## Space Charge

JAI Graduate Course

Dr. Suzie Sheehy
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John Adams Institute, University of Oxford

## Table of contents

1. Introduction
2. Space Charge Forces
3. Space Charge in Transport Line
4. Image Effects
5. Incoherent vs Coherent Effects
6. Examples
7. Conclusion

Introduction

## Space Charge

The basic idea behind space charge is very simple. We impose electromagnetic fields on a beam of particles, but we must also take into account the EM fields produced by the beam itself.

These fields can consist of:

1. Direct self fields
2. Image self fields
3. Wakefields

- not discussed in this lecture



## Space Charge

Consider two point charges, $q$, spaced a distance $r$ apart. They experience a repulsive Coulomb force,

$$
\begin{equation*}
F_{\text {elec. }}=\frac{q^{2}}{4 \pi \epsilon_{0} r^{2}} \tag{1}
\end{equation*}
$$

In an accelerator the particles are moving with some velocity, v. This is equivalent to a current carrying wire with $I=q v$. Recall that between two current carrying wires, there is in fact an attractive force,

$$
\begin{equation*}
F_{\text {mag. }}=\frac{\mu_{0} I^{2}}{4 \pi r^{2}}=\frac{\mu_{0} q^{2} v^{2}}{4 \pi r^{2}}=\frac{v^{2}}{c^{2}} F_{\text {elec. }} \tag{2}
\end{equation*}
$$



## Space Charge

Combining Eqns. [1] and [2], the overall force is repulsive

$$
\begin{equation*}
F_{\text {total }}=\left(1-v^{2} / c^{2}\right) F_{\text {elec. }} \tag{3}
\end{equation*}
$$

This cancels to almost zero in the case $v \approx c$, i.e. for electrons travelling near to the speed of light. For hadron (proton or ion) machines, often $\beta=v / c \approx 0.5$ so the space charge repulsion becomes significant.

Of course, this is only for two charges. In reality we have a full beam with some intensity.

## Space Charge

If we take a cross-section through the beam:


Repulsive Coulomb force


Attractive magnetic force

Note that the force on a test particle at the centre of the beam is zero and the force increases nearer the beam edge.

## Aside: What does 'space charge' mean?

There are two 'regimes' to describe the net effects of Coulomb interactions in a system with many particles.

Collisional regime: dominated by particle-on-particle collisions and described by single particle effects.

Space Charge regime: dominated by the self fields of the distribution of particles themselves, which varies over distances which are larger than the average particle separation and described by collective effects.

## Aside: What does 'space charge' mean?

To tell which regime we're in, it is useful to consider the Debye length $\lambda_{D}$.
In a beam moving at relativistic velocity, but assuming the transverse motion is non-relativistic,

$$
\begin{equation*}
\lambda_{D}=\sqrt{\frac{\epsilon_{0} \gamma^{2} k_{B} T}{q^{2} n}} \tag{4}
\end{equation*}
$$

$k_{B}$ is the Boltzmann constant, T is temperature, thus $k_{B} T$ is the average kinetic energy of the particles, and $n$ is the particle density $N / V$.

If the $\lambda_{D} \ll$ a (beam radius), collective effects due to self fields play an important role and we can use smooth functions of the charge and field distributions. For most beams of practical interest ${ }^{1}$, collisional forces are small and can be neglected.
${ }^{1}$ See M. Reiser, Chapter 4 for more discussion on this. Note that intrabeam scattering in high energy storage rings is an exception where collisional forces play a key role.

## Space Charge Forces

## Unbunched Uniform Beam

Consider a beam as a continuous cylinder of charge, length I, beam radius $a$, charge density

$$
\begin{equation*}
\rho(r)=q n(r)=\frac{I_{\text {beam }}}{\pi a^{2} v} \tag{5}
\end{equation*}
$$



The electric field is radial and inside the beam is given by Gauss' Flux theorem:

$$
\begin{equation*}
\int \epsilon_{0} E \cdot d S=\int \rho d V \tag{6}
\end{equation*}
$$

