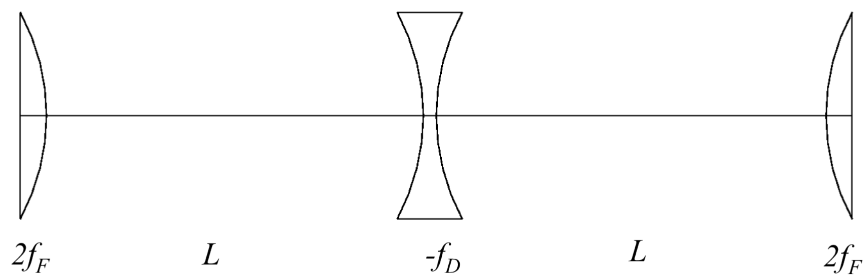
Transverse Beam Dynamics

JUAS 2018 - tutorial 3 (solutions)

1 Exercise: Chromaticity in a FODO cell

Consider a ring made of N_{cell} identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1. Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)



Answer. First we calculate the transfer matrix for a FODO cell (see figure). We start from the centre of the focusing quadrupole where the betatron function is maximum. This exercise considers a general case where f_F is not necessarily equal to f_D. Using the thin lens approximation for the FODO cell with drifts of length L we get the following matrix:

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \end{aligned} \tag{1}$$

Remember that, in terms of betatron functions and phase advance, the matrix of a FODO cell is given by:

$$M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{2}$$

Since β has a maximum at the centre of the focusing quadrupole, then $\alpha = -\beta'/2 = 0$, and we can also write:

$$M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Equating Eq. (1) to Eq. (2) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{3}$$

Where we have applied the following trigonometric identity: $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function β_{max} occurs at the focusing quadrupole. Since Eq. (1) is for a periodic cell starting at the centre of the focusing quadrupole, the m_{12} component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu} \tag{4}$$

On the other hand, the minimum for the betatron function occurs at the defocusing quadrupole position. Therefore, interchanging f_F with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} \tag{5}$$

2. Calculate the natural chromaticities for this ring.

   Answer. Let us remember the definition of natural chromaticity. The so-called "natural" chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta P/P_0} \tag{6}$$

   where ΔQ is the tune shift due to the chromaticity effects and $\Delta P/P_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum p_0.

   The natural chromaticity is defined as (remember from Lecture 4):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{7}$$

   Sometimes, especially for small accelerators, the chromaticity is normalised to the machine tune Q and defined also as:

$$\xi' = \frac{\Delta Q/Q}{\Delta P/P_0} \tag{8}$$

$$\xi'_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{9}$$

   For this exercise, either you decide to use Eq. (7) or Eq. (9) it is fine! From now on let us use Eq. (7):

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi} \times N_{cell}\int_{cell}\beta(s)k(s)ds \\
&= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}}\beta_i(kl_q)_i
\end{aligned}$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i(kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi} \sum_{i\in\{quads\}} \beta_i(kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right) + \beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(2L + \frac{L^2}{f_D}\right)\frac{1}{f_F} - \left(2L - \frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{N_{cell}L}{2\pi\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]
\end{aligned}$$

Here we have used the expressions (4) and (5) for β_{max} and β_{min}.

3. Show that for short quadrupoles, if $f_F \simeq f_D$,

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}.$$

Answer. If $f_F \simeq f_D$, we have

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{2\pi\sin\mu}\frac{L^2}{f_F f_D} \\
&= -\frac{N_{cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}} 4\sin^2\frac{\mu}{2}
\end{aligned}$$

where we have used the trigonometric identity: $\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$

Considering Eq. (3), we have

$$4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}$$

which finally gives:

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}$$

Q.E.D.!

4. Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

Answer. Lattice parameters: $L = 10$ m, $\theta = 5$ degrees= 0.087266 rad, $f = \frac{1}{\sqrt{2}}\frac{L}{2} = 3.535$ m

Maximum and minimum betatron functions:

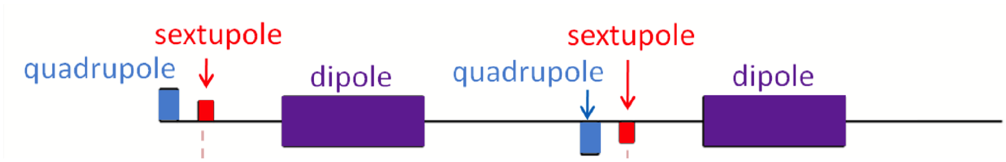
$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f + \frac{L}{2}\right)\theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f - \frac{L}{2}\right)\theta = 0.28207 \text{ m}$$

5. Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).

Answer. By locating sextupoles with strength $K_s > 0$ where β_x is large and β_y is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity. Similarly, by locating sextupoles with $K_s < 0$ where β_y is large and β_x is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity. See figure below.



