deviation is given by,

   $$\frac{\Delta p}{p_0} = \sqrt{\frac{\sigma^2 - \sigma_\beta^2}{D_x^2}} = \sqrt{\frac{0.00966^2 - 0.00121^2}{(-2)^2}} = 4.8 \cdot 10^{-3}$$

2. Perform the same calculation with the beam at collision energy ($E_b = 7\,\text{TeV}$). [1pt]

   Answer.

   We have to perform the same calculation but taking into account that the energy higher and the physical aperture will be larger.

$$\gamma = \frac{E}{m_p c^2} = \frac{7000 \text{ GeV}}{0.938 \text{ GeV}} = 7462.7$$

$$8\sigma_\beta = 8 \cdot \sqrt{3.5 \cdot 10^{-6} \cdot 200/7462.7} = 2.45 \text{ mm}$$

$$\frac{\Delta p}{p_0} = \sqrt{\frac{\sigma^2 - \sigma_\beta^2}{D_x^2}} = \sqrt{\frac{0.00245^2 - 0.00031^2}{(-2)^2}} = 1.2 \cdot 10^{-3}$$

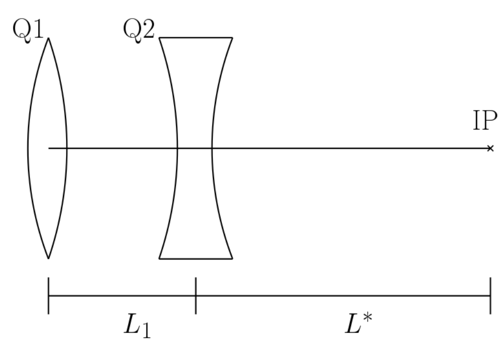
3. In the real machine, do you think the collimator aperture is kept at the same gap at injection and collision energies? Why? [0.5pt]

   Answer.

   No. The collimator gap varies during the ramp. In reality, the largest amount of off-momentum particles must be removed right after the injection in order to avoid later problems. For this reason, the collimator settings at injection energy are tighter than at collision energy. In addition, when the energy ramp starts, if the collimation at injection is not tight enough then a natural drift of the off-momentum particles outside the bucket (see the lectures on longitudinal dynamics) may end up in hitting the cold aperture.

2 Exercise: CLIC Final Focus System [8pt]

The CLIC Final Focus System focalises 3 TeV electron and positron beams at the interaction point (IP) to produce high-luminosity collisions. This is achieved by means of a quadrupole doublet. The two quadrupoles forming the doublet, Q1 and Q2 and are separated by a distance L_1 and the distance from the second quadrupole to the IP is L^*. The sketch of the electron beam line is,



1. Compute the transfer matrix between the first quadrupole and the IP using the thin lens approximation. [2.5pt]

   Answer.

   The transfer matrix of the system as it is shown in the figure is,

$$M = M_{L^\star} \cdot M_{\text{Q2}} \cdot M_{L_1} \cdot M_{\text{Q1}}$$

$$M = \begin{pmatrix} 1 & L^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{-f_{\text{Q2}}} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{-f_{\text{Q1}}} & 1 \end{pmatrix}$$

   where $f_{\text{Q1}} = \frac{1}{(k_1 l)_{\text{Q1}}}$ and $f_{\text{Q2}} = \frac{1}{(k_1 l)_{\text{Q2}}}$

$$M = \begin{pmatrix} -\frac{L^*}{f_1} - \frac{L^*}{f_2} - \frac{L_1}{f_1} + \frac{L_1 L^*}{f_1 f_2} & L_1 + 1 + L^* - \frac{L_1 L^*}{f_2} \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{L_1}{f_1 f_2} & 1 - \frac{L_1}{f_2} \end{pmatrix}$$

2. At the IP, which is a dispersion-free point, the β-functions have a minimum, with $\beta_x^* = 7.0$ mm $\beta_y^* = 0.07$ mm. The integrated quadrupole strengths are $(k_1 l)_{\mathrm{Q1}} = 0.14\ \mathrm{m}^{-1}$ and $(k_1 l)_{\mathrm{Q2}} = -0.32\ \mathrm{m}^{-1}$, and the drift lengths are $L_1 = 2.0$ m and $L^* = 6.0$ m. Compute β_x and β_y at the entrance of Q1. [2.5pt]

Answer.

