CERN

Basic Mathematics and Units

Rende Steerenberg – BE/OP

Contents

- Vectors & Matrices
- Differential Equations
- Some Units we use

- **Vectors & Matrices**
- Differential Equations
- Some Units we use

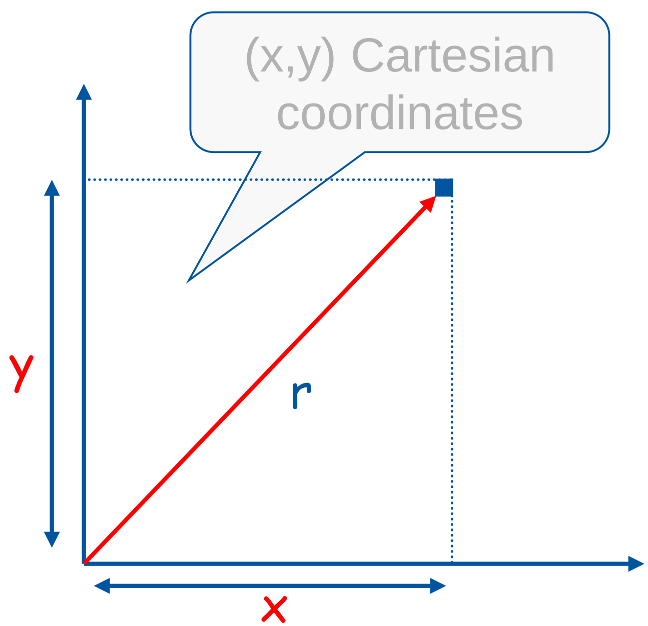
Scalars & Vectors

Scalar, a single quantity or value

Vector, (origin,) length, direction

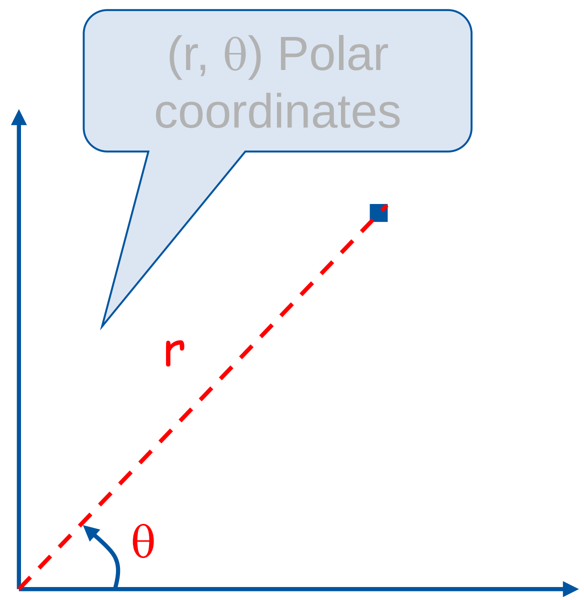
Coordinate systems

A **vector** has 2 or more quantities associated with it



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

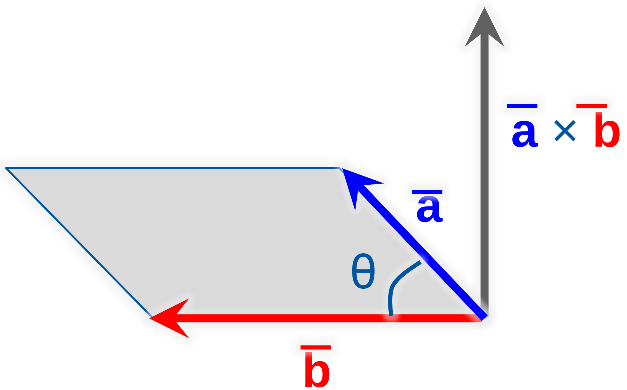


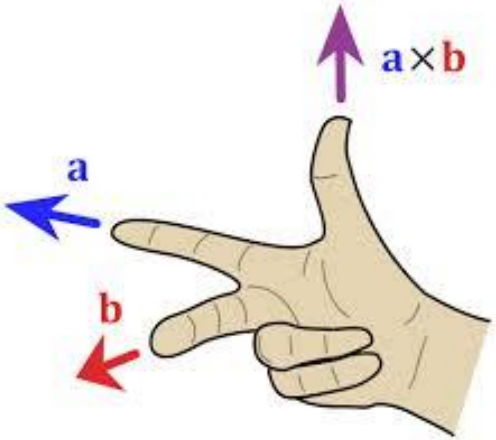
θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Vector Cross Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ





The cross product $\bar{a} \times \bar{b}$ is defined by:

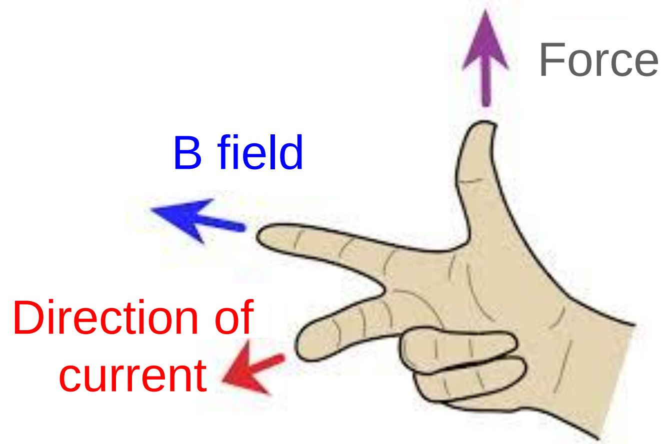
- **Direction**: $\bar{a} \times \bar{b}$ is perpendicular (normal) on the plane through $\bar{a}$ and $\bar{b}$
- The **length** of $\bar{a} \times \bar{b}$ is the surface of the parallelogram formed by $\bar{a}$ and $\bar{b}$

