A full set of TMD splitting functions: real contributions

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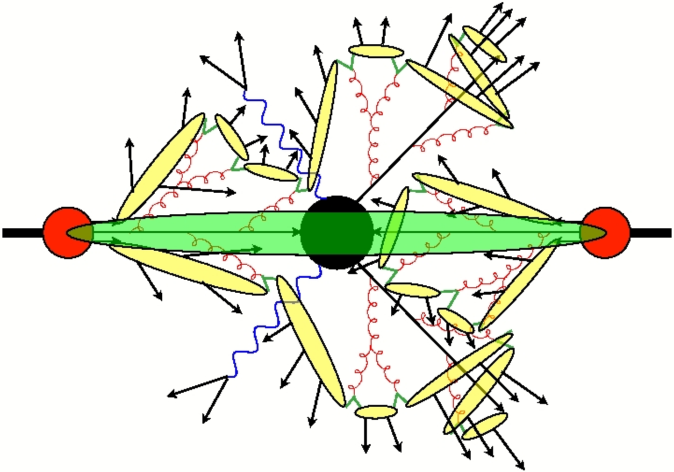
Work in collaboration with
Martin Hentschinski, Aleksander Kusina and Krzysztof Kutak
arXiv: 1711.04587

Motivation for our work

The overall appearance of a QCD process



One (or more) step away from the hard scattering: the parton shower

Apart from the hard matrix element, an exact description of all the parton radiation is not feasible: an approximation is needed

Hard scattered partons are heavily accelerated, which means that they radiate (colour) charge, i.e. mostly gluons, which radiate in turn, since they are colour charged too, unlike photons. This is why the emission pattern is so intricate.

Evolution in energy ≈ evolution in time
up to the hard collision (Initial state shower) and thereafter (final state shower)

Tools :

Real and resolvable emissions: universal collinear splitting

$$d\sigma_{n+1} = d\sigma_n \, \frac{\alpha_s}{2\pi} \, \frac{d\theta^2}{\theta^2} \, P_{ji}(z,\phi)\, dz\, d\phi + \mathcal{O}(\theta) \qquad \theta : \text{emission angle}$$

Virtual part and unresolvable emissions: the Sudakov form factor

$$\Delta_i(q_1^2, q_2^2) = \exp\left\{ -\frac{\alpha_s}{2\pi} \int_{q_2^2}^{q_1^2} \frac{dq^2}{q^2} \int_{Q_0^2/q^2}^{1-Q_0^2/q^2} dz \int_0^{2\pi} d\phi \, P_{ji}(z,\phi) \right\} \qquad Q_0 : \text{soft scale}$$

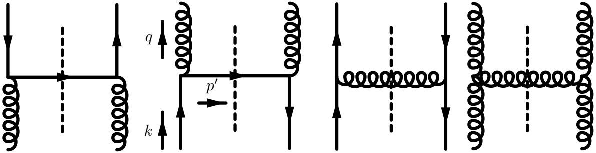
Take-home message:
the splitting function is the foundational tool for a parton shower Monte Carlo

The usual tool for a parton shower: the DGLAP evolution equations

Gribov, Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438
Altarelli, Parisi, Nucl. Phys. B126 (1977) 298;
Dokshitzer, Sov. Phys. JETP 46 (1977) 641

$$\frac{df_i(x,\boldsymbol{q}^2)}{d\log \boldsymbol{q}^2} = \sum_j \frac{\alpha_s}{2\pi}\int \frac{dz}{z}\,P_{ij}(\alpha_s,z)\,f_j\left(\frac{x}{z},\boldsymbol{q}^2\right)$$

$P_{ij} \equiv$ **splitting functions** $\approx$ **probability that** $i(k) \to j(q) +$ **something**(p')



$$k^\mu = yp^\mu \quad q^\mu = zk^\mu + q_\perp^\mu + \frac{q^2+\boldsymbol{q}^2}{2xp\cdot n}n^\mu \quad p' = q-k\,, \quad p^\mu = (1,0,0,1)\,, n^\mu = (1,0,0,-1)$$

(Main) DGLAP-based Monte Carlo programs:

- Sjostrand et al., Pythia
- Webber et al. HERWIG
- Krauss et al. SHERPA

Other evolution equations (for TMD gluon PDF): BFKL and CCFM

High-energy (small-z) limit of gluon emissions: BFKL kernel

$$P_{gg}^{BFKL}(\alpha_s, \boldsymbol{q}, \boldsymbol{k}) \approx \frac{\alpha_s}{|\boldsymbol{q}-\boldsymbol{k}|^2} - \alpha_s\,\delta(\boldsymbol{q}^2-\boldsymbol{k}^2)\,\frac{1}{\pi}\int^k \frac{d^2\boldsymbol{q}}{\boldsymbol{q}^2}$$