## Unbunched Uniform Beam



The electric field:

$$
2 \pi r \left\lvert\, \epsilon_{0} E_{r}= \begin{cases}\rho \pi r^{2} l, & \text { if } r \leq a \\ \rho \pi a^{2} l, & \text { if } r>a\end{cases}\right.
$$

Therefore:

$$
E_{r}= \begin{cases}\frac{I_{\text {beam }}}{2 \pi \epsilon_{0} c} \frac{r}{a^{2}}, & \text { if } r \leq a \\ \frac{I_{\text {beam }}}{2 \pi \epsilon_{0} \beta c} \frac{1}{r}, & \text { if } r>a\end{cases}
$$

## Unbunched Uniform Beam



The magnetic field is angular, $\vec{B}=B_{\phi}$ from Ampére's law:

$$
\begin{gather*}
\int B . d I=\mu_{0} \times\{\text { current flowing through a loop }\}  \tag{7}\\
2 / B_{\rho}=\mu_{0} J / r \tag{8}
\end{gather*}
$$

Where $J=\frac{I_{\text {beam }}}{\pi a^{2}}$.

$$
\begin{equation*}
\therefore B_{\phi}=\frac{\mu_{0} I_{\text {beam }} r}{2 \pi a^{2}} \text { for } r \leq a \tag{9}
\end{equation*}
$$

## Unbunched Uniform Beam

The force experienced by a test particle in the beam is given (as always...) by the Lorentz force. Taking the $E$ and $B$ fields from previous slides,

$$
\begin{gather*}
F_{r}=e\left(E_{r}-v_{s} B_{\phi}\right)  \tag{10}\\
F_{r}=\frac{e l_{\text {beam }}}{2 \pi \epsilon_{0} \beta c}\left(1-\beta^{2}\right) \frac{r}{a^{2}}=\frac{e l_{\text {beam }}}{2 \pi \epsilon_{0} \beta c^{2}} \frac{1}{\gamma^{2}} \frac{r}{a^{2}} \tag{11}
\end{gather*}
$$

## Unbunched Uniform Beam

$$
\begin{equation*}
F_{x}=\frac{e l_{\text {beam }}}{2 \pi \epsilon_{0} \beta c^{2}} \frac{1}{\gamma^{2}} \frac{x}{a^{2}}, \quad F_{y}=\frac{e l_{\text {beam }}}{2 \pi \epsilon_{0} \beta c^{2}} \frac{1}{\gamma^{2}} \frac{y}{a^{2}} \tag{12}
\end{equation*}
$$

Quadrupole
(Hor. Foc.)



Uniform



Gaussian

In the x and y directions for a circular beam, uniform charge density this gives a linear force in $x, y$, decreasing with energy. Note this is a defocusing lens in BOTH planes.

## Unbunched Gaussian Beam

If instead we assume the bunch has a transverse Gaussian profile (a bit more realistic):

$$
\begin{equation*}
n(r)=A \exp \left(-\frac{r^{2}}{2 \sigma^{2}}\right) \tag{13}
\end{equation*}
$$

Where $A=N / 2 \pi \sigma^{2}$ and $N$ is particles per unit length. Working this through gives us the space charge force as:

$$
\begin{equation*}
F(r)=\frac{N q^{2}}{2 \pi \epsilon_{0} \gamma^{2}} \frac{1}{r}\left(1-\exp \left(-\frac{r^{2}}{2 \sigma^{2}}\right)\right) \tag{14}
\end{equation*}
$$

## Space Charge in Transport Line

## Hill's equation

In a FODO transport line, we know the motion is described by Hill's equation, where we can add a perturbation term from the force due to space charge:

$$
\begin{equation*}
x^{\prime \prime}+\left(k(s)+k_{S C}(s)\right) x=0 \tag{15}
\end{equation*}
$$

$k_{S C}$ is derived by expressing $x^{\prime \prime}$ in terms of transverse acceleration $d^{2} x / d t^{2}$ and thus of the force $F_{x}$

$$
\begin{equation*}
x^{\prime \prime}=\frac{2 r_{0} I_{\text {beam }}}{e e^{2} \beta^{3} \gamma^{3} c} x \tag{16}
\end{equation*}
$$