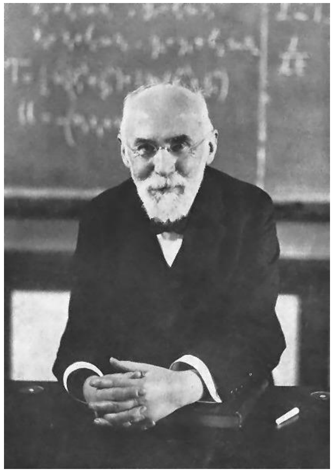
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

Cross Product & Magnetic Field

The Lorentz force is a pure magnetic field

$$F = e(\vec{v} \times \vec{B})$$





The reason why our particles move around our "circular" machines under the influence of the magnetic fields

Tuesday

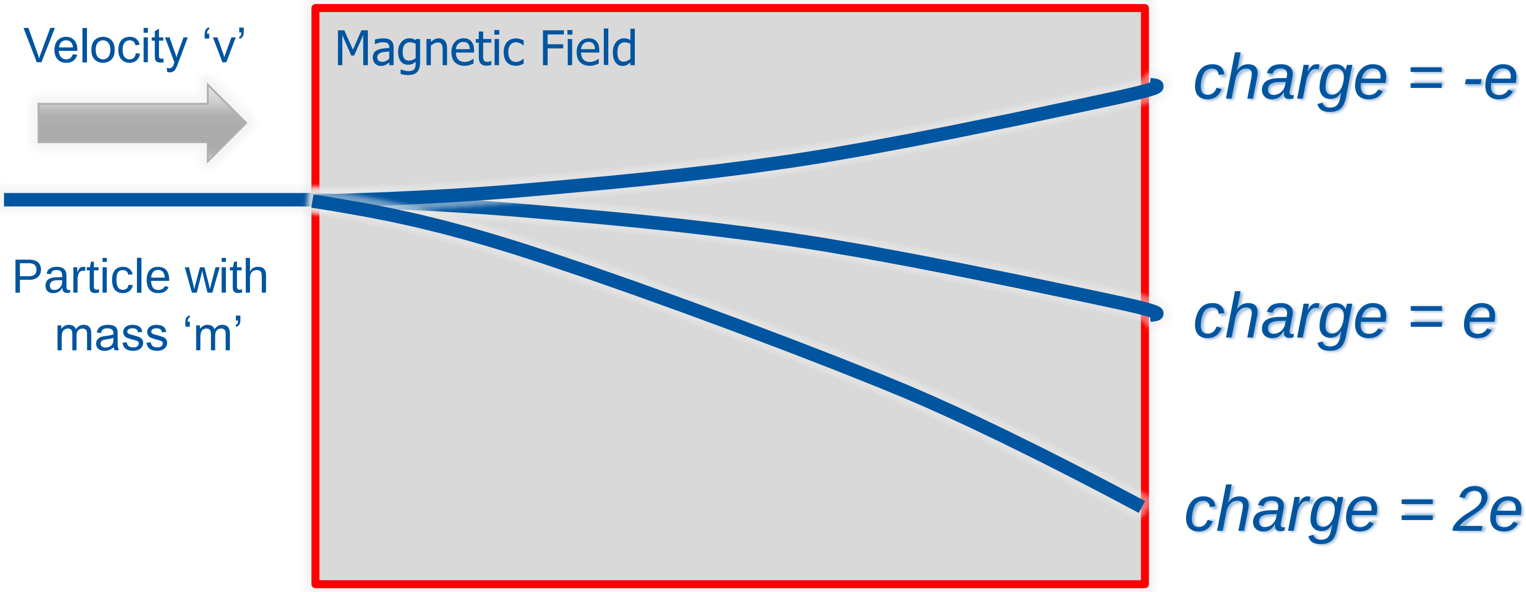
"Electro Magnetic Theory" by Werner Herr

