# *Stability questions in dissipative hydrodynamics*

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1. Dissipative fluids
   – Criteria of extension: stability
2. Flow-frames and thermodynamics
   – Necessary and sufficient?

## *Dissipative relativistic fluids*

heavy ion collisions
cosmology

- Kinetic theory
  Do we need anything else? Dense matter?
  Role of heterogenities?
- Causality
  Hyperbolic or parabolic?
- Flow-frames
  Is there a freedom? (Eckart, Landau-Lifshitz, ...)
  What is ideal?
- Stability – second law
  What is equilibrium? Consistent extension.

# Nonrelativistic hydrodynamics I:

$$\dot{\rho} + \rho \partial_i v^i = 0$$
$$\rho \dot{v}^i + \partial_j P^{ij} = 0$$
$$\dot{e} + e \partial_i v^i + \partial_i q^i + P^{ij} \partial_i v_j = 0$$

balance of mass

balance of momentum

balance of internal energy

$i, j = 1,2,3 ; \quad \dot{e} = v^i \partial_i e$

$\rho$ – density
$v^i$ – velocity field
$e$ – internal energy density
$q^i$ – heat flux
$P^{ij}$ – pressure
$j^i$ – extra mass flux (?)
$p^i$ – momentum density (?)

Missing parts, relativistic knowledge:

$$\dot{\rho} + \rho \partial_i v^i + \partial_i j^i = 0$$
$$\dot{p}^i + p^i \partial_j v^j + \rho \dot{v}^i + \partial_j P^{ij} + j^k \partial_k v^i = 0^i$$
$$\dot{e} + e \partial_i v^i + \partial_i q^i + P^{ij} \partial_i v_j + p_i \dot{v}^i = 0$$

5 equations, 14/20 variables

# Nonrelativistic hydrodynamics II:

$$\dot{\rho}+\rho\,\partial_i v^i+\partial_i j^i=0$$
$$\dot{p}^i+p^i\partial_j v^j+\rho\dot{v}^i+\partial_j P^{ij}+j^k\partial_k v^i=0^i$$
$$\dot{e}+e\,\partial_i v^i+\partial_i q^i+P^{ij}\partial_i v_j+p_i\dot{v}^i=0$$

$\rho$ – density
$v^i$ – velocity field
$e$ – internal energy density
$q^i$ – heat flux
$P^{ij}$ – pressure

$j^i$ – particle current (?)
$p^i$ – momentum density (?)
$p$ – thermostatic pressure
T, s, μ – temperature, ...

$\upsilon^{ijkl}$ – viscosities
$\lambda$ – heat condictuvity

Constitutive theory – second law compatibility.
Static: *s(e,ρ)*
Dynamic: entropy production

$$P^{ij}=p\,\delta^{ij}+\Pi^{ij},\qquad \Pi^{ij}=-\nu^{ijkl}\partial_k v_l,\qquad q^i=\lambda\partial^i\beta$$

Stability: concave entropy and
nonnegative entropy production

Extension? Gradient and memory…

$$\Pi^{ij}(\partial^j v^i,\partial^{jk}v^i,\ldots,\rho,\partial_i\rho,\partial_{ij}\rho,\ldots)$$

Higher grade fluids and Korteweg fluids: instable

Capillarity, qfluids, second sound, etc..., the question is the universality.

# Special relativistic hydrodynamics:

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$

energy-momentum density

$$N^a = nu^a + j^a.$$

particle number density

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

$$T^{ab} = \begin{pmatrix} e & q^i \\ q^j & P^{ij} \end{pmatrix}, \qquad N^a = \begin{pmatrix} n \\ j^i \end{pmatrix}$$

$$a, b \in \{0,1,2,3\}; \quad i, j \in \{1,2,3\}; \quad diag(1,-1,-1,-1)$$

$$\dot{e} = u^a \partial_a e$$

- $u^a$ – velocity field
- $e$ – energy density
- $q^a$ – momentum density or energy current??
- $P^{ab}$ – pressure
- $n$ – particle number density
- $j^a$ – particle current

