Evolution of the moments of a multiplicity distribution

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6. 12. 2017

Motivation

- Overall observed multiplicity of different types of particles agrees with the statistical model at temperatures above 160 MeV.
- The phase transition temperature can be determined also by lattice QCD methods $\rightarrow$ susceptibilities as functions of temperature are changing fastest $\rightarrow$ 150 MeV.
- Susceptibilities manifest themselves in higher moments of the multiplicity distribution.
- The main aim of this work is to know how fast different moments of the multiplicity distribution approch their equilibrium value and how they evolve if the system slips off equilibrium.
- The evolution of multiplicity distribution out of equilibrium is described by a master equation.

Master equation

- We consider a binary process $a_1 a_2 \to b_1 b_2$ with $a \neq b$, eg. $\pi N \to K\Lambda$
- The master equation for $P_n(\tau)$, the probability of finding n pairs $b_1 b_2$ at time τ has the following form

$$\frac{dP_n}{d\tau} = \epsilon \left[P_{n-1} - P_n\right] - \left[n^2 P_n - (n+1)^2 P_{n+1}\right] \tag{1}$$

  where $n = 0, 1, 2, 3...$, $\epsilon = G \langle N_{a_1} \rangle \langle N_{a_2} \rangle / L$, $\tau = t \; L/V$ - dimensionless time variable, $V/L = \tau_0^c$ - relaxation time, V - proper volume of the raction
- For thermal distribution of particle momentum $\to$ G - "creation term", L - "anihilation term" $\Rightarrow$ thermal averaged cross section

Time evolution of the factorial moments

- The scaled second factorial moment

$$F_2(\tau) = \langle N(N-1)\rangle \,/\, \langle N\rangle^2\,, \tag{2}$$

- the scaled third factorial moment

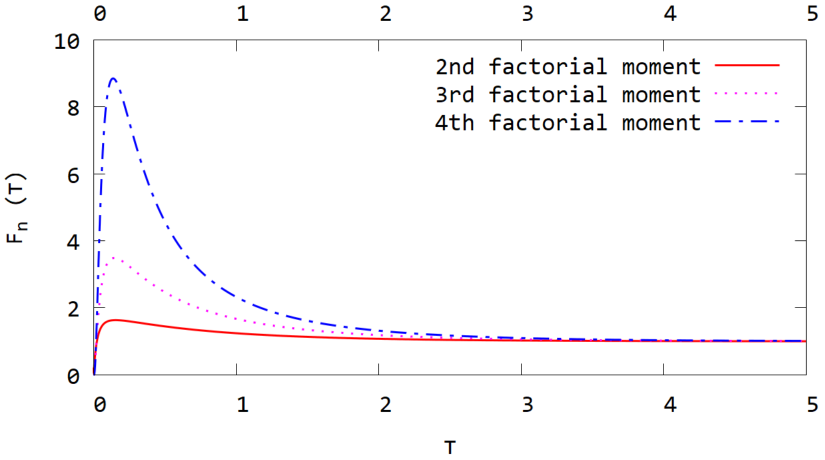
$$F_3(\tau) = \langle N(N-1)(N-2)\rangle \,/\, \langle N\rangle^3 \tag{3}$$

- and the scaled fourth factorial moment

$$F_4(\tau) = \langle N(N-1)(N-2)(N-3)\rangle \,/\, \langle N\rangle^4\,. \tag{4}$$

- We let the distribution of the multiplicity evolve in time according to the master equation.
- For numerical calculations were used binomial initial conditions.