Let us assume the case of a FODO lattice where $f_F = f_D = f$. Then the natural chromaticity of this FODO cell is given by the expression (exercise 1.3):

$$\xi_N \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}$$

For $\mu = 90$ it is $\xi_N \simeq -1/\pi$ in both horizontal and vertical plane. Therefore, we need to adjust the strength of the sextupoles to cancel this chromaticity:

$$-\frac{1}{4\pi}\left[K_{2F}D_{max}\beta_{max} + K_{2D}D_{min}\beta_{min}\right] \simeq -\frac{1}{\pi}$$

where $K_{2F} = k_{2F}l_s$ is the normalised integrated strength of the sextupole located near the focusing quadrupole, and $K_{2D} = k_{2D}l_s$ the normalised integrated strength of the sextupole near the defocusing quadrupole (with l_s the effective length of the sextupole). For an effective cancellation of the chromaticity in both planes, usually $K_{2F} > 0$ and $K_{2D} < 0$. Substituting the values for the maximum and minimum dispersion and betatron function in terms of the total length of the lattice L and the focal length of the quadrupoles f, one obtains the following expression:

$$-\frac{1}{4\pi}\frac{f}{L}\theta\left[K_{2F}\left(4f + \frac{L}{2}\right)\left(L + \frac{L^2}{4f}\right) + K_{2D}\left(4f - \frac{L}{2}\right)\left(L - \frac{L^2}{4f}\right)\right] \simeq -\frac{1}{\pi}$$

Considering the same absolute value for the strength of the sextupoles, $K_{2F} = -K_{2D} = K_s$, we can write then:

$$\frac{3}{4\pi}K_sLf\theta = \frac{1}{\pi}$$

The strength of the sextupole is given then by:

$$K_s = \frac{4}{3Lf\theta}$$

Then, substituting all the numerical values for the lattice parameters:

$K_{2F} = 0.865 \text{ m}^{-2}$

$K_{2D} = -0.865 \text{ m}^{-2}$

6. If the gradient of all focusing quadrupoles in the ring is wrong by $+10\%$, how much is the tune-shift with and without sextupoles?

Answer.

If the gradient of the focusing quadrupole has and error of 10%, then the corresponding quad. strength error is also 10%. We calculate the number of cells of a ring made of these FODO cells, $N_{cell} = 72$ cells, and then we calculate the total tune-shift in both planes:

$\Delta Q_x = N_{cell}\frac{\Delta K_F \beta_{max}}{4\pi} = 9.78$

$\Delta Q_y = N_{cell}\frac{\Delta K_F \beta_{min}}{4\pi} = 1.68$

When the sextupoles correct for the chromaticity, the particles have, in principle, no tune-shift with energy. In real machines, one wants to have a non-zero residual chromaticity to stabilise the beam against resonant imperfections.

2 Exercise: Measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalised gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β and the α functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV/c. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m.

Hint: G is proportional to the current. **Advice:** use thin-lens approximation.

1. How does the normalised focusing strength K vary with I_Q?

   Answer. The quadrupole gradient G is proportional to the current flowing through the coils I_Q

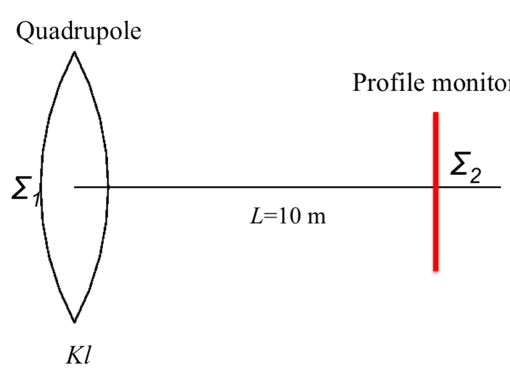
   $$G = C \cdot I_Q,$$

   C is the proportionality coefficient. We know that $G = 1$ T/m when $I_Q = 100$ A, therefore $C = 0.01$ T/(A·m). The normalised focusing strength is

   $$K = \frac{G}{P/q} \quad \text{therefore} \quad K = \frac{C \cdot I_Q}{P/q}$$

2. Give the expression Σ_2 as function of α_1, β_1, and γ_1

   Answer. Let Σ_1 and Σ_2 be the 2×2 matrices with the twiss parameters, $\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, at the quadrupole entrance and at the wire scanner, respectively.