After this

"Transverse Beam Dynamics" by Bernhard Holzer

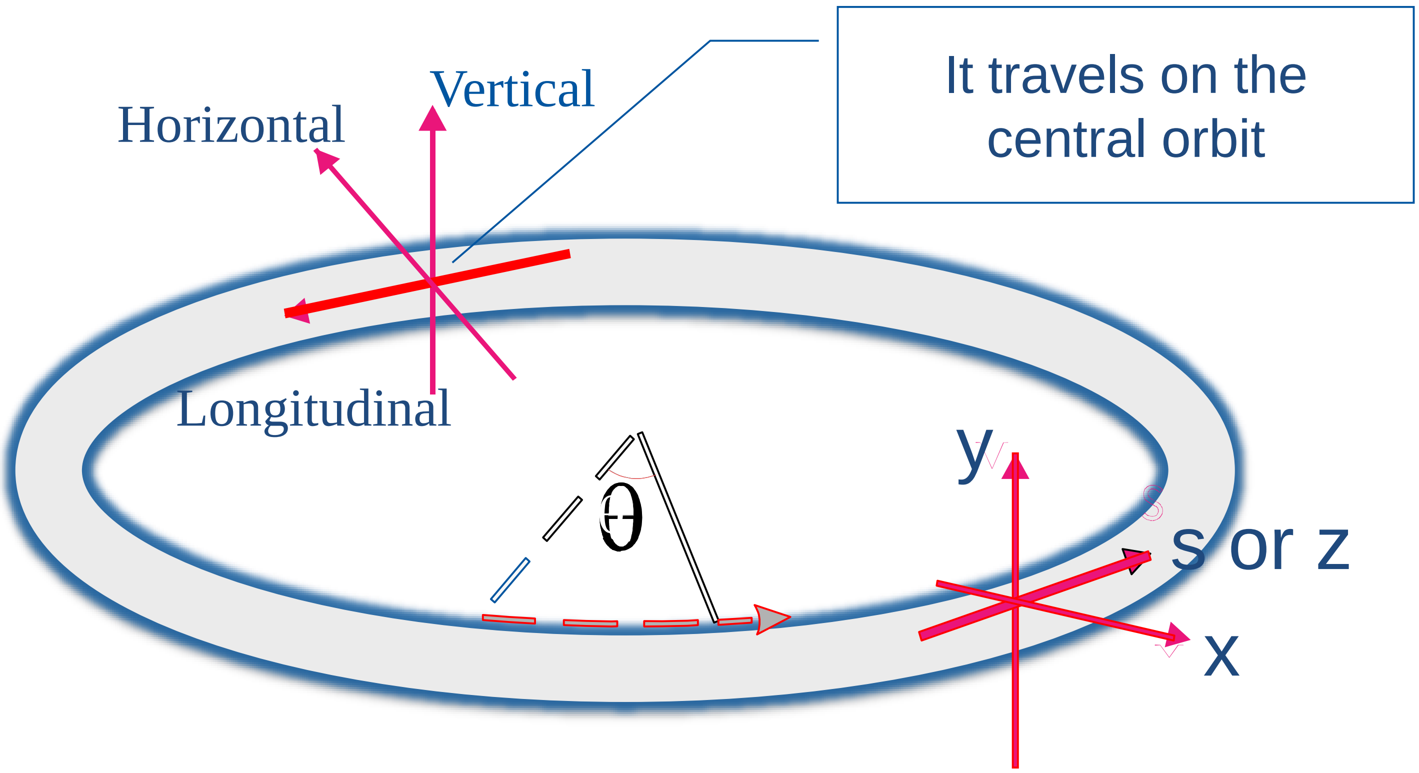
Lorentz Force in Action

$$F = e(\vec{v} \times \vec{B})$$



The larger the energy of the beam the larger the radius of curvature

A Rotating Coordinate System



Magnetic Rigidity

$$F = e(\vec{v} \times \vec{B})$$

- As a formula this is:

$$F = evB = \frac{mv^2}{\rho}$$

Radius of curvature

Like for a stone attached to a rotating rope

- Which can be written as:

$$B\rho = \frac{mv}{e} = \frac{p}{e}$$

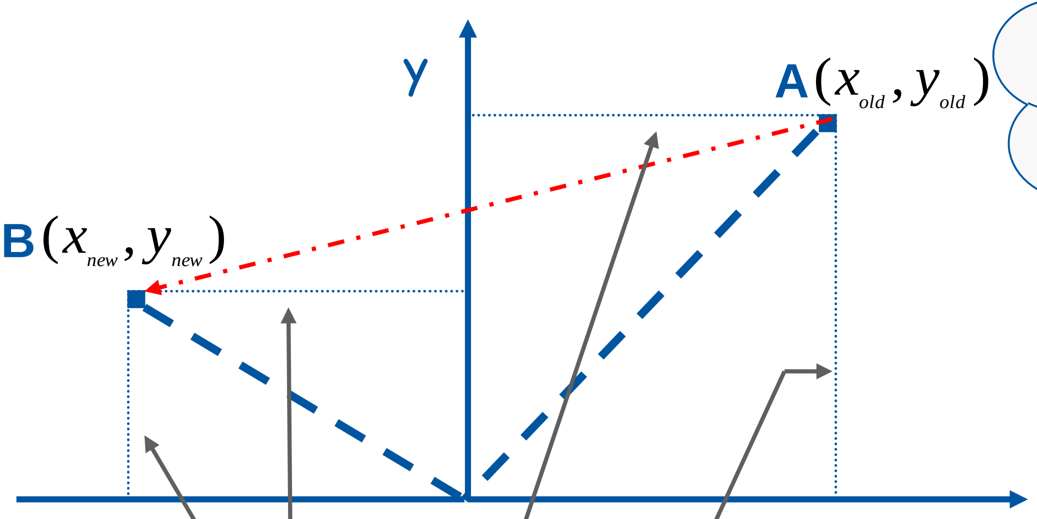
Momentum p=mv

- Bρ is called the magnetic rigidity, and if we put in all the correct units we get:
- Bρ = 33.356·p [KG·m] = 3.3356·p [T·m] (if p is in [GeV/c])

Moving a Point in a Coordinate System

To move from one point (A) to any other point (B) one needs control of both **Length** and **Direction**

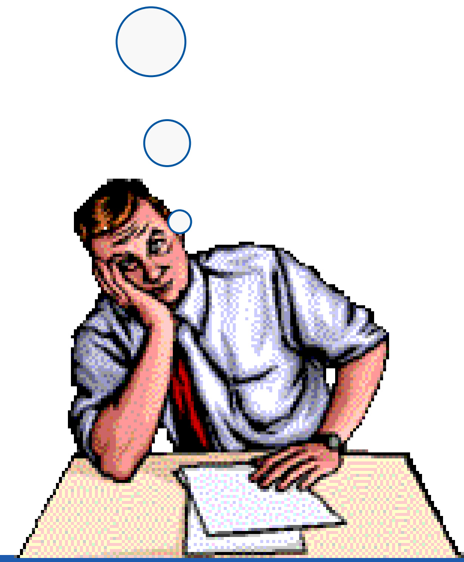
Rather clumsy !
Is there a more efficient way of doing this ?



$$x_{new} = a x_{old} + b y_{old}$$

$$y_{new} = c x_{old} + d y_{old}$$

2 equations needed !!!



Matrices & Vectors

So, we have:

$$\begin{cases} x_{new} = a x_{old} + b y_{old} \\ y_{new} = c x_{old} + d y_{old} \end{cases}$$

Lets write this as **<u>one</u>** equation:

$$\overline{B} = M\overline{A}$$

Rows

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Columns

- $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are **<u>Vectors</u>** or **<u>Matrices</u>**
- $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ have 2 rows and 1 column
- **M** is a **<u>Matrix</u>** and has 2 rows and 2 columns

Matrix Multiplications

This implies:

$$\left.\begin{aligned} x_{new} &= ax_{old} + by_{old} \\ y_{new} &= cx_{old} + dy_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