Time evolution of the 2nd, 3rd and 4th factorial moment divided by its equilibrium value for $\epsilon = 0.1$ a $N_0 = 0.005$



Real time and temperature dependent master equation

- For further study purposes we want to add temperature and real time dependence.
- In case of constant temperature $\rightarrow$ equation formulated in dimensionless time.
- We will calculate the evolution for given chemical reaction $\pi^+ + n \rightarrow K^+ + \Lambda$
- Real time and temperature dependent master equation has the form

$$
\begin{aligned}
\frac{dP_n}{dt}(t/\tau_0^c) = \frac{G}{V} \langle N_{a_1} \rangle \langle N_{a_2} \rangle \left[P_{n-1}(t/\tau_0^c) - P_n(t/\tau_0^c)\right] \\
-\frac{L}{V}\left[n^2 P_n(t/\tau_0^c) - (n+1)^2 P_{n+1}(t/\tau_0^c)\right]
\end{aligned}
\tag{5}
$$

where $G \equiv \langle \sigma_G v \rangle$ and $L \equiv \langle \sigma_L v \rangle$.

Reaction $\pi^+ + n \longrightarrow K^+ + \Lambda^0$.

- For masses and spins we have

$$m_{\pi^+} = 139,570 \text{ MeV}, \quad m_n = 939,565 \text{ MeV}, \quad m_{\Lambda^0} = 1116 \text{ MeV}, \\ m_{K^+} = 493,667 \text{ MeV}, \tag{6}$$

$$d_{\pi^+} = 0, \quad d_n = 2, \quad d_{\Lambda^0} = 2, d_{K^+} = 0. \tag{7}$$

- Volume of the reaction is $V = 125 \text{ fm}^3$.
- Cross section for this reaction is

$$\sigma_{\pi N}^{\Lambda K} = \frac{0,054 \cdot (s^{1/2} - 1,61)}{0,091} \text{ fm}^2, \qquad 1,7 \geq s^{1/2} \geq 1,61 \text{ GeV}, \tag{8}$$

$$\sigma_{\pi N}^{\Lambda K} = \frac{0,0045}{s^{1/2} - 1,6} \text{ fm}^2, \qquad s^{1/2} \geq 1,7 \text{ GeV}, \tag{9}$$

$$\sigma_{\pi N}^{\Lambda K} = 0 \text{ fm}^2, \qquad s^{1/2} \leq 1,61 \text{ GeV}. \tag{10}$$

Real time and temperature dependent master equation - gradual change of temperature

- After complete thermalization of the factorial moments, the temperature decreases according to the Bjorken model from the initial temperature $T_0 = 165 \text{ MeV}$ according to the relation

$$T = T_0 \frac{t_0}{t} \tag{11}$$

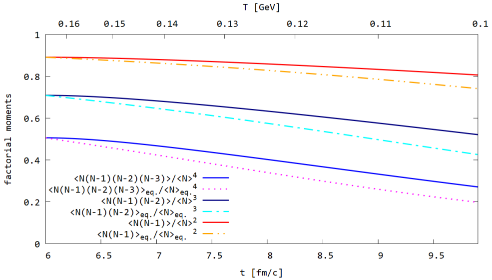
  up to temperature $T = 100 \text{ MeV}$, t_0 is hadronisation time for $T = 165 \text{ MeV} \rightarrow t_0 = 6 \text{ fm/c}$.

- System volume varies according to the relationship

$$V = V_0 \frac{t}{t_0}. \tag{12}$$

- We want to obtain the thermalisation time of quark-gluon plasma (around 10 fm/c) $\rightarrow$ we vary the cross-sections.

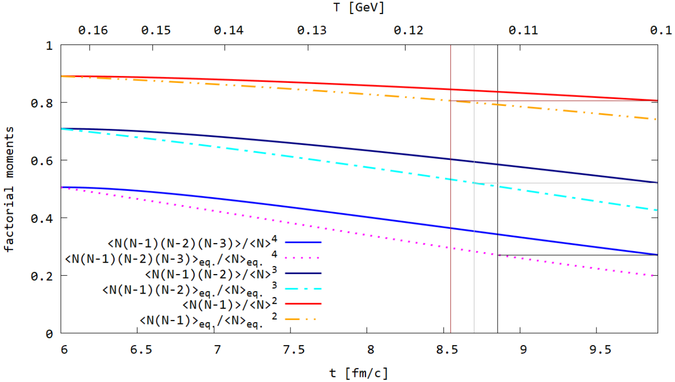
Scaled factorial moments for gradual change of temperature