It is worth explaining that the matrix Σ multiplied by the emittance ϵ is the covariance matrix of the beam distribution:

$$\Sigma\epsilon = \begin{pmatrix} \beta\epsilon & -\alpha\epsilon \\ -\alpha\epsilon & \gamma\epsilon \end{pmatrix} = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle \end{pmatrix}$$

The transverse beam size of the beam is given by $\sigma_x = \sqrt{\langle x^2\rangle} = \sqrt{\beta_x\epsilon_x}$ (horizontal beam size), and $\sigma_y = \sqrt{\langle y^2\rangle} = \sqrt{\beta_y\epsilon_y}$ (vertical beam size). Here we will simply use the following notation: $\sigma_1 = \sqrt{\beta_1\epsilon}$ for the beam size (horizontal or vertical) at position 1, and $\sigma_2 = \sqrt{\beta_2\epsilon}$ for the beam size (horizontal or vertical) at position 2.The matrix Σ propagates from position 1 to position 2 as follows:

$$\Sigma_2 = M\Sigma_1 M^T$$

where M is the transfer matrix of the system and M^T its transpose. We have:

$$\begin{aligned} \Sigma_2 &= \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} = \begin{pmatrix} 1-KlL & L \\ -Kl & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1-KlL & -Kl \\ L & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 L^2(Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2 & \beta_1 L(Kl)^2 + (2\alpha_1 L - \beta_1)Kl + \gamma_1 L - \alpha_1 \\ \beta_1 L(Kl)^2 + (2\alpha_1 L - \beta_1)Kl + \gamma_1 L - \alpha_1 & \beta_1(Kl)^2 + 2\alpha_1 Kl + \gamma_1 \end{pmatrix} \end{aligned} \tag{10}$$

3. Show that β_2 can be written in the form: $\beta_2 = A_2\,(Kl)^2 + A_1\,(Kl) + A_0$, and express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1.

   Answer. We can see from Eq. (10) that:

   $$\beta_2 = \beta_1 L^2(Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2$$

   and therefore:

   $$\begin{aligned} A_2 &= \beta_1 L^2 \\ A_1 &= 2L(\alpha_1 L - \beta_1) \\ A_0 &= \beta_1 - 2\alpha_1 L + \gamma_1 L^2 \end{aligned}$$

   Hint for the next questions: show that if one expresses β_2 as

   $$\beta_2 = B_0 + B_1\,(Kl - B_2)^2$$

   one has:

   $$\begin{aligned} B_0 &= A_0 - A_1^2/4A_2^2 = L^2/\beta_1 \\ B_1 &= A_2 = L^2\beta_1 \\ B_2 &= -A_1/A_2 = 1/L - \alpha_1/\beta_1 \end{aligned}$$

4. Express the final beam size, σ_2, as a function of Kl, and find its minimum, which will correspond to $(Kl)_{\mathrm{min}}$.

   Answer. The transverse r.m.s. beam size is $\sigma = \sqrt{\epsilon\beta}$, where ϵ is the transverse (geometric) emittance. As we have seen in the previous questions β_2 depends quadratically on Kl: $\beta_2 = B_0 + B_1\,(Kl - B_2)^2$. Since ϵ is constant, if we want to minimise σ_2, we have to minimise β_2:

   $$\frac{\mathrm{d}\beta_2}{\mathrm{d}(Kl)} = 0 \longrightarrow 2B_1(Kl - B_2) = 0 \longrightarrow (Kl)_{min} = B_2 = \frac{1}{L} - \frac{\alpha_1}{\beta_1} \tag{11}$$

   We can write:

   $$\sigma_2^2 = \beta_2\epsilon = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon$$

Why is this useful? By means of a quadrupole scan (i.e. changing the quadrupole strength) we identify the strength Kl which minimises the value σ_2^2. We fit a parabola to the measurements σ_2^2 vs. Kl, and select then $\sigma_2^2((Kl)_{min})$. The minimum beam size is given by:

$$\mathrm{Min}(\sigma_2) = L\sqrt{\frac{\epsilon}{\beta_1}} = \sqrt{B_0\epsilon} \tag{12}$$

5. How does σ_2 vary with Kl when $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$?

   Answer. Under this condition:

$$\sigma_2^2 = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon \longrightarrow \sigma_2 \simeq L\sqrt{\beta_1\epsilon}(Kl - (Kl)_{min})$$

   For $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$, σ_2 depends linearly on Kl, with slope

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = \frac{L^2\beta_1}{\sigma_2}(Kl - (kl)_{min})\epsilon.$$

6. Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

   Answer. We know that

$$Kl = \frac{G \cdot l}{p/e} = \frac{C \cdot l \cdot I_Q}{p/e} = \frac{0.01[\mathrm{T/(Am)}]\cdot 0.2[\mathrm{m}]}{(0.3[\mathrm{GeV}]/0.3)[\mathrm{Tm}]} \cdot I_Q = 2 \times 10^{-3} \cdot I_Q$$

   If we measure σ_2 as a function of the quadrupole current I_Q, from the minimum value we can get β_1 (Eq. (12)), and since from the measurement we obtain $(Kl)_{min} = 2 \times 10^{-3}(I_Q)_{min}$, using Eq. (11) we can calculate α_1. Once we know β_1 and α_1, it is then straightforward to calculate $\gamma_1 = (1 + \alpha_1^2)/\beta_1$.