BFKL-based Monte Carlo programs:

- **Andersen, Smillie, Lonnbland et al., HEJ (High Energy Jets) JHEP 1106 (2011) 010, JHEP 1107 (2011) 110...**
- **Agustin Sabio Vera and Grigorios Cachamis (unpublished)**

Soft limit ($p' \Rightarrow 0$) of gluon emissions: CCFM kernel

$$P_{gg}^{CCFM}(\alpha_s, \boldsymbol{q}, \boldsymbol{k}) \approx z\left[\frac{\alpha_s(\boldsymbol{q}^2(1-z)^2)}{1-z} + \frac{\alpha_s(\boldsymbol{k}^2)}{z}\,(1+\dots)\right]$$

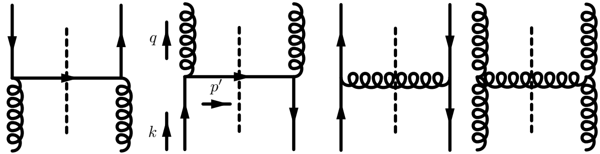
CCFM-based Monte Carlo program: CASCADE

- **Hannes Jung, Gavin Salam et al.**
  Eur.Phys.J. C19 (2001) 351-360
  Comput.Phys.Commun. 143 (2002), Eur.Phys.J. C70 (2010)

What is your purpose: the TMD splitting kernels

Short term: connecting DGLAP and low-x evolutions.
Medium/long-term: Monte Carlo evolution bridging between DGLAP, CCFM and BFKL

Squared matrix elements for the determination of the real contributions to the splitting functions à-la Curci-Furmanski-Petronzio (CFP)



$$k^\mu = yp^\mu + k_\perp^\mu \quad q^\mu = xp^\mu + q_\perp^\mu + \frac{q^2 + \boldsymbol{q}^2}{2x\, p\, n} n^\mu \quad p' = q - k$$

$$p^\mu = (1, 0, 0, 1) \quad n^\mu = (1, 0, 0, -1)$$

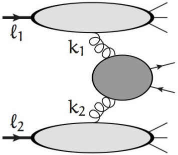
First three computed and consistent with DGLAP
Catani and Hautmann Nucl.Phys. B427 (1994)
Gituliar, Hentschinski, Kutak JHEP 1601 (2016) 181

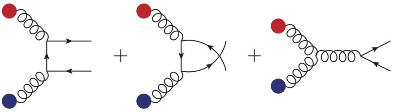
So far, a computation of the $\tilde{P}_{gg}$ kernel which could successfully reproduce the DGLAP AND BFKL limit not done: let me (briefly) show why !

Gauge invariant amplitudes in High Energy Factorisation

High Energy Factorization: more degrees of freedom

High Energy Factorization *(Catani,Ciafaloni,Hautmann, 1991 / Collins,Ellis, 1991)*





$$\sigma_{h_1,h_2\to q\bar{q}} = \int d^2k_{1\perp} d^2k_{2\perp} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\, \mathcal{F}_g(x_1,k_{1\perp},\mu^2)\,\mathcal{F}_g(x_2,k_{2\perp},\mu^2)\,\hat{\sigma}_{gg}\,(m,x_1,x_2,s,k_{1\perp},k_{2\perp})$$

$\mathcal{F}_g$'s: unintegrated gluon densities, $\int d^2k_T\mathcal{F}_g(x,k_t,\mu^2) = f_g(x,\mu^2)$.

Non negligible transverse momentum $\Leftrightarrow$ small x physics.

Exact initial state kinematics $\Rightarrow$ collinear higher order effects *ab initio*.

Momentum parameterization:

$$k_1^\mu = x_1\, l_1^\mu + k_{1\perp}^\mu \quad , \quad k_2^\mu = x_2\, l_2^\mu + k_{2\perp}^\mu$$
$$l_i^2 = 0, \quad l_i\cdot k_i = 0, \quad k_i^2 = -k_{i\perp}^2, \quad i = 1,2$$

Gauge invariant amplitudes with off-shell gluons

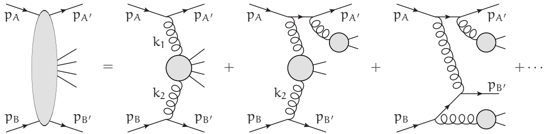
Kutak, Kotko, van Hameren, JHEP 1301 (2013) 078

Problem: general partonic processes must be described by gauge invariant amplitudes
$\Rightarrow$ ordinary Feynman rules are not enough !