- Decrease from 165 MeV to 100 MeV
- Thermalisation time around 10 $\mathrm{fm/c}$
- For 15 pions and 10 neutrons
- 200times enlarged cross section

Freeze-out temperature

- At the beginning we set the moments to equilibrium values $\rightarrow$ we let them evolve $\rightarrow$ we are looking for a temperature at which the thermalized system would lead to a given value of the factorial moment in the equilibrium state $\rightarrow$ reverse determination of the apparent freeze-out temperature

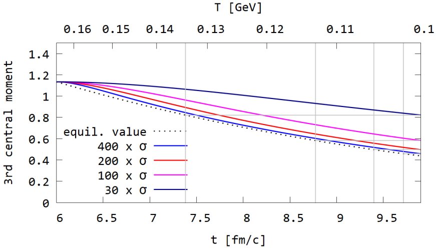


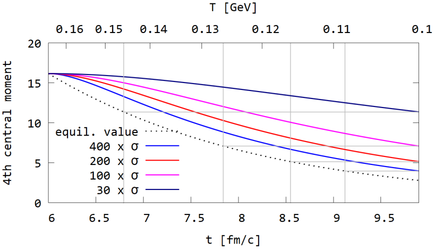
Central moments

- For data processing $\rightarrow$ central moments, event. their combination.
- 2nd central moment $\mu_2 = \left\langle N^2 \right\rangle - \langle N \rangle^2$.
- 3rd central moment $\mu_3 = \left\langle (N - \langle N \rangle)^3 \right\rangle$.
- 4th central moment $\mu_4 = \left\langle (N - \langle N \rangle)^4 \right\rangle$.
- Coefficient of skewness $S = \dfrac{\mu_3}{\sigma^3} = \dfrac{\left\langle (N - \langle N \rangle)^3 \right\rangle}{\left\langle (N - \langle N \rangle)^2 \right\rangle^{3/2}}$.
- Coefficient of kurtosis $\kappa = \dfrac{\mu_4}{\sigma^4} - 3 = \dfrac{\left\langle (N - \langle N \rangle)^4 \right\rangle}{\left\langle (N - \langle N \rangle)^2 \right\rangle^2} - 3$.

Apparent freeze-out temperature for 3rd (left) and 4th (right) central moment for gradual change of temperature

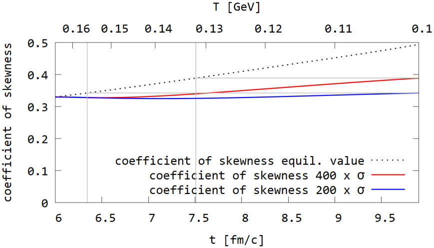
- Decrease from 165 MeV to 100 MeV for different cross sections, for 15 pions and 10 neutrons
- Different apparent freeze-out temperatures for every moment

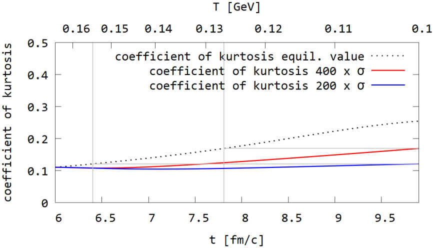




Apparent freeze-out temperature for coefficient of skewness (left) and kurtosis (right) for gradual change of temperature

- Decrease from 165 MeV to 100 MeV for different cross sections, for 15 pions and 10 neutrons





Conclusion - 1st part

- The phase transition temperature can also be determined by measuring the higher moments of the proton multiplicity $\rightarrow$ comparison with the results for the susceptibilities of the baryon number
- Fluctuations in the baryon number usually lead to a seemingly lower phase transition temperature than examining the number of particles $\rightarrow$ perhaps because the higher moments seem to show a different temperature than what we really have.
- In non-equilibrium state, higher factorial moments differ more from their equilibrium values than the lower moments.
- The behavior of the combination of the central moments depends on the combination of moments we choose.