There are several options to solve this part. The most compact solution is obtained using the Ω-matrix (pag. 53 of the lectures):

$$\Omega = \begin{pmatrix} \beta & -\alpha \\ \alpha & \gamma \end{pmatrix}$$

and its transformation between two points 0 and 1,

$$\Omega_1 = M \cdot \Omega_0 \cdot M^T$$

where in our case, Ω_1 is given by the Twiss functions at the IP,

$$\Omega_1 = \begin{pmatrix} \beta^* & 0 \\ 0 & 1/\beta^* \end{pmatrix}$$

therefore,

$$\Omega_0 = M^{-1} \cdot \Omega_1 \cdot (M^T)^{-1}$$

using the fact that $(M^T)^{-1} = (M^{-1})^T$ and given that,

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

we obtain,

$$\Omega_0 = \begin{pmatrix} 1 - \frac{L_1}{f_2} & -L_1 - L^* + \frac{L_1 L^*}{f_2} \\ +\frac{1}{f_1} + \frac{1}{f_2} - \frac{L_1}{f_1 f_2} & -\frac{L^*}{f_1} - \frac{L^*}{f_2} - \frac{L_1}{f_1} + \frac{L_1 L^*}{f_1 f_2} + 1 \end{pmatrix} \cdot \begin{pmatrix} \beta^* & 0 \\ 0 & 1/\beta^* \end{pmatrix} \cdot \begin{pmatrix} 1 - \frac{L_1}{f_2} & +\frac{1}{f_1} + \frac{1}{f_2} - \frac{L_1}{f_1 f_2} \\ -L_1 - L^* + \frac{L_1 L^*}{f_2} & -\frac{L^*}{f_1} - \frac{L^*}{f_2} - \frac{L_1}{f_1} + \frac{L_1 L^*}{f_1 f_2} + 1 \end{pmatrix}$$

At this stage, it is useful to substitute the parameters by the given values in order to avoid algebraic mistakes. For the x-plane,

$$L_1 (k_1 l)_{\mathrm{Q2}} = -0.64, \quad L_1 L^* (k_1 l)_{\mathrm{Q2}} = -3.84\ \mathrm{m}, \quad L_1 (k_1 l)_{\mathrm{Q1}} (k_1 l)_{\mathrm{Q2}} = -0.09\ \mathrm{m}^{-1}$$

$$L^* (k_1 l)_{\mathrm{Q1}} = 0.84, \quad L^* (k_1 l)_{\mathrm{Q2}} = -1.92, \quad L^* L_1 (k_1 l)_{\mathrm{Q1}} (k_1 l)_{\mathrm{Q2}} = -0.53, \quad L_1 (k_1 l)_{\mathrm{Q1}} = 0.28$$

$$\Omega_0^x = \begin{pmatrix} 1.64 & -11.84 \\ -0.09 & 1.26 \end{pmatrix} \begin{pmatrix} 7 \cdot 10^{-5} & 0 \\ 0 & 14285 \end{pmatrix} \begin{pmatrix} 1.64 & -0.09 \\ -11.84 & 1.26 \end{pmatrix} = \begin{pmatrix} 20026 & -2135 \\ -2135 & 227.6 \end{pmatrix}$$

$$\beta_x^{\mathrm{Q1}} \approx 20\ \mathrm{km}$$

For the y-plane we must reverse the sign of the quadurpole strengths:

$$\Omega_0^x = \begin{pmatrix} 0.36 & -4.16 \\ 0.27 & -0.34 \end{pmatrix} \begin{pmatrix} 0.007 & 0 \\ 0 & 142.6 \end{pmatrix} \begin{pmatrix} 0.36 & 0.27 \\ -4.16 & -0.34 \end{pmatrix} = \begin{pmatrix} 247222 & 20063 \\ 20063 & 1628 \end{pmatrix}$$

$$\beta_y^{\mathrm{Q1}} \approx 247\ \mathrm{km}$$

3. In a linear collider the Final Doublet is the main source of chromaticity. Compute the natural vertical chromaticity ξ_y of the doublet. [2pt]

   Answer.