*General,* expressed by comoving splitting

$$_a\partial_b T^{ab} = \dot{e} + e\,\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0^a$$

energy balance

$$\Delta^a_c \partial_b T^{cb} = e\,\dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta^a_c \dot{q}^c + \Delta^a_c \partial_b P^{bc} = 0^a$$

momentum balance

$$\partial_b N^b = \dot{n} + n\,\partial_a u^a + \partial_a j^a = 0$$

particle number balance

# Constitutive theory:

*Fields:*

| | |
|---|---|
| $N^a$ | 4 |
| $T^{ba}$ | 10 |
| $u^a$ | 3 |
| | Σ17 |

| | |
|---|---|
| $j^a$ | 3 |
| $q^a$ | 3 |
| $\Pi^{ab}$ | 6 |
| | Σ 12 |

$$q^a u_a = j^a u_a = 0, \;\; \Pi^{ba} u_a = \Pi^{ab} u_a = 0^b$$

*Equations:*

| | |
|---|---|
| $\partial_a N^a = 0,$ | 1 |
| $\partial_b T^{ab} = 0^a,$ | 4 |

$N^a$ – particle number vector
$T^{ab}$ – energy-momentum density
$u^a$ – velocity field

$j^a$ – particle flux/current
$q^a$ – energy flux and momentum density
$\Pi^{ab}$ – viscous pressure

$n, e, u^a$ – basic fields

flow-frames, stability and second law

# Relativistic thermodynamics:

$$\partial_a S^a = \dot{s} + s\,\partial_a u^a + \partial_a J^a \geqslant 0$$

Eckart (1940), theory and flow-frame:

$$S^a(T^{ab}, N^a) = s(e, n)\,u^a + \frac{q^a}{T}$$

(Müller)-Israel-Stewart (1969-72) theory in Eckart flow-frame:

$$S^a(T^{ab}, N^a) = \left( s(e, n) - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_b q^b - \frac{\beta_2}{2T}\pi^{bc}\pi_{bc} \right) u^a +$$

$$+ \frac{1}{T}\left( q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b \right)$$

isotropic, Grad compatible

# *Concept of dissipation*

entropy balance constrained by the other balances
constitutive theory – closure by linear relations

$$\Sigma = -j^a \partial_a \alpha - \beta \Pi^{ab} \partial_b u_a + q^a \left( \partial_a \beta + \beta \dot{u}_a \right) \geqslant 0$$

Closure:

$$j^a = \chi \Delta^{ab} \partial_b \alpha ,$$
$$\Pi^{ab} = \nu \partial_c u^c \Delta^{ab} + \eta \Delta^{ac} \Delta^{bd} \left( \partial_c u_d + \partial_d u_c \right)/2,$$
$$q^a = \lambda \Delta^{ab} \left( \partial_b \beta + \beta \dot{u}_b \right)$$

\+ balances

Background:

Ideal fluid: $q^a = 0^a , j^a = 0^a , \Pi^{ab} = 0^{ab}$

Entropy flux and Gibbs relation:

$$J^a = \beta q^a ,$$
$$ds = \beta de - \alpha dn$$

## Kinetic theory – hydrodynamics

6th problem of Hilbert, minimal version: unsolved

Only for rarefied gases. Dense matter: doubled hierarchy

Open problems: <u>instabilities,</u> systematic closure.

## Causality and stability

Hyperbolicity <u>is a kind of stability</u>. Finite propagation speeds.

## Flow-frames (and covariance)

Is there a freedom? (Eckart, Landau-Lifshitz, ...) <u>What is ideal?</u>

## Stability – second law (dissipation: cannot be avoided)

Stability of equilibrium: homogeneous in space.

<u>EOS: concave entropy</u>

<u>Dissipation leads stability</u>

<u>Linear stability</u> is a necessary condition.

Nonlinear stability:

entropy is Ljapunov functional (energy methods).

# *Hyperbolicity: convenient beliefs*

(Müller)-Israel-Stewart theory:

It is not proved to be symmetric hyperbolic.

The linearized version is hyperbolic in Eckart frame with conditions.