Master equation for reaction $p + \pi^- \to \Delta^0 \to n + \pi^0$

- For masses and spins we have

$$m_{\pi^-} = 139,570\ \mathrm{MeV}, \quad m_{\pi^0} = 134,977\ \mathrm{MeV}, \quad m_n = 939,565\ \mathrm{MeV},$$
$$m_p = 938,272\ \mathrm{MeV}, \tag{13}$$

$$d_{\pi^-} = 0, \quad d_{\pi^0} = 0, \quad d_n = 2, d_p = 2. \tag{14}$$

- Volume of the reaction is $V = 125\ \mathrm{fm}^3$ and temperature drops from $T = 165\ \mathrm{MeV}$ to $T = 100\ \mathrm{MeV}$.

- For pion-nucleon cross section we have

$$\sigma(\pi^+ p \to \Delta^{++}) = \frac{326,5}{1 + 4\left(\frac{\sqrt{s} - 1,215}{0,110}\right)^2} \frac{q^3}{q^3 + (0,18)^3} \quad [\mathrm{mb}], \quad (15)$$

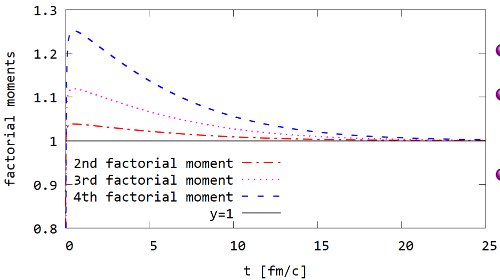
- where q is the cm momentum

$$q = \left[\frac{(s - (m_\pi + m_p)^2)(s - (m_\pi - m_p)^2)}{4s}\right]^{1/2} = \frac{m_p}{\sqrt{s}} p_{lab} \quad [\mathrm{GeV/c}]. \quad (16)$$

- Then cross section for this reaction is

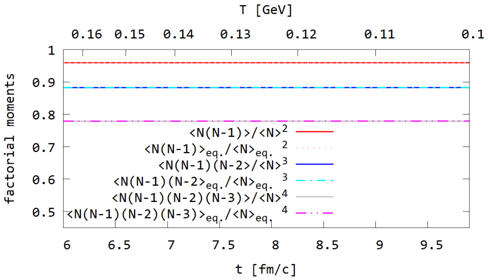
$$\sigma(\pi^- p \to \Delta^0 \to n + \pi^0) = \frac{1}{3} \cdot \frac{2}{3} \sigma(\pi^+ p \to \Delta^{++}) = \frac{2}{9} \sigma(\pi^+ p \to \Delta^{++}). \quad (17)$$

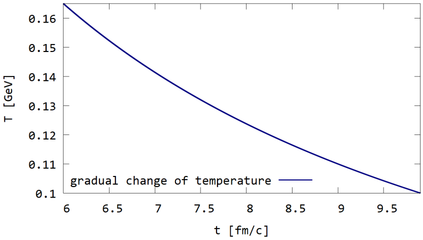
Scaled factorial moments for constant temperature



- For temperature 165 MeV
- Thermalisation time around 10 $\mathrm{fm/c}$
- For 15 protons and 10 pions

Scaled factorial moments for gradual change of temperature





Conclusion - 2nd part

- Factorial moments do not change in time for the gradual change of temperature $\rightarrow$ no fluctuations in the proton and neutron number.
- The same conclusion $\rightarrow$ M. Kitazawa and M. Asakawa in articles:
    - M. Kitazawa, M. Asakawa, *Revealing baryon number fluctuations from proton number fluctuations in relativistic heavy ion collisions*, Phys. Rev. C **85** (2012) 021901(R)
    - M. Kitazawa, M. Asakawa, *Relation between baryon number fluctuations and experimentally observed proton number fluctuations in relativistic heavy ion collisions*, Phys. Rev. C **86** (2012) 024904