THE IDEA:

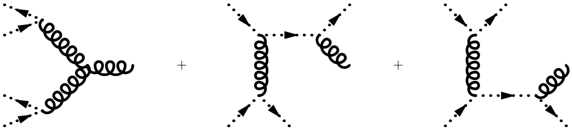
on-shell amplitudes are gauge invariant, so off-shell gauge-invariant amplitudes could be got by embedding them into on-shell processes...

...result for gluons...: represent g^* as coming from a $\bar{q}qg$ vertex



- embed the scattering of the off-shell gluons in the scattering of two quark pairs carrying momenta $p_A^\mu = k_1^\mu$, $p_B^\mu = k_2^\mu$, $p_{A'}^\mu = 0$, $p_{B'}^\mu = 0$
- Assign the spinors $|p_1\rangle, |p_1]$ to the A-quark and $\frac{i\not{k}}{k^2} \rightarrow \frac{i\,\not{p}_1}{p_1 \cdot k}$ to the A-propagators; same for the B-quark line.
- ordinary Feynman elsewhere and factor $x_1\sqrt{-k_\perp^2/2}$ to match to the collinear limit
- **Big advantage: Spinor helicity formalism**
- **for off-shell quakrs: Kutak, Salwa, van Hameren, Phys.Lett. B727 (2013)**

Open-index vertices in HEF: $\Gamma^{\mu}_{g^*g^*g}(q,k,p')$



HEF gluon $q = y\,p + q_\perp$, **HEF gluon** $k = x\,p + k_\perp$, **radiated gluon** $p' = k - q \quad p'^2 = 0$

$$
\begin{aligned}
\mathcal{A}(q,k,p') \quad &= \quad (\sqrt{2})\,\frac{p_{\mu_1}\,n_{\mu_2}\,\epsilon_{\mu_3}(p')}{q^2\,k^2}\Big\{\mathcal{V}^{\lambda\kappa\mu_3}(q,k,p')\,d^{\mu_1}{}_{\lambda}(q)\,d^{\mu_2}{}_{\kappa}(k) \\
&+ d^{\mu_1\mu_2}(k)\,\frac{q^2 n^{\mu_3}}{n\cdot p'} - d^{\mu_1\mu_2}(q)\,\frac{k^2 p^{\mu_3}}{p\cdot p'}\Big\} \\
&\equiv \quad (\sqrt{2})\,\frac{p_{\mu_1}\,n_{\mu_2}\,\epsilon_{\mu_3}(p')}{q^2\,k^2}\,\Gamma^{\mu_1\mu_2\mu_3}(q,k,p')
\end{aligned}
$$

$\mathcal{V}^{\lambda\kappa\mu_3}(q,k,p')$: ordinary 3-gluon vertex

$d_{\mu\nu}$ **not invertible in light-cone gauge $\Rightarrow$ it has to be kept everywhere !**

$$d_{\mu\nu}(p) = -g_{\mu\nu} + \frac{n^\mu p^\nu + n^\nu p^\mu}{n\cdot p}$$

Open-index vertices in HEF: all of them, with color

Full set of gauge invariant 3-point off-shell vertices

$$
\begin{aligned}
\Gamma^{\mu}_{q^* g^* q}(q,k,p') &= i\, g\, t^a\, d^{\mu}{}_{\nu}(k) \left(\gamma^{\nu} - \frac{n^{\nu}}{k\cdot n}\, \not{q}\right) \\
\Gamma^{\mu}_{g^* q^* q}(q,k,p') &= i\, g\, t^a\, d^{\mu}{}_{\nu}(q) \left(\gamma^{\nu} - \frac{p^{\nu}}{p\cdot q}\, \not{k}\right) \\
\Gamma^{\mu}_{q^* q^* g}(q,k,p') &= i\, g\, t^a \left(\gamma^{\mu} - \frac{p^{\mu}}{p\cdot p'}\, \not{k} + \frac{n^{\mu}}{n\cdot p'}\, \not{q}\right) \\
\Gamma^{\mu}_{g^* g^* g}(q,k,p') &= i\, g\, f^{abc} \Big\{ \mathcal{V}^{\lambda\kappa\mu_3}(q,k,p')\, d^{\mu_1}{}_{\lambda}(q)\, d^{\mu_2}{}_{\kappa}(k) \\
&\quad + d^{\mu_1\mu_2}(k)\, \frac{q^2 n^{\mu_3}}{n\cdot p'} - d^{\mu_1\mu_2}(q)\, \frac{k^2 p^{\mu_3}}{p\cdot p'} \Big\}
\end{aligned}
$$