Entropy inequality (normal, covariant):

$$\partial_a N^a = 0, \;\; \partial_a T^{ab} = 0^b$$

$$\partial_a S^a + \alpha \partial_a N^a + \beta_b \partial_a T^{ab} \geqslant 0.$$

Divergence type theories – built in hyperbolicity.

$$\partial_a N^a = 0, \;\; \partial_a T^{ab} = 0^b, \;\; \partial_a A^{abc} = I^{bc}$$

$$\partial_a S^a + \zeta \partial_a N^a + \zeta_b \partial_a T^{ab} + \zeta_{bc} \partial_a A^{abc} = 0.$$

$$\partial_a S^a = -\zeta_{bc} I^{bc} \geqslant 0.$$

# *Flow-frames: more than two of them*

Landau-Lifshitz:

$$N^a = \hat{n}\hat{u}^a + \hat{j}^a$$

$$T^{ab} = \hat{e}\hat{u}^b\hat{u}^a + \hat{P}^{ab} = \hat{e}\hat{u}^b\hat{u}^a - \hat{p}\hat{\Delta}^{ab} + \hat{\Pi}^{ab}$$

$\Longleftrightarrow$

Eckart:

$$N^a = nu^a$$

$$T^{ab} = eu^b u^a + q^b u^a + q^a u^b - p\Delta^{ab} + \Pi^{ab}$$

Transformation:

$$\boxed{\hat{u}^a = \frac{u^a + w^a}{\zeta}}$$

Stability: what is ideal?

$$N_0^a = nu^a$$

$$T_0^{ab} = eu^b u^a - p\Delta^{ab}$$

$\Longrightarrow$

$$N_0^a = \hat{n}\hat{u}^a + j^a$$

$$T_0^{ab} = \hat{e}\hat{u}^b\hat{u}^a + q^b\hat{u}^a + q^a\hat{u}^b - p\hat{\Delta}^{ab} + \Pi^{ab}$$

$$\hat{n} = \frac{n}{\zeta}, \quad j^a = n\frac{\hat{w}^a}{\zeta}, \quad \hat{e} = \frac{h}{\zeta^2} - p, \quad q^a = h\hat{w}^a, \quad \Pi^{ab} = \frac{\hat{w}^a\hat{w}^b}{h}$$

Ideal fluid is a class of $N^a, T^{ab}$

# Thermodynamics in arbitrary frames I.

$$S^a + \alpha N^a - \beta_b T^{ab} = \Phi^a$$ timelike part: Gibbs relation

$$\boxed{\beta^a = \beta\left(u^a + w^a\right)}$$ temperature vector

Thermodynamics:

a) $\Phi^a = p\beta^a$ matching $S_0^a + \alpha N_0^a - \beta_b T_0^{ab} = \beta^a p_0$

b) $$ds + \alpha\, dn = \beta_a dE^a + \beta\, p\, w_a du^a$$
$$= \beta\left(u_a + w_a\right) d\left(eu^a + q^a\right) + \beta\, p\, w_a du^a$$
$$= \beta\left(de + w_a dq^a + \underline{\left[\left(e+p\right) w_a - q_a\right] du^a}\right)$$

Kinetic theory compatible frame: $q^a = \left(e+p\right) w^a$

$$w^\mu = 0 \;\Rightarrow\; ds + \alpha\, dn \neq \beta\, de \qquad w^\mu = 0 \wedge q^a = 0 \;\Rightarrow\; ds + \alpha\, dn = \beta\, de$$

# Thermodynamics in arbitrary frames II.

$$S^a + \alpha N^a - \beta_b T^{ab} = \Phi^a$$

spacelike part: entropy flux

$$\boxed{\beta^a = \beta\left(u^a + w^a\right)}$$

temperature vector

Entropy flux: $s^a = \beta q^a - \alpha j^a + \beta w^a \Pi^{ab}$

Constrained entropy inequality:

$$0 \leq \partial_a S^a + \alpha \partial_a N^a - \beta_b \partial_a T^{ab} = \ \ldots$$

$$= \left(\Pi^{ab} - q^a w^b\right)\partial_b \beta_a + \left(q^a - h w^a\right)\left(\partial_a \beta - \beta w_b \partial_a u^b\right) + \left(n w^a - j^a\right)\partial_a \alpha$$

Linearization is possible.