This defines the rules for **matrix multiplication**

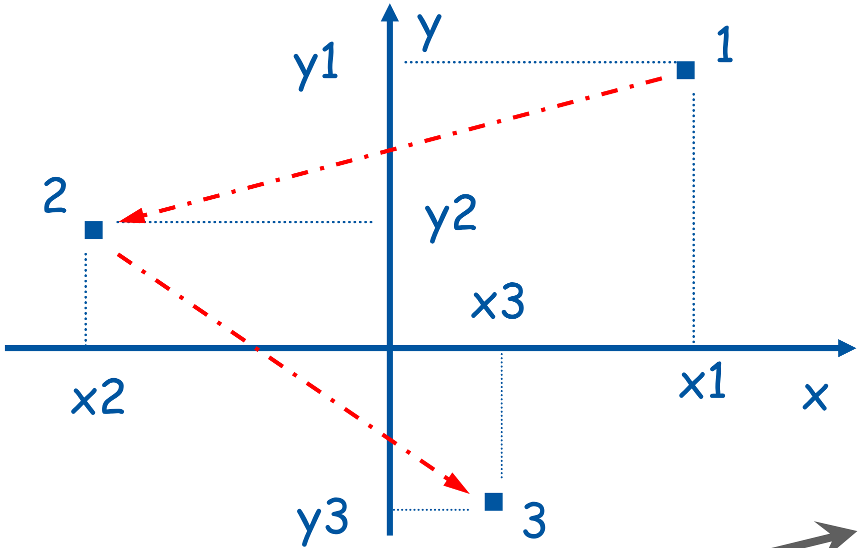
$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

This matrix multiplication results in:

$$i = ae + bg,\ j = af + bh,\ k = ce + dg,\ l = cf + dh$$

Moving a Point & Matrices

Lets apply what we just learned and move a point around:



- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = M1\begin{pmatrix}x_1\\y_1\end{pmatrix}$$

$$\begin{pmatrix}x_3\\y_3\end{pmatrix} = M2\begin{pmatrix}x_2\\y_2\end{pmatrix} = M2.M1\begin{pmatrix}x_1\\y_1\end{pmatrix}$$

$$\begin{pmatrix}x_3\\y_3\end{pmatrix} = M3\begin{pmatrix}x_1\\y_1\end{pmatrix}$$

Matrices & Accelerators

- We use matrices to describe the various magnetic elements in our accelerator.
  - The **x** and **y** co-ordinates are the **position** and **angle** of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we **multiply all the matrices** describing the magnetic elements between the two points to give a single matrix

- Now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

The Unit Matrix

There is a special matrix that when multiplied with an initial point will result in the same final point.

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

The **<u>Unit matrix</u>** has **<u>no effect</u>** on x and y

Going Backwards

What about **going back** from a **final** point to the corresponding **initial** point ?

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \quad \text{or} \quad \overline{B} = M\overline{A}$$

For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B} \quad \text{such that} \quad \overline{B} = MM^{-1}\overline{B}$$

The combination of M and M^{-1} does have no effect

$$MM^{-1} = Unit\ Matrix$$

M^{-1} is the "**inverse**" or "**reciprocal**" matrix of **M.**

Calculating the Inverse Matrix

If we have a 2 x 2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

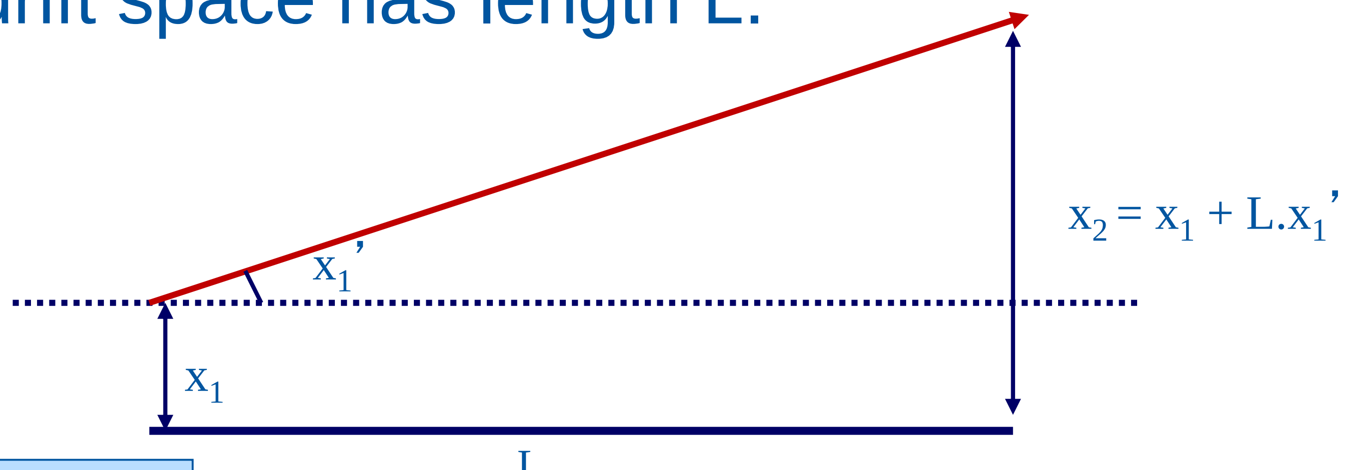
Then the **inverse matrix** is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term **(ad – bc)** is called the **determinate**, which is just a **number** (scalar).

Example: Drift Space Matrix

- A drift space contains no magnetic field.
- A drift space has length L.



x_1' small

$$\left.\begin{aligned} x_2 &= x_1 + Lx_1' \\ x_2' &= 0 + x_1' \end{aligned}\right\} \longrightarrow \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

A Practical Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M\,\overline{\Delta I}$$