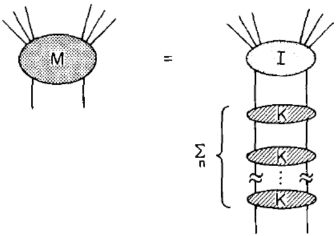
We still need a suitable procedure to extract splitting functions from these vertices... to which we turn next !

Calculation of TMD splitting kernels

The fundamental result to start from

Why does the parton model work so well ?
Ellis, Georgi, Machacek, Politzer, Ross, Nucl. Phys. B152 (1979)

Proof of collinear factorisation based on
2 Particle Irreducible expansion of the scattering process

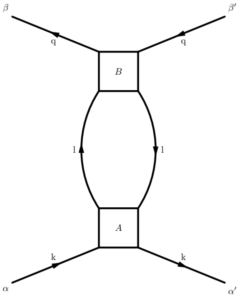


In light-cone gauge, ALL the IR divergences come only from the convolution integral over the intermediate momenta connecting the 2PI kernels K

How to put at work this result to get an evolution equation ?

The CFP approach to factorisation: isolating the IR poles

Crucial point: introduce a projector to isolate the infrared poles in ϵ



Intermediate quark:

$$A(q,l)_{\dots\alpha\alpha'}\,\mathbb{P}^{s\,\alpha\alpha'}_{\;\beta\beta'}\,B^{\dots\beta\beta'}(l,k)$$

$$\equiv\quad A(q,l)_{\dots\alpha\alpha'}\,\frac{(\not{l})^{\alpha\alpha'}}{2}\,\frac{(\not{n})_{\beta\beta'}}{2n\cdot l}\,B^{\dots\beta\beta'}(l,k)\,,$$

Intermediate gluon

$$A(q,l)_{\dots\mu'\nu'}\,\mathbb{P}^{s\,\mu'\nu'}_{\;\mu\nu}\,B^{\dots\mu\nu}(l,k)$$

$$\equiv\quad A(q,l)_{\dots\mu'\nu'}\,\frac{d^{\mu'\nu'}(l)}{d-2}\,(-g_{\mu\nu})\,B^{\dots\mu\nu}(l,k)\,,$$

- **We can split the spin projectors into an "in" and "out" component**

$$\mathbb{P}^s \equiv \mathbb{P}^s_{\text{in}} \otimes \mathbb{P}^s_{\text{out}}$$

- **The momentum $\mathbb{P}^{\epsilon}$ projector extracts the poles in ϵ from the $\int d^d l$ after setting the incoming particle on-shell**

Modifying the collinear projectors

Collinear spin projectors:

$$\mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{in}} = \frac{1}{d-2}\left(-g^{\mu\nu} + \frac{l^\mu n^\nu + n^\mu l^\nu}{l\cdot n}\right), \qquad \mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{out}} = -g^{\mu\nu}$$

$$\mathbb{P}^{s}_{q,\,\mathrm{in}} = \frac{\not{l}}{2}, \qquad \mathbb{P}^{s}_{q,\,\mathrm{out}} = \frac{\not{n}}{2\,n\cdot l}$$

Important to keep in mind:

- **The "In" projector is a d-dim average over incoming particle helicities**
- **Projectors are defined only modulo finite terms (renormalization scheme)**
- **In order to move on to TMDs, one needs gauge-invariant generalisation of the collinear vertices**
- **need some projectors with something retaining TMD dependence and reducing to the CFP projectors (modulo finite terms) in the collinear limit and squares to itself all the way through**

Someone did this already (more on this later):
Catani and Hautmann, Nucl.Phys. B427 (1994) 475-524

Modifying the Catani-Hautmann projectors

The modified projectors for HEF
gluons: Catani and Hautmann, Nucl.Phys. B427 (1994) 475-524
quarks: Gituliar, Hentschinski, Kutak - JHEP 1601 (2016) 181

$$l = y\,p + l_\perp\,, \quad \Rightarrow \quad \begin{cases} \mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{in}} = -\frac{l_\perp^\mu\, l_\perp^\nu}{l_\perp^2} & \mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{out}} = -g^{\mu\nu} \\ \mathbb{P}^{s}_{q,\,\mathrm{in}} = \frac{y\,\not{p}}{2} & \mathbb{P}^{s}_{q,\,\mathrm{out}} = \frac{\not{n}}{2\,n\cdot l} \end{cases}$$