Correct equilibrium/ideality conditions

Compatible with kinetic equilibrium: $q^a = h w^a$

# Entropy production:

$$\left(\Pi^{ab}-q^a w^b\right)\partial_b\beta_a+\left(q^a-hw^a\right)\left(\partial_a\beta-\beta w_b\partial_a u^b\right)+\left(nw^a-j^a\right)\partial_a\alpha\geqslant 0$$

Frames:

$$\beta^a=\beta\left(u^a+w^a\right)$$

Kinetic (EOS): $w^a=\frac{q^a}{h}$

Thermometer: $w^a=0$

Eckart: $j^a=0$

Landau-Lifshitz: $q^a=0$

| | | | |
|---|---|---|---|
| $N^a$ | 4 | $\partial_a N^a=0,$ | 1 |
| $T^{ba}$ | 10 | $\partial_b T^{ab}=0^a,$ | 4 |
| $u^a,\ w^a$ | 6 | | |
| | Σ20 | | Σ5 |

Constitutive theory, solution of the entropy inequality??

$j^a, q^a, \Pi^{ab}, w^a$ 15 components

## Entropy production:

$$(\Pi^{ab}-q^a w^b)\partial_b\beta_a+(q^a-hw^a)(\partial\beta_a-\beta w_b\partial_a u^b)+(nw^a-j^a)\partial_a\alpha=$$

$$\Pi^{ab}\underline{\partial_b\beta_a}+q^a\partial_a\beta_a-j^a\partial_a\alpha+\underline{\beta w^a\partial_a p}\geqslant 0$$

Stability of the linear solution: not stable/conditional

Kinetic : $q^a=hw^a$
undetermined

Thermometer: $w^a=0$
not stable

Eckart: $j^a=0$
not stable

Landau-Lifshitz: $q^a=0$
stable

Consequences:

- dissipation evolves toward a stable manifold,
- ideality and equilibrium (see, Gorban and Karlin, Geroch),
- one may develop higher order keeping the stable manifold,
- consistent with LL flow-frame and ideal hydro success.

# Summary

Thermodynamics is a kind of stability theory.

Temperature and flow are not necessarily parallel.

Generic stability is the benchmark.

$$\beta^a = \beta\left(u^a + w^a\right)$$

Generalizations beyond local equilibrium.

Temperature!

*VP, Biró, TS., EPJ–ST, 155:201–212, 2008, (arXiv:0704.2039v2).*
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*Biró, TS., VP. EPL, 89:30001, 2010.*
*VP, EPJ WoC, 13:07004, 2011, (arXiv:1102.0323).*
*VP, Biró, TS., PLB, 709(1-2):106–110, 2012, (arXiv:1109.0985).*
*VP-Pavelka-Grmela, JNET, 42/2, 133-142, 2017 (arXiv:1508.00121)*
*VP, CMT, 29/2, 133/151, 2017 (arXiv:1510.03900)*

Thank you for the attention!

# *Velocity – flow-frames*

What is a fluid? What is moving?

Eckart (material) frame: $$u^a = \frac{N^a}{\sqrt{-N^b N_b}} \rightarrow N^a = n u^a$$

Landau-Lifshitz (energy) frame: $$\hat{u}^a = \frac{E^a}{\sqrt{-E^b E_b}} \rightarrow T^{ab} = \hat{e}\,\hat{u}^a \hat{u}^b + \hat{P}^{ab}$$

Jüttner (thermometer) frame: $$\hat{u}^a = \frac{\beta^a}{\sqrt{-\beta^b \beta_b}} \rightarrow \beta^a = \hat{\beta}\,\hat{u}^a$$

Do we have a choice?