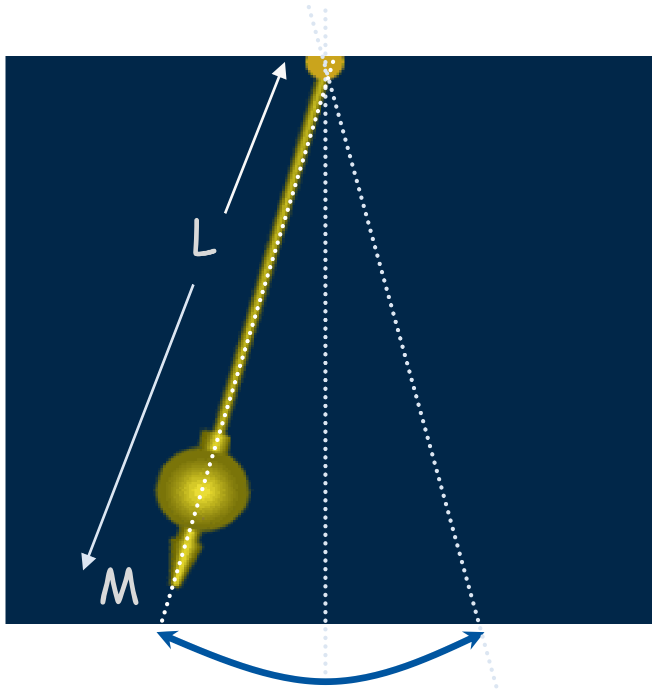
- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1}\,\overline{\Delta Q}$$

- Vectors & Matrices
- **Differential Equations**
- Some Units we use

The Pendulum

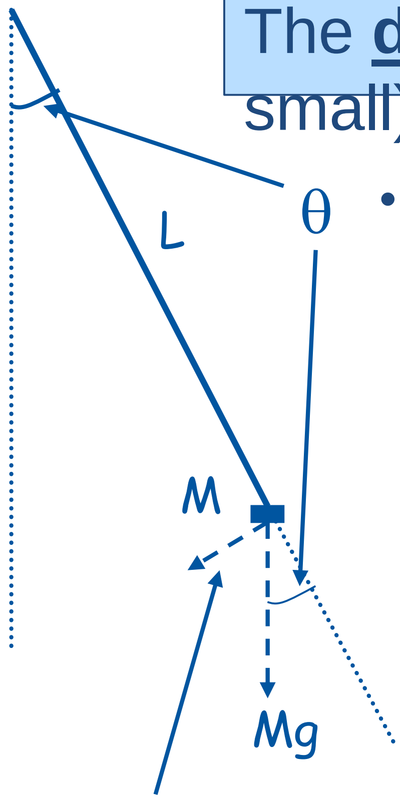
- Lets use a pendulum as example
- The **length** of the pendulum is **L**
- It has a **mass m** attached to it
- It moves back and forth under the **influence of gravity**



- Lets try to find an **equation** that **describes** the **motion** of the mass **m** makes.
- This will result in a **Differential Equation**

Differential Equation

The **distance** from the centre = **Lθ** (since θ is small)



- The **velocity** of mass M is: $v = \dfrac{d(L\theta)}{dt}$
- The **acceleration** of mass M is: $a = \dfrac{d^2(L\theta)}{dt^2}$
- Newton: **F**orce = **m**ass x **a**cceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity is

-M g sinθ

(force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

θ is small

L is constant

Solving a Differential Equation

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

Differential equation describing the motion of a pendulum at small amplitudes.

Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t)$$ and $$\frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving a Differential Equation

Now we have to find the solution for the following equation:

$$-\omega^2\cos(\omega t)+\left(\frac{g}{L}\right)\cos(\omega t)=0$$

Solving this equation gives: $\omega=\sqrt{\frac{g}{L}}$

The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}t$$

Oscillation amplitude

Oscillation frequency

Position & Velocity

The differential equation that describes the transverse motion of the particles as they move around our accelerator.

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

The solution of this second order describes **oscillatory motion**

For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

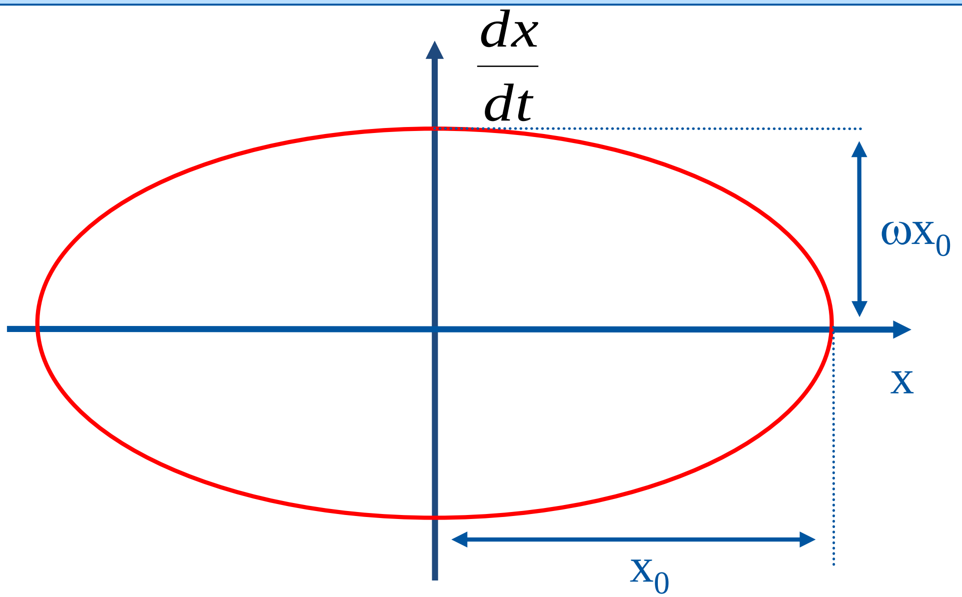
$$x = x_0 \cos(\omega t) \qquad \frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Phase Space Plot