   The chromaticity of the final doublet is given by,

$$\xi_y = -\frac{1}{4\pi}\sum_{\text{quad}}(k_1 l_q \beta_y)_{\text{quad}}$$

   The β-function at the first quadrupole was obtained at the previous exercise and the β-function at the second quadrupole is simply given by the expression,

$$\beta_2 = \beta_y^* + \frac{L^{*2}}{\beta_{y^*}} = 0.07 \cdot 10^{-3} + \frac{6^2}{0.07 \cdot 10^{-3}} = 514 \text{ km}$$

   Putting all together we obtain,

$$\xi_y = -\frac{1}{4\pi}(0.14 \cdot 247 \cdot 10^3 - 0.32 \cdot 514 \cdot 10^3) \approx 10337$$

4. What is the effect of chromaticity at the IP? [1pt]

   Answer.

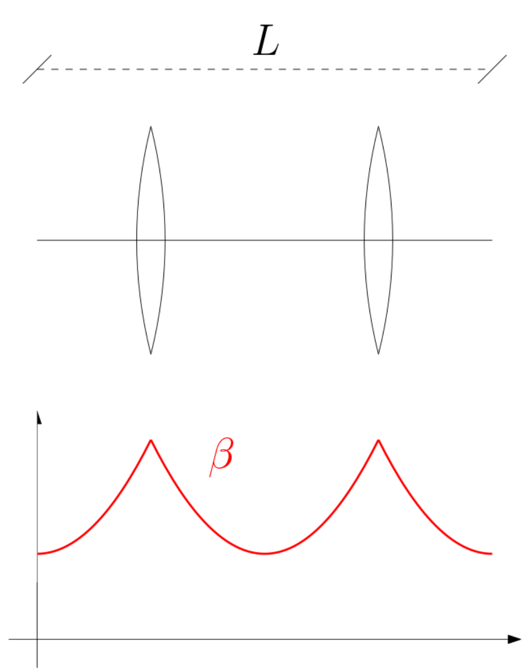
   When chromaticity is not compensated, particles with different energies focalise at different points. Therefore, the focal point is not a unique point at the IP. This translates into a larger beam size at the IP and, in consequence, a reduction of the luminosity of the collider.

3 Exercise: FOFO lattice [8pt]

The on-going R&D effort to create accelerators driven by wafefields in plasmas, foresees the use of plasma cells not just to accelerate the beam but also to focus it. In contrast to standard quadrupole magnets, which focus in one direction and defocus in the other, a "plasma quadrupole" focuses simultaneously in both transverse axes, x and y. Imagine, then, to design a FOFO cell using two focusing quadrupoles, each with focal length $f = 50$ cm, and a total cell length of $L = 1$ m (the two quadrupoles are then at distance $L/2$ from each other).

1. Sketch the FOFO cell and draw the beta function. In between two quadrupoles, what functional form does the β-function follow? [1pt]

   Answer 1.



Answer 2. Parabolic.

2. Compute the transfer matrix of the FOFO cell [3pt]

   Answer. A FOFO cell is composed by two periodic FO lattices, that is:

$$M_{\text{FOFO}} = M_{\text{FO}} \cdot M_{\text{FO}}$$

   The FO cell can be written in two forms:

$$\begin{aligned} M_{\text{FO-1}} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} && \textit{(starting with a half quadrupole)} \\ &= \begin{pmatrix} 1 - \frac{L}{4f} & \frac{L}{2} \\ -\frac{8f-L}{8f^2} & 1 - \frac{L}{4f} \end{pmatrix} \\ M_{\text{FO-2}} &= \begin{pmatrix} 1 & L/4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/4 \\ 0 & 1 \end{pmatrix} && \textit{(starting with a half drift)} \\ &= \begin{pmatrix} \frac{4f-L}{4f} & \frac{8Lf-L^2}{16f} \\ -\frac{1}{f} & \frac{4f-L}{4f} \end{pmatrix} \end{aligned}$$

   If one substitute $L = 1$ m and $f = 0.5$ m, then:

$$M_{\text{FOFO}} = \begin{pmatrix} 0 & 0.5 \\ -2 & -1 \end{pmatrix}$$

3. Compute: the phase advance, β_{max}, and β_{min} of the FOFO cell [2.5pt]

   Answer. We know that

$$\text{trace}\left(M_{\text{FOFO}}\right) = 2\cos\mu$$

   therefore:

$$\mu = \arccos\left(\text{trace}\left(M_{\text{FOFO}}\right)/2\right) = 120 \text{ deg}$$

   - β_{max} is the periodic β when one starts the cell from half a quadrupole,

$$\beta_{\text{max}} = M_{\text{FO-1},\,12}/\sin\left(\mu\right) = 0.58 \text{ m}$$