Where the classical particle radius $r_{0}=e^{2} /\left(4 \pi \epsilon_{0} m_{0} c^{2}\right)=1.54 \times 10^{-18}$ for protons. Which yields the new Hill's equation:

$$
\begin{equation*}
x^{\prime \prime}+\left(k(s)-\frac{2 r_{0} I_{\text {beam }}}{e a^{2} \beta^{3} \gamma^{3} c}\right) x=0 \tag{17}
\end{equation*}
$$

## Incoherent tune shift

Space charge leads to defocusing in both planes, so we would expect that there will be a shift in betatron tune, $\Delta Q$. If we take the simplest case of an unbunched beam, with uniform circular cross section, we find by calculating the (effective) gradient errors around the ring:

$$
\begin{equation*}
\Delta Q_{x}=\frac{1}{4 \pi} \int_{0}^{2 \pi r} k_{s c} \beta_{x}(s) d s \tag{18}
\end{equation*}
$$

Using $k_{S C}$ from before:

$$
\begin{equation*}
\Delta Q=-\frac{1}{4 \pi} \int_{0}^{2 \pi R} \frac{2 r_{0} I_{b}}{e \beta^{3} \gamma^{3} c} \frac{\beta_{x}(s)}{a^{2}} d s=-\frac{r_{0} R I_{b}}{e \beta^{3} \gamma^{3} c}\left\langle\frac{\beta_{x}(s)}{a^{2}(s)}\right\rangle . \tag{19}
\end{equation*}
$$

If we use that $\left\langle\frac{\beta_{x}(s)}{a^{2}(s)}\right\rangle=\frac{1}{\epsilon_{0}}$, the $100 \%$ emittance, and replace $I=N e \beta c /(2 \pi R)$, we get for the direct space charge tune shift:

$$
\begin{equation*}
\Delta Q_{x, y}=-\frac{r_{0} N}{2 \pi \epsilon_{x, y} \beta^{2} \gamma^{3}} \tag{20}
\end{equation*}
$$

## Incoherent tune shift

Things to note about the tune shift:

$$
\begin{equation*}
\Delta Q_{x, y}=-\frac{r_{0} N}{2 \pi \epsilon_{x, y} \beta^{2} \gamma^{3}} \tag{21}
\end{equation*}
$$

- 'Direct' space charge, unbunched beam in a synchrotron
- Vanishes for $\gamma \gg 1$
- Important for low-energy machines
- Independent of machine size $2 \pi R$ for a given $N$
- Incoherent motion - particle moves within the beam.



## Incoherent tune shift

$$
\begin{equation*}
\Delta Q_{x, y}=-\frac{r_{0} N}{2 \pi \epsilon_{x, y} \beta^{2} \gamma^{3}} \tag{22}
\end{equation*}
$$

Taking the values for the CERN PS Booster, (assume unbunched), calculate the tune shift:

- $N=1 \times 10^{13}$ protons
- $\epsilon_{x, y}=80,27 \mu \mathrm{rad} \mathrm{m}$
- $E=50 \mathrm{MeV}$, i.e. $\gamma=1.053, \beta=0.314$

Image Effects

## Parallel Conducting Plates

Perfectly conducting plate parallel to beam pipe, produces an infinite system of images.

Field created by a line charge at distance $d$
 is

$$
\begin{equation*}
E_{y}=\frac{\lambda}{2 \pi \epsilon_{0}} \frac{1}{d} \tag{23}
\end{equation*}
$$

From first pair of images:

$$
\begin{align*}
& E_{1 y}=\frac{\lambda}{2 \pi \epsilon_{0}}\left(\frac{1}{2 h-y}-\frac{1}{2 h+y}\right)  \tag{24}\\
& E_{2 y}=\frac{\lambda}{2 \pi \epsilon_{0}}\left(\frac{1}{4 h-y}-\frac{1}{4 h+y}\right) \tag{25}
\end{align*}
$$