Plot the **velocity** as a function of **displacement**:

$$x = x_0 \cos(\omega t)$$

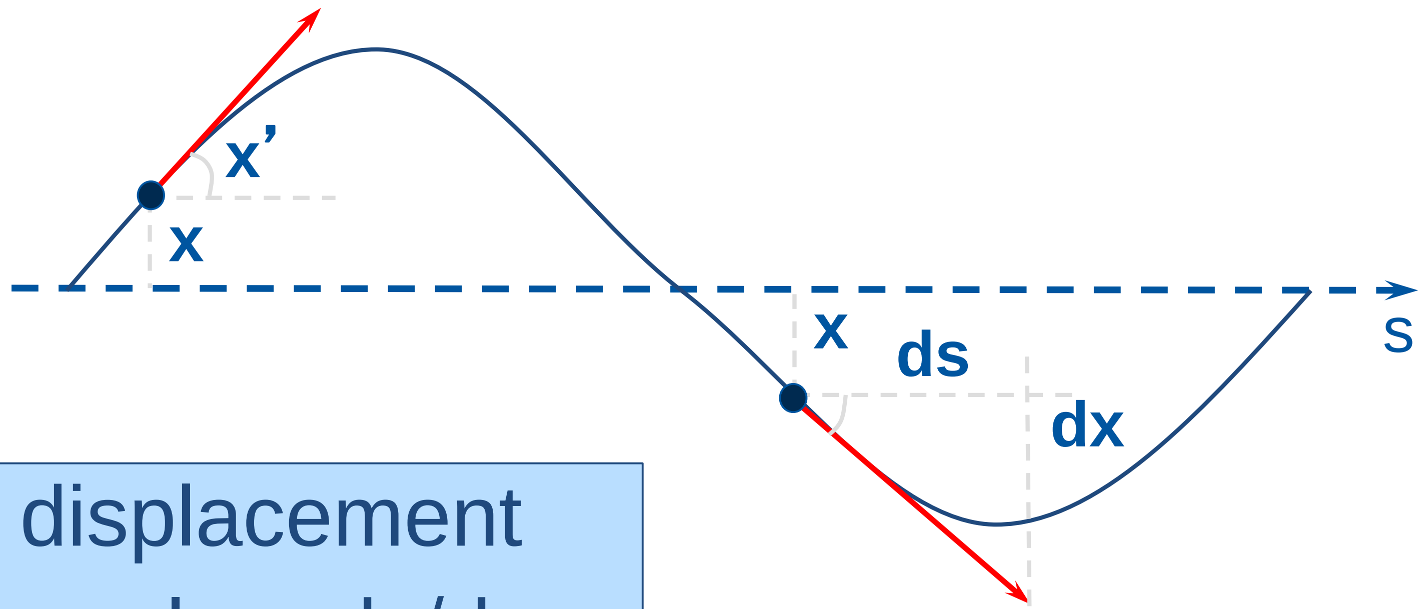
$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$



- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

Oscillations in Accelerators

Under the influence of the **magnetic fields** the **particle oscillate**



x = displacement

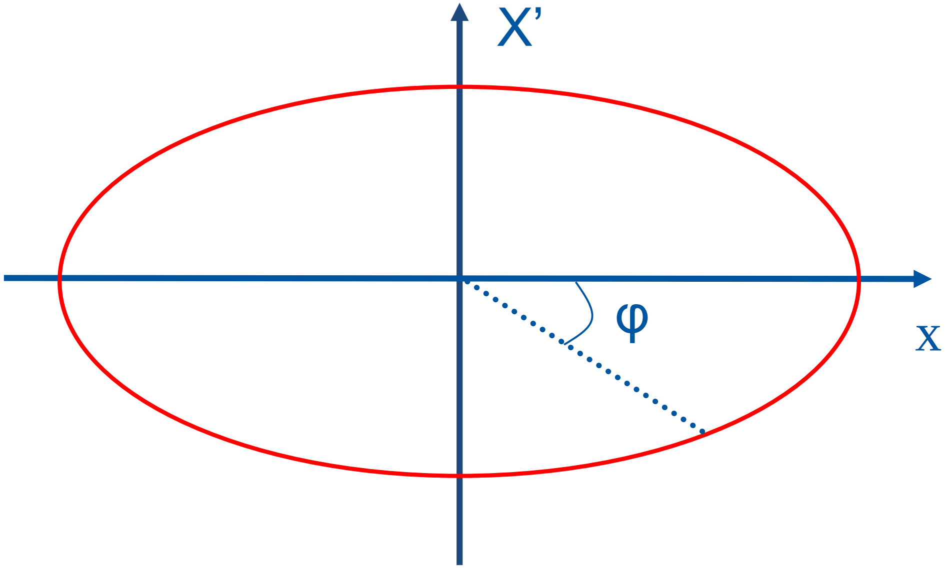
x' = angle = dx/ds

Transverse Phase Space Plot

This changes slightly the Phase Space plot

Position x

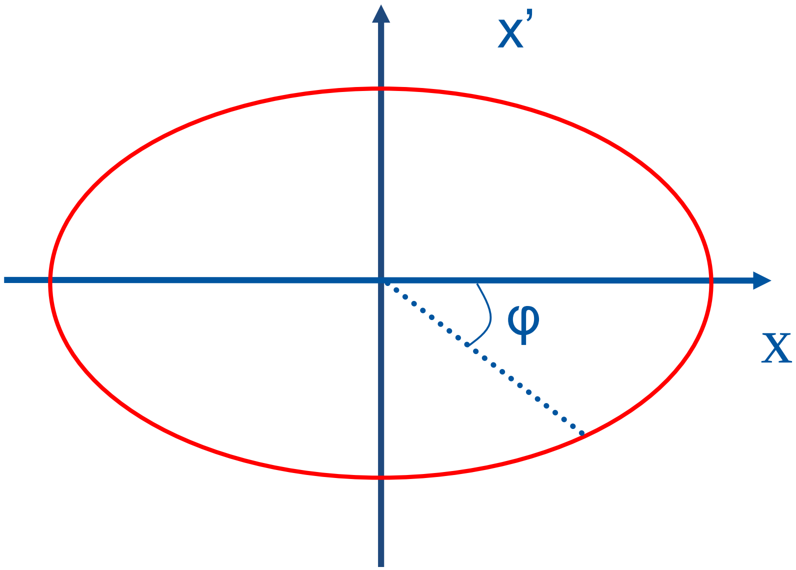
Angle $x' = \dfrac{dx}{ds}$



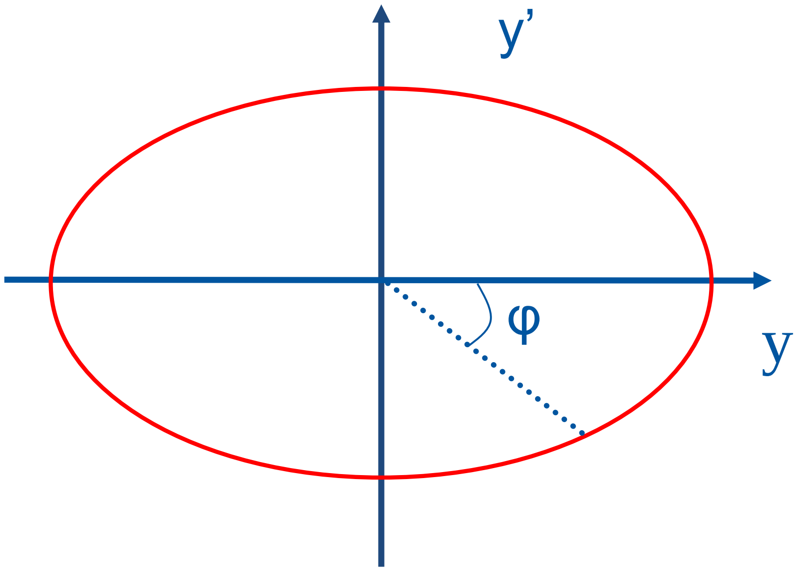
- φ = ωt is called the **phase angle**
- X-axis is the horizontal or vertical position (or time).
- Y-axis is the horizontal or vertical phase angle (or energy).

Transverse Phase Space Plot

We distinguish motion in the Horizontal & Vertical Plane



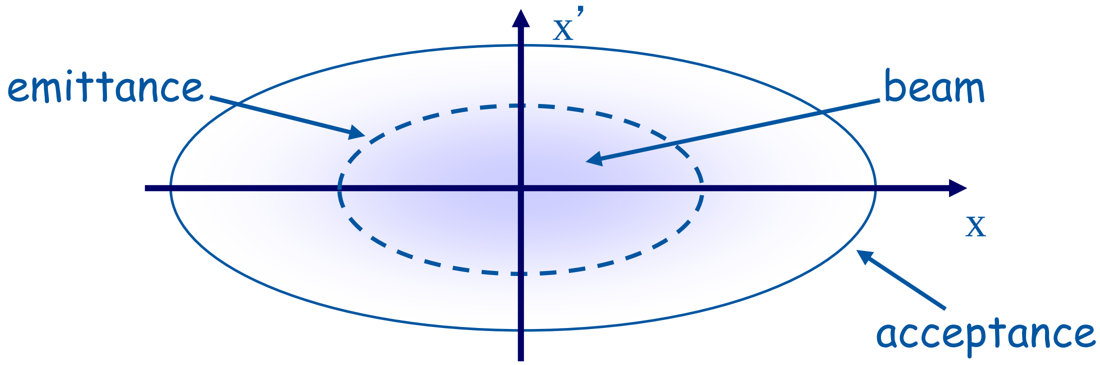
Horizontal Phase Space



Vertical Phase Space

Transverse Emittance

- To be rigorous we should define the emittance slightly differently.
- Observe all the particles at a single position on one turn and measure both their position and angle.
- This will give a large number of points in our phase space plot, each point representing a particle with its co-ordinates x, x'.