CH prescription derived from analysis of heavy quark production
$\Rightarrow$ both numerators of gluon propagators factorize
Our modified set of projectors

$$l = y\,p + l_\perp\,, \quad \Rightarrow \quad \begin{cases} \mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{in}} = -y^2\,\frac{p^\mu\, p^\nu}{l_\perp^2} & \mathbb{P}^{s\,\mu\nu}_{g,\,\mathrm{out}} = -g^{\mu\nu} + \frac{l^\mu n^\nu + l^\nu n^\mu}{l\cdot n} - k^2\,\frac{n_\mu n_\nu}{(l\cdot n)^2}\,, \\ \mathbb{P}^{s}_{q,\,\mathrm{in}} = \frac{y\,\not{p}}{2} & \mathbb{P}^{s}_{q,\,\mathrm{out}} = \frac{\not{n}}{2\,n\cdot l} \end{cases}$$

The gluon TMD

For quark TMDs, we exactly recover the result of Catani and Hautmann, Nucl.Phys. B427 (1994) 475-524 Gituliar, Hentschinski, Kutak - JHEP 1601 (2016) 181

The central new result of this work: the gluon TMD splitting function

$$
\begin{aligned}
\tilde{P}_{gg}^{(0)}(z,\tilde{\boldsymbol{q}},\boldsymbol{k}) \quad = \quad & C_A \left(\frac{\tilde{\boldsymbol{q}}^2}{\tilde{\boldsymbol{q}}^2 + z(1-z)\boldsymbol{k}^2}\right)^2 \frac{\tilde{\boldsymbol{q}}^2}{(\tilde{\boldsymbol{q}}-(1-z)\boldsymbol{k})^2} \\
& \times \Bigg[-\frac{4z^2-4z+2}{z(1-z)} - z(1-z)(4z^4-12z^3+9z^2+1)\frac{\boldsymbol{k}^4}{\tilde{\boldsymbol{q}}^4} \\
& -4z(1-z)\frac{\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}^2}{\boldsymbol{k}^2\tilde{\boldsymbol{q}}^2} + 2(4z^3-6z^2+6z-3)\frac{\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}}{\tilde{\boldsymbol{q}}^2} \\
& -4z(1-z)^2(3-5z)\frac{\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}^2}{\tilde{\boldsymbol{q}}^4} - (4z^4-8z^3+5z^2-3z-2)\frac{\boldsymbol{k}^2}{\tilde{\boldsymbol{q}}^2} \\
& +8z(1-z)^2\frac{\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}^3}{\boldsymbol{k}^2\tilde{\boldsymbol{q}}^4} - 2z^2(1-z)(3-4z)(3-2z)\frac{\boldsymbol{k}^2\,\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}}{\tilde{\boldsymbol{q}}^4}\Bigg] \\
& -\epsilon\, C_A\, z(1-z)\frac{\tilde{\boldsymbol{q}}^2}{\boldsymbol{k}^2}\left(\frac{(2z-1)\boldsymbol{k}^2 + 2\boldsymbol{k}\cdot\tilde{\boldsymbol{q}}}{\tilde{\boldsymbol{q}}^2 + z(1-z)\boldsymbol{k}^2}\right)^2 \\
\tilde{\boldsymbol{q}} \quad = \quad & \boldsymbol{q} - z\,\boldsymbol{k}
\end{aligned}
$$

Kinematical limits of the $\tilde{P}_{gg}$ TMD: DGLAP

1. Introduce angular averaging

$$\bar{P}_{ij}^{(0)} = \frac{1}{\pi} \int_0^{\pi} d\phi \, \sin^{2\epsilon} \phi \, \tilde{P}_{ij}^{(0)}$$

and easily get the DGLAP limit $\Rightarrow$ al least not everything is wrong !

$$\lim_{|\vec{k}_\perp| \to 0} \bar{P}_{gg}^{(0)} = 2 \, C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z \, (1-z) \right]$$

Kinematical limits of the $\tilde{P}_{gg}$ TMD: BFKL

1. Introduce angular averaging

$$\bar{P}_{ij}^{(0)} = \frac{1}{\pi}\int_0^{\pi} d\phi\, \sin^{2\epsilon}\phi\, \tilde{P}_{ij}^{(0)}$$

and easily get the DGLAP limit $\Rightarrow$ al least not everything is wrong !