## Parallel Conducting Plates

$$
\begin{gather*}
E_{i n y}=\frac{(1)^{n+1} \lambda}{2 \pi \epsilon_{0}}\left(\frac{1}{2 n h-y}-\frac{1}{2 n h+y}\right)=(1)^{n+1} \frac{\lambda}{4 \pi \epsilon_{0}} \frac{y}{n^{2} h^{2}}  \tag{26}\\
E_{i y}=\sum_{n=1}^{\infty}=\frac{\lambda}{4 \pi \epsilon_{0} h^{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2}} y=\frac{\lambda}{4 \pi \epsilon_{0} h^{2}} \frac{\pi^{2}}{12} y  \tag{27}\\
\therefore F_{y}^{i}=\frac{q \lambda}{\pi \epsilon_{0} h^{2}} \frac{\pi^{2}}{48} y, F_{x}^{i}=-\frac{q \lambda}{\pi \epsilon_{0} h^{2}} \frac{\pi^{2}}{48} x \tag{28}
\end{gather*}
$$

- The vertical image field vanishes at $y=0$
- Field is linear in $y$, vertically defocusing
- Field is large if vacuum chamber is small


## Incoherent Tune Shift

The total incoherent tune shift for a round beam between parallel conducting walls:

$$
\begin{align*}
& \Delta Q_{x}=-\frac{2 r_{0} I_{b} R\left\langle\beta_{x}\right\rangle}{q c \beta^{3} \gamma}(\underbrace{\frac{1}{2\left\langle a^{2}\right\rangle \gamma^{2}}}_{\text {direct }}-\underbrace{\frac{\pi^{2}}{48 h^{2}}}_{\text {image }})  \tag{29}\\
& \Delta Q_{y}=-\frac{2 r_{0} I_{b} R\left\langle\beta_{y}\right\rangle}{q c \beta^{3} \gamma}(\underbrace{\frac{1}{2\left\langle a^{2}\right\rangle \gamma^{2}}}_{\text {direct }}+\underbrace{\frac{\pi^{2}}{48 h^{2}}}_{\text {image }}) \tag{30}
\end{align*}
$$

- Image effects $\propto 1 / \gamma$
- They do not vanish for large $\gamma$ so not negligible for electron machines
- Electrical image effects are normally focusing in horizontal, defocusing in vertical plane
- Note there are also image effects from the ferromagnetic boundary

Incoherent vs Coherent Effects

## Incoherent and Coherent Motion

## Incoherent motion



Test particle in a beam whose centre of mass does not move

The beam environment does not "see" any motion

Each particle features its individual amplitude and phase

## Coherent motion



The centre of mass moves doing betatron oscillation as a whole

The beam environment (e.g. a position monitor "sees" the "coherent motion")

On top of the coherent motion, each particles has still its individual one

## Coherent Tune Shift, Round Beam Pipe


$\overline{\mathrm{X}}$. .hor. beam position (centre of mass)
a...beam radius
$\rho$...beam pipe radius ( $\rho \gg$ a)
$\mathrm{b} \overline{\mathrm{x}}=\rho^{2} \quad$ (mirror charge on a circle)

$$
\mathrm{E}_{\mathrm{ix}}(\overline{\mathrm{x}})=\frac{\lambda}{2 \pi \varepsilon_{0}} \frac{1}{\mathrm{~b}-\overline{\mathrm{x}}} \approx \frac{\lambda}{2 \pi \varepsilon_{0}} \frac{1}{\mathrm{~b}}=\frac{\lambda}{2 \pi \varepsilon_{0}} \frac{1}{\rho^{2}} \overline{\mathrm{x}}
$$

$$
\mathrm{F}_{\mathrm{ix}}(\overline{\mathrm{x}})=\frac{\mathrm{e} \lambda}{2 \pi \varepsilon_{0}} \frac{1}{\rho^{2}} \overline{\mathrm{x}} \quad \begin{aligned}
& \square \text { same in vertical plane }(\mathrm{y}) \text { due to symmetry } \\
& \square \text { force linear in } \overline{\mathrm{X}}
\end{aligned}
$$

$$
\Delta \mathrm{Q}_{\mathrm{x}, \mathrm{y} \text { coh }}=-\frac{\mathrm{r}_{0} \mathrm{R}\left\langle\beta_{\mathrm{x}, \mathrm{y}}\right\rangle \mathrm{I}}{\operatorname{ec} \beta^{3} \gamma \rho^{2}}=-\frac{\mathrm{r}_{0}\left\langle\beta_{\mathrm{x}, \mathrm{y}}\right\rangle}{2 \pi \beta^{2}} \frac{\mathrm{~N}}{\gamma \rho^{2}}
$$