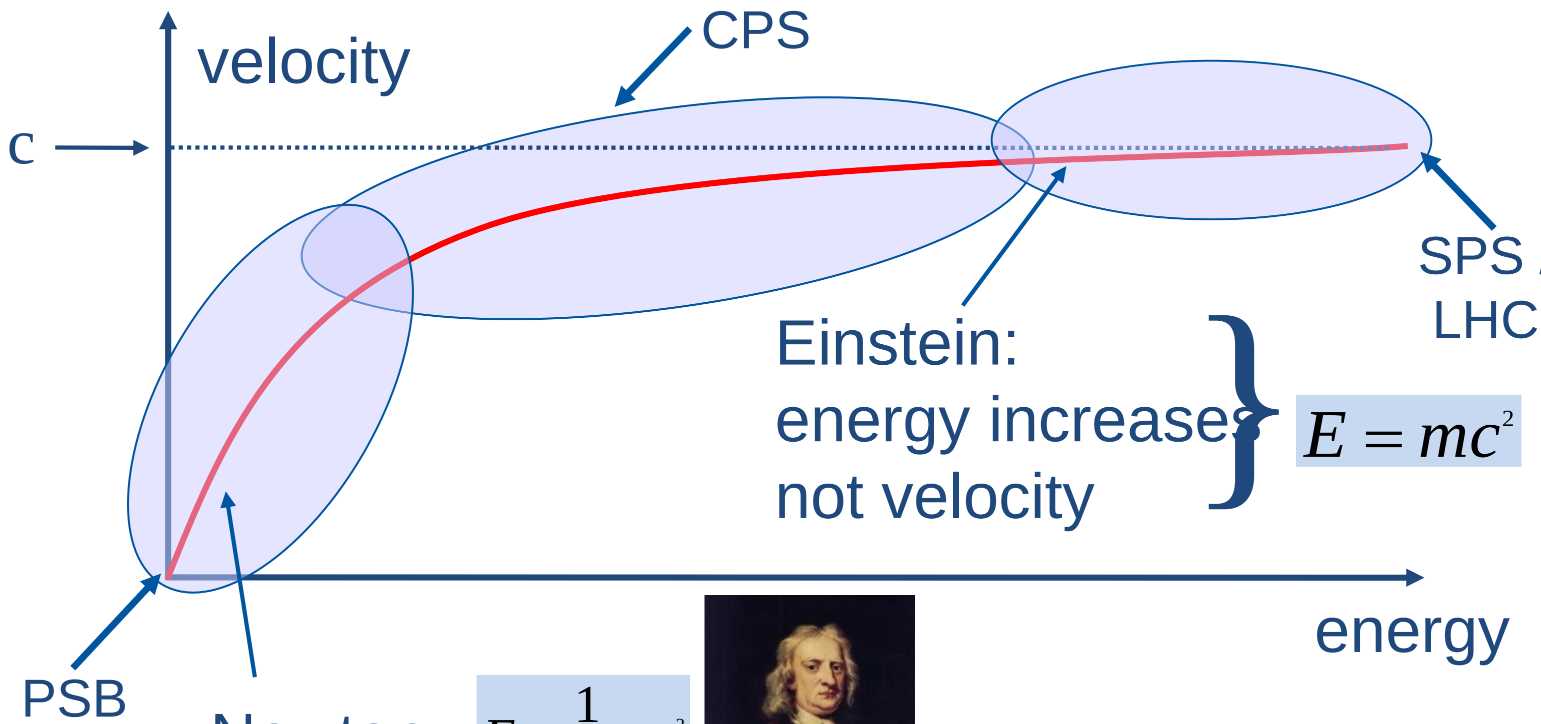


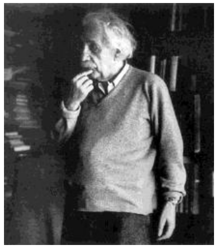
Symbol: ε
Expressed in 1σ, 2σ,..
Units: mm mrad

- The **emittance** is the **area** of the ellipse, which contains all, or a defined percentage, of the particles.
- The **acceptance** is the maximum **area** of the ellipse, which the emittance can attain without losing particles

- Vectors & Matrices
- Differential Equations
- **Some Units we use**

Relativity





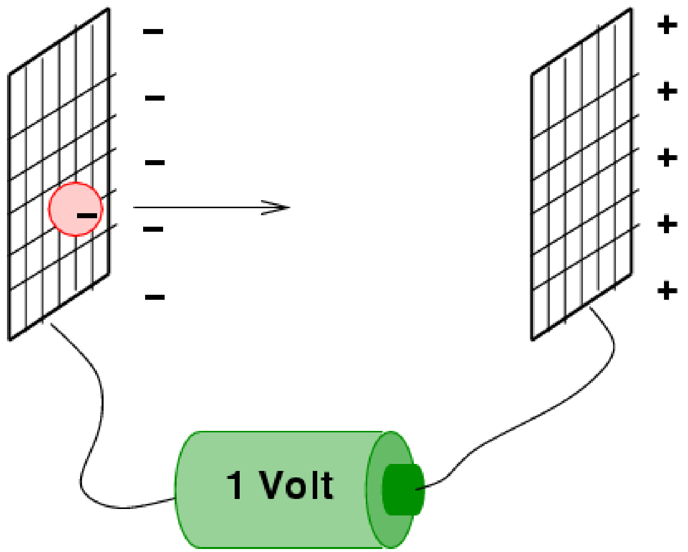
Newton: $E = \frac{1}{2} m v^2$

More about "Relativity" by Werner Herr

This afternoon

The Units we use for Energy

- The energy acquired by an electron in a potential of 1 Volts is defined as being 1 eV



- The unit eV is too small to be used today, we use:

  1 KeV = 10^3, MeV = 10^6, GeV = 10^9, TeV = 10^{12}

Energy: eV versus Joules

- The unit most commonly used for **Energy** is **Joules [J]**
- In accelerator and particle physics we talk about **eV**….!?
- The **energy** acquired by an **electron** in a potential of **1 Volt** is defined as being **1 eV**
- **1 eV** is **1 elementary charge** ‘pushed’ by **1 Volt**.

$$\mathbf{1\ eV = 1.6 \times 10^{-19}\ Joules}$$

The Energy in the LHC beam

- The energy in one LHC beam at high energy is about 320 Million Joules
- This corresponds to the energy of a TGV engine going at 150 km/h

...... but then concentrated in the size of a needle

Energy versus Momentum

Einstein's formula:

$E = mc^2$ which for a mass at rest is: $E_0 = m_0 c^2$

The ratio between the total energy and the rest energy is

$$\gamma = \frac{E}{E_0}$$

The ratio between the real velocity and the velocity of light is

$$\beta = \frac{v}{c}$$

Then the mass of a moving particle is: $m = \gamma m_0$

We can write: $\beta = \dfrac{mvc}{mc^2}$

Momentum is: $p = mv$

$$\beta = \frac{pc}{E} \quad or \quad p = \frac{E\beta}{c}$$

Energy versus Momentum

Energy

$$p = \frac{E\beta}{c}$$

Momentum

- Therefore the **units** for
  - **momentum** are: MeV/c, GeV/c, …etc.
  - **Energy** are: MeV, GeV, …etc.

Attention:

when **β=1** **energy** and **momentum** are **equal**

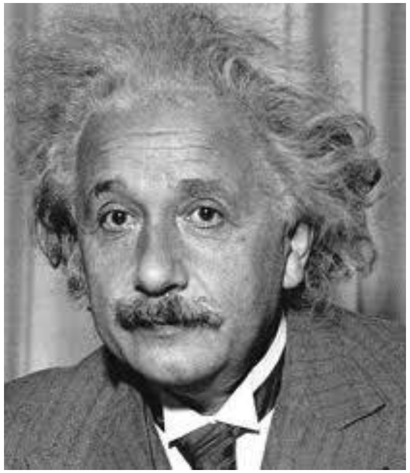
when **β<1** the **energy** and **momentum** are **not equal**

A Practical Example (PSB- PS)

- Kinetic energy at injection $E_{kinetic}$ = 1.4 GeV
- Proton rest energy E_0=938.27 MeV
- The total energy is then: $E = E_{kinetic} + E_0$ = **2.34 GeV**
- We know that $\gamma = \frac{E}{E_0}$, which gives γ = 2.4921
- We can derive $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, which gives **β = 0.91597**
- Using $p = \frac{E\beta}{c}$ we get p = **2.14 GeV/c**

In this case: **Energy ≠ Momentum**

Pure mathematics is, in its way, the poetry of logical ideas.



Albert Einstein