   - β_{min} it the periodic β when one starts the cell from the middle point between two quadrupoles,

$$\beta_{\text{min}} = M_{\text{FO-2},\,12}/\sin\left(\mu\right) = 0.43 \text{ m}$$

4. Compute the chromaticity of the FOFO cell [1pt]

   Answer. Chromaticity comes from the quadrupoles. Each FOFO cell has two focusing quadrupoles:

$$\xi_{\text{FOFO}} = -\frac{1}{4\pi}\sum_{\text{two quads}}\frac{\beta_{\text{max}}}{f} = -\frac{1}{2\pi}\frac{0.58 \text{ m}}{0.5 \text{ m}} = -0.18$$

5. Would this cell provide a stable lattice? [0.5pt]

   Answer. Yes. Stability occurs when

$$\left|\text{trace}\left(M\right)\right| < 2$$

   and in our case,

$$\text{trace}\left(M_{\text{FOFO}}\right) = -1.$$

4 Exercise: Orbit control [5pt]

Two kickers are located at the two ends of a FODO cell with phase advance 90 degrees and length $L_{\text{cell}} = 1$ m (the two kickers are therefore located at L_{cell} distance from each other). The two kickers have strengths K_1 and K_2 respectively.

1. Write the transfer matrix of the FODO cell [Hint. Write the transfer matrix in Twiss form, in such a way that the initial α parameter is zero.] [1pt]

   Answer. The FODO cell has transfer matrix:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

   with

$$\begin{aligned} L_{\text{cell}} &= 1 \text{ m} \\ \mu &= 90 \text{ degrees} \\ \alpha &= 0 \\ \beta &= \frac{L_{cell}\left(1 + \sin\frac{\mu}{2}\right)}{\sin\mu} = L_{\text{cell}}\left(1 + \frac{\sqrt{2}}{2}\right) = \left(1 + \frac{\sqrt{2}}{2}\right) \text{ m} \end{aligned}$$

   Therefore:

$$M_{\text{FODO}} = \begin{pmatrix} 0 & 1 + \frac{\sqrt{2}}{2} \\ -\frac{1}{1+\frac{\sqrt{2}}{2}} & 0 \end{pmatrix}$$

2. Write the full transfer matrix, inclusive of the two kickers. [Hint. Each kicker acts on the beam as a thin dipole.] [2pt]

   Answer. We can write the complete transfer matrix, inclusive of the two kickers, as:

$$\begin{aligned} M_{\text{K1-FODO-K2}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & K_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 + \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{1+\frac{\sqrt{2}}{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & K_1 \\ 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 1 + \frac{\sqrt{2}}{2} & \left(1 + \frac{\sqrt{2}}{2}\right) K_1 \\ -\frac{1}{1+\frac{\sqrt{2}}{2}} & 0 & K_2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

3. Compute the strengths of the kickers (in radians), in order to give the beam, initially at $(x_i,\, x_i')$, an offset $\left(x_f,\, x_f'\right) =$ (50 cm, 0) at the end of the system. Give the expression of K_1 and K_2 as a function of $(x_i,\, x_i')$ [2pt]

   Answer. In order to send a particle initially at $(x_i,\, x_i')$ to (50 cm, 0) offset one must solve the following system of equations:

$$\begin{pmatrix} 0.5 \text{ m} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 + \frac{\sqrt{2}}{2} & \left(1 + \frac{\sqrt{2}}{2}\right) K_1 \\ -\frac{1}{1+\frac{\sqrt{2}}{2}} & 0 & K_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ x_i' \\ 1 \end{pmatrix}$$

   which gives:

$$\begin{pmatrix} 0.5 \text{ m} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{\sqrt{2}}{2}\right) x_i' + \left(1 + \frac{\sqrt{2}}{2}\right) K_1 \\ -\frac{1}{1+\frac{\sqrt{2}}{2}} x_i + K_2 \\ 1 \end{pmatrix}$$

   that is:

$$\begin{aligned} K_1 &= \left(0.5 \text{ m} - \left(1 + \frac{\sqrt{2}}{2}\right) x_i'\right) / \left(1 + \frac{\sqrt{2}}{2}\right) \\ K_2 &= \frac{1}{1 + \frac{\sqrt{2}}{2}} x_i \end{aligned}$$