Coherent tune shift, round pipe
negative (defocusing) both planes
only weak dependence on $\gamma$
$\square \Delta Q_{\text {coh }}$ always negative

## The "Laslett"* Coefficients



$$
\begin{array}{r}
\Delta \mathrm{Q}_{\mathrm{y}, \text { inc }}=-\frac{\mathrm{Nr}_{0}\left\langle\beta_{\mathrm{y}}\right\rangle}{\beta^{2} \gamma \pi}\left(\frac{\varepsilon_{0}^{\mathrm{y}}}{\mathrm{~b}^{2} \gamma^{2}}+\frac{\varepsilon_{1}^{\mathrm{y}}}{\mathrm{~h}^{2}}+\beta^{2} \frac{\varepsilon_{2}^{\mathrm{y}}}{\mathrm{~g}^{2}}\right) \\
\text { direct electr. magnet. } \\
\Delta \mathrm{Q}_{\mathrm{y}, \text { coh }}=-\frac{\mathrm{Nr}_{0}\left\langle\beta_{\mathrm{y}}\right\rangle}{\beta^{2} \gamma \pi}\binom{\text { image image }}{\frac{\xi_{1}^{\mathrm{y}}}{\mathrm{~h}^{2}}+\beta^{2} \frac{\xi_{2}^{\mathrm{y}}}{\mathrm{~g}^{2}}}
\end{array}
$$

Uniform, elliptical beam in an elliptical beam pipe. Similar formulae for $\Delta \mathbf{Q}_{x}$ In general, $\Delta \mathrm{Q}_{\mathrm{y}}>\Delta \mathrm{Q}_{\mathrm{x}}$
*L.J. Laslett, 1963

| Laslett <br> coefficients | Circular <br> $(a=b, w=h)$ | Elliptical <br> $($ e.g. $w=2 h)$ | Parallel plates <br> $(h / w=0)$ |
| :---: | :---: | :---: | :---: |
| $\varepsilon_{0}^{\mathrm{X}}$ | $1 / 2$ | $\frac{b^{2}}{a(a+b)}$ |  |
| $\varepsilon_{0}^{\mathrm{y}}$ | $1 / 2$ | $\frac{b}{a+b}$ |  |
| $\varepsilon_{1}^{\mathrm{x}}$ | 0 | -0.172 | -0.206 |
| $\varepsilon_{1}^{\mathrm{y}}$ | 0 | 0.172 | 0.206 |
| $\xi_{1}^{\mathrm{x}}$ | $1 / 2$ | 0.083 | 0 |
| $\xi_{1}^{\mathrm{y}}$ | $1 / 2$ | 0.55 | $0.617\left(\pi^{2} / 16\right)^{8}$ |
| $\varepsilon_{2}^{\mathrm{x}}$ | $-0.411\left(-\pi^{2} / 24\right)$ | -0.411 | -0.411 |
| $\varepsilon_{2}^{\mathrm{y}}$ | $0.411\left(\pi^{2} / 24\right)$ | 0.411 | 0.411 |
| $\xi_{2}^{\mathrm{x}}$ | 0 | 0 | 0 |
| $\xi_{2}^{\mathrm{y}}$ | $0.617\left(\pi^{2} / 16\right)$ | 0.617 | 0.617 |

## Bunched Beam in a Synchrotron



What's different with bunched beams?
$\square$ Q-shift much larger in bunch centre than in tails
$\square$ Q-shift changes periodically with $\omega_{\mathrm{s}}$
$\square$ peak Q-shift much larger than for unbunched beam with same $N$ (number of particles in the ring)
$\square$ Q-shift $\Rightarrow$ Q-spread over the bunch