$$\lim_{|\vec{k}_\perp|\to 0} \bar{P}_{gg}^{(0)} = 2\, C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z\,(1-z)\right]$$

2. In the high energy limit $z \to 0$:

$$\begin{aligned}\lim_{z\to 0} \hat{P}_{gg}\left(z, \frac{\boldsymbol{k}^2}{\mu^2}, \epsilon, \alpha_s\right) &= \frac{\alpha_s C_A}{\pi (e^{\gamma_E}\mu^2)^{\epsilon}} \int \frac{d^{2+2\epsilon}\tilde{\boldsymbol{p}}}{\pi^{1+\epsilon}} \Theta\left(\mu_F^2 - (\boldsymbol{k} - \tilde{\boldsymbol{p}})^2\right) \frac{1}{\tilde{\boldsymbol{p}}^2} \\ &= \int \frac{d^{2+2\epsilon}\boldsymbol{q}}{\pi^{1+\epsilon}} \Theta\left(\mu_F^2 - \boldsymbol{q}^2\right) \frac{\alpha_s C_A}{\pi (e^{\gamma_E}\mu^2)^{\epsilon}} \frac{1}{(\boldsymbol{q} - \boldsymbol{k})^2},\end{aligned}$$

the real part of the LO BFKL kernel $\Rightarrow$ something is probably correct !

Kinematical limits of the $\tilde{P}_{gg}$ TMD: CCFM

Last we are interested in the limit $\tilde{\boldsymbol{p}} \to 0$,
i.e. vanishing transverse momentum of the produced gluon

For $\tilde{\boldsymbol{p}} \to 0$ we get the real CCFM kernel

$$\begin{aligned}
\hat{P}_{gg}\left(z, \frac{\boldsymbol{k}^2}{\mu^2}, 0, \alpha_s\right) &= z \cdot \int_0 \frac{d\tilde{\boldsymbol{p}}^2}{\tilde{\boldsymbol{p}}^2} \frac{\alpha_s C_a}{\pi} \left[\frac{1}{z} + \frac{1}{1-z} + \mathcal{O}\left(\frac{\tilde{\boldsymbol{p}}^2}{\boldsymbol{k}^2}\right)\right] \\
\tilde{\boldsymbol{p}} &= \frac{\boldsymbol{k} - \boldsymbol{q}}{1-z}
\end{aligned}$$

Comes "for free"...no *ad hoc* assumptions made *a priori*

Kinematical limits of the $\tilde{P}_{gg}$ TMD: CCFM

Last we are interested in the limit $\tilde{\boldsymbol{p}} \to 0$,
i.e. vanishing transverse momentum of the produced gluon

For $\tilde{\boldsymbol{p}} \to 0$ we get the real CCFM kernel

$$\hat{P}_{gg}\left(z, \frac{\boldsymbol{k}^2}{\mu^2}, 0, \alpha_s\right) = z \cdot \int_0 \frac{d\tilde{\boldsymbol{p}}^2}{\tilde{\boldsymbol{p}}^2} \frac{\alpha_s C_a}{\pi} \left[\frac{1}{z} + \frac{1}{1-z} + \mathcal{O}\left(\frac{\tilde{\boldsymbol{p}}^2}{\boldsymbol{k}^2}\right)\right]$$

$$\tilde{\boldsymbol{p}} = \frac{\boldsymbol{k} - \boldsymbol{q}}{1-z}$$

Comes "for free"...no *ad hoc* assumptions made *a priori*
$\Rightarrow$ something must be RIGHT !

Conclusions and perspectives

- The method by Curci, Furmanski and Petronzio was successfully extended to the TMD case using gauge invariant vertices. Subtleties preventing the Catani-Hautmann generalisation from being directly extended to the P_{gg} case were uncovered and worked out.
- All the three limits, DGLAP, BFKL and CCFM are consistently satisfied: very non trivial consistency check ! For other early attempts, see **Ciafaloni, Colferai, Salam, Stasto Phys. Lett. B587 (2004)** **Kwiecinski, Martin, Stasto Phys. Rev. D56 (1997)**
- The next step are the virtual corrections, for which a systematic method has recently been proposed in **A. van Hameren, arXiv:1710.07609**...
- Higher orders will come next, once the complete consistency of the LO formalism has been fully established
- A Monte Carlo implementation is the main medium-term goal on the agenda...so stay tuned, please !