# *Causality*

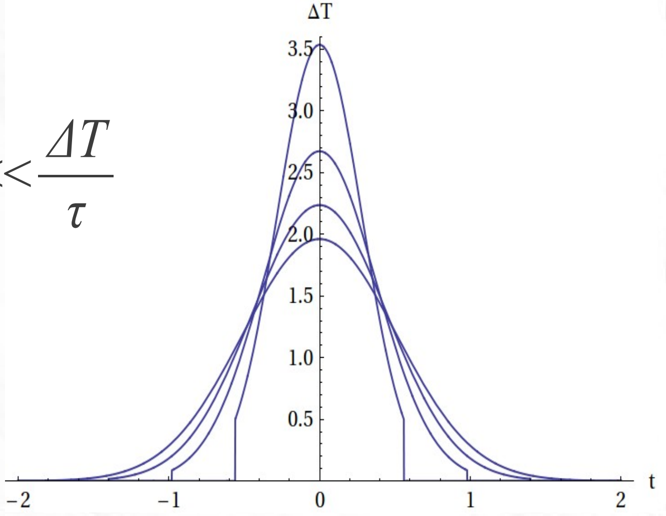
– infinite speed of signal propagation
– second order time derivatives
– hyperbolic system of equations

Divergence type theories – finite speed is material
(Liu-Ruggeri-Müller, Geroch, Lindblom, Calzetta)

Physical:
Propagation speed of *continuum limit.*
Propagation speed of observable signals.

Example:

$$\partial_t T = -\kappa \partial_x^2 T \qquad \partial_x T << \frac{\Delta T}{\xi}, \quad \partial_t T << \frac{\Delta T}{\tau}$$




water at room temperature:

$$v_{max} \approx \frac{\kappa}{\xi} = 14\, m/s$$

# Stability conditions of the Israel-Stewart theory

(Hiscock-Lindblom 1983,1987)

$$\Omega_1 = \frac{1}{e+p}\frac{\partial e}{\partial p}\Big|_{\frac{s}{n}} = \frac{T}{(e+p)\frac{\partial p}{\partial e}\Big|_n - n\frac{\partial p}{\partial n}\Big|_e} \geq 0,$$

$$\Omega_2 = \frac{1}{e+p}\frac{\partial e}{\partial(s/n)}\Bigg|_p \frac{\partial p}{\partial(s/n)}\Bigg|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

Conditons for the
- EOS
- IS coefficients
- both
- Eckart frame

 + usual

$$\Omega_5 = \beta_0 \geq 0, \qquad \Omega_8 = \beta_2 \geq 0, \qquad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \qquad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2 T}\frac{\partial T}{\partial(s/n)}\Bigg|_n \geq 0,$$

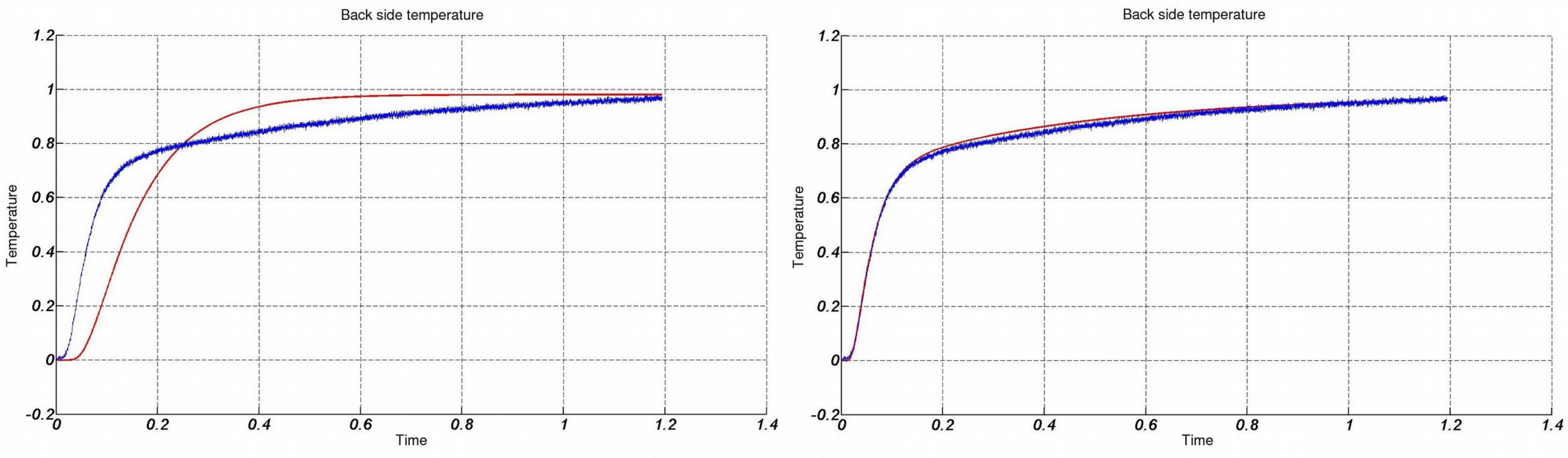
$$\Omega_3 = (e+p)\left(1 - \frac{\partial p}{\partial e}\Bigg|_{\frac{s}{n}}\right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0, \qquad K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\frac{\partial T}{\partial n}\Bigg|_{s/n} \geq 0.$$