## Incoherent $\Delta \mathrm{Q}:$ A Practical Formula

$$
\begin{gathered}
\Delta Q_{y}=-\frac{r_{0}}{\pi}\left(\frac{q^{2}}{A}\right) \frac{N}{\beta^{2} \gamma^{3}} \frac{F_{y} G_{y}}{B_{f}}\left\langle\frac{\beta_{y}}{b(a+b)}\right\rangle \quad\left\langle\frac{\beta_{y}}{b(a+b)}\right\rangle=\left\langle\frac{\beta_{y}}{b^{2}\left(1+\frac{a}{b}\right)} \approx \frac{1}{E_{y}\left(1+\sqrt{\frac{E_{x} Q_{y}}{E_{y} Q_{x}}}\right)}\right. \\
\Delta Q_{x, y}=-\frac{r_{0}}{\pi}\left(\frac{q^{2}}{A}\right) \frac{N}{\beta^{2} \gamma^{3}} \frac{F_{x, y} G_{x, y}}{B_{f}} \frac{1}{E_{x, y}\left(1+\sqrt{\frac{E_{y, x} Q_{x, y}}{E_{x, y} Q_{y, x}}}\right)}
\end{gathered}
$$

q/A...... charge/mass number of ions (1 for protons, e.g. 6/16 for ${ }_{16} \mathrm{O}^{6+}$ )
$F_{x, y} \ldots . .$. .Form factor" derived from Laslett's image coefficients $\varepsilon_{1}{ }^{x}, \varepsilon_{1}{ }^{y}, \varepsilon_{2}^{x}, \varepsilon_{2}^{y}(F \approx 1$ if dominated by direct space charge)
$G_{x, y}$.....Form factor depending on particle distribution in $x, y$. In general, $1<G \leq 2$ Uniform $G=1$ ( $E_{x, y} 100 \%$ emittance) Gaussian G=2 ( $E_{x, y} 95 \%$ emittance)
$B_{f . . . . . . ~ " B u n c h i n g ~ F a c t o r ": ~ a v e r a g e / p e a k ~ l i n e ~ d e n s i t y ~} B_{f}=\frac{\bar{\lambda}}{\hat{\lambda}}=\frac{\bar{I}}{\hat{I}}$

## Examples

## A Space-Charge Limited Accelerator



## How to Remove the Space-Charge Limit?

Direct space charge

$$
\Delta Q_{y} \approx \frac{N}{E_{y} \beta^{2} \gamma^{3}} \hat{\bar{I}}
$$

Problem: A large proton synchrotron is limited in N because $\Delta Q_{y}$ reaches $0.3 \ldots 0.5$ when filling the (vertical) acceptance. Solution: Increase $N$ by raising the injection energy and thus $\beta^{2} \gamma^{3}$ while keeping to the same $\Delta Q$. Ways to do this:

Make a longer (higher-energy) Linac (by adding tanks as has been done in Fermilab)


Add a small "Booster"
synchrotron of radius $r=R / n$ with $n$ the number of batches (BNL) or rings (CERN)


|  | Linac <br> $(\mathrm{MeV})$ | Booster <br> $(\mathrm{GeV})$ | $n=R / r$ | Potential <br> gain in $N$ | Achieved |
| :--- | :---: | :---: | :---: | :---: | :---: |
| CERN PS | 50 | 1 | 4(rings) | 59 | $\sim 15$ |
| BNL AGS | 200 | 1.5 | 4(batches) | 26 | $\sim 8$ |

High Intensity Proton Beam in a FODO Line

Transverse phase planes rad vs. $m$
horizontal vertical

Transverse envelopes mm vs. m


50 MeV



Courtesy of Alessandra Lombardi/ CERN, 8/04

## Summary

Coherent and incoherent tune shift in a synchrotron:
A high-intensity, un-bunched beam experiences a small deflection by a kicker magnet in one plane and performs betatron oscillations. The machine tune for vanishing intensity is known to be $Q_{0}$. A position detector measures the oscillations from which an effective tune $Q$ is derived. Is it:

1. equal to $Q_{0}$ ?
2. equal to $Q_{0}+Q_{\text {coherent }}$ ?
3. equal to $Q_{0}+Q_{\text {incoherent }}$ ?
4. equal to $Q_{0}+Q_{\text {coherent }}+Q_{\text {incoherent }}$ ?

Questions?

## Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the appendixnumberbeamer package in your preamble and call \appendix before your backup slides. metropolis will automatically turn off slide numbering and progress bars for slides in the appendix.