# Improved continuum theories I.

Heat conduction in dense, heterogeneous media

Compatibility with kinetic theory.

Experiment: beyond Fourier, different materials




$$\tau_q \partial_i q + q = \lambda \partial_i T + a \partial_{ii} q$$

see R. Kovács

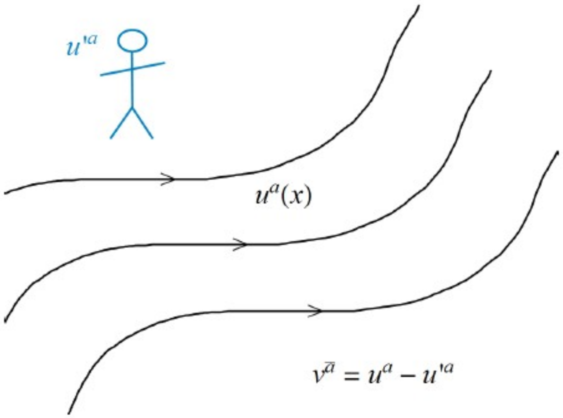
# Improved continuum theories II.

Galilean-relativistic space-time: time is absolute

Transformation rules: kinetic energy

Flow-frames: extra mass flux, Brenner

$$
\begin{aligned}
\hat{\rho} &= \rho, \\
\hat{p}^i &= p^i + \rho v^i, \\
\hat{e} &= e + p^i v_i + \rho \frac{v^2}{2}, \\
\hat{j}^i &= j^i + \rho v^i, \\
\hat{P}^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\
\hat{q}^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i)\frac{v^2}{2}.
\end{aligned}
$$



# Nonrelativistic hydrodynamics:

$$\dot{\rho}+\rho\partial_i v^i=0$$
$$\rho\dot{v}^i+\partial_j P^{ij}=0$$
$$\dot{e}+e\partial_i v^i+\partial_i q^i+P^{ij}\partial_i v_j=0$$

balance of mass

balance of momentum

balance of internal energy

$v^i$ – velocity field
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$q^i$ – heat flux
$P^{ij}$ – pressure
$\rho$ – particle number density
$j^i$ – particle current (?)
$p^i$ – momentum density (?)

$$i.j=1,2,3; \quad \dot{e}=v^i\partial_i e$$

Missing parts, relativistic knowledge:

$$\dot{\rho}+\rho\partial_i v^i+\partial_i j^i=0$$
$$\dot{p}^i+p^i\partial_j v^j+\rho\dot{v}^i+\partial_j P^{ij}+j^k\partial_k v^i=0^i$$
$$\dot{e}+e\partial_i v^i+\partial_i q^i+P^{ij}\partial_i v_j+p_i\dot{v}^i=0$$

Constitutive theory – second law compatibility:

$$P^{ij}=p\delta^{ij}+\Pi^{ij}, \qquad \Pi^{ij}=-\nu^{ijkl}\partial_k v_l, \qquad q^i=\lambda\partial^i\beta$$

$p$ – thermostatic pressure
$\upsilon^{ijkl}$ – viscosities
$\lambda$ – heat condictuvity

Extension? Gradient and memory…

$$\Pi^{ij}(\partial^j v^i, \partial^{jk} v^i, \ldots, \rho, \partial_i\rho, \partial_{ij}\rho, \ldots)$$

Higher grade fluids and Korteweg fluids: instable

Capillarity, qfluids, second sound, etc..., the question is the universality.