References

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- C. M. Ko, V. Koch, Z.-W. Lin, K. Redlich, M. Stephanov, X.-N. Wang, *Kinetic Equation with Exact Charge Conservation*, Phys. Rev. Lett. **86** (2001) 5438.
- C. M. Ko and L. Xia, *K^+/π^+ enhancement in heavy-ion collisions*, Phys. Rev. C **38** (1988) 179.
- B. Tomášik and E. E. Kolomeitsev, *Strangeness dynamics in heavy-ion collisions: The K/π ratios and the lifetime of a fireball*, arXiv:nucl-th/0512088v1 (2005).
- Particle Data Group: http://pdg.lbl.gov/2016/tables/contents_tables.html [online 17/04/2017].
- J. Cugnon, D. L'Hôte, J. Vandermeulen, *Simple parametrization of cross-sections for nuclear transport studies up to GeV range*, Nucl. Instr. in Phys. Res. B **111** (1996) 215-220.

Backup slides

Generating equation

- The master equation can be converted into the partial differential equation for the generating function

$$g(x,\tau) = \sum_{n=0}^{\infty} x^n P_n(\tau) \tag{18}$$

- From the derivative of the generating function we can easily determine the moments.
- Multiplying eq. (18) by x^n and summing over n, we find

$$\frac{\partial g(x,\tau)}{\partial \tau} = \frac{L}{V}(1-x)(xg'' + g' - \epsilon g), \tag{19}$$

where $g' \equiv \partial g / \partial x$.
- $g(1,\tau)$ does not change with time, which is equivalent to the conservation of total probability.

- The equilibrium solution, $g_{eq.}(x)$, thus obeys the following equation:

$$xg_{eq.}^{''} + g_{eq.}^{'} - \epsilon g_{eq.} = 0. \tag{20}$$

- The solution that is regular at $x = 0$ is then given by

$$g_{eq.}(x) = \frac{I_0(2\sqrt{\epsilon x})}{I_0(2\sqrt{\epsilon})} \tag{21}$$

- The average number of $b_1 b_2$ pairs per event in equilibrium is given by

$$\langle N \rangle_{eq.} = g_{eq.}'(1) = \sqrt{\epsilon}\frac{I_1(2\sqrt{\epsilon})}{I_0(2\sqrt{\epsilon})} \tag{22}$$

Higher factorial moments in equilibrium state

- We can express higher factorial moments by the derivative of the generating function $g(x, \tau)$, which is given by eq. (18)
- I also used these relations for modified Bessel functions

$$I_0'(z) = I_1(z) \tag{23}$$

$$I_1'(z) = \frac{1}{2}(I_2(z) + I_0(z)) \tag{24}$$

$$I_2'(z) = \frac{1}{2}(I_3(z) + I_1(z)) \tag{25}$$

$$I_3'(z) = \frac{1}{2}(I_4(z) + I_2(z)) \tag{26}$$

2nd factorial moment

- The second derivative of the generating function is given by

$$g''_{eq.}(x) = -\frac{1}{2}\sqrt{\varepsilon}x^{-3/2}\frac{I_1(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} + \varepsilon\frac{1}{x}\frac{I_2(2\sqrt{\varepsilon x}) + I_0(2\sqrt{\varepsilon x})}{2I_0(2\sqrt{\varepsilon})} \tag{27}$$

- And then the equilibrium value of the second factorial moment has the form

$$\langle N(N-1)\rangle_{eq.} = g''_{eq.}(1) = -\frac{1}{2}\sqrt{\varepsilon}\frac{I_1(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} + \frac{1}{2}\varepsilon\frac{I_2(2\sqrt{\varepsilon}) + I_0(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} \tag{28}$$

- The third derivative of the generating function is given by

$$g'''_{eq.}(x) = \frac{3}{4}x^{-5/2}\sqrt{\varepsilon}\frac{I_1(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} - \frac{5}{4}\varepsilon\frac{1}{x^2}\frac{I_2(2\sqrt{\varepsilon x}) + I_0(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} + \frac{1}{2}\varepsilon^{3/2}\frac{1}{x^{3/2}}\frac{I_3(2\sqrt{\varepsilon x}) + 3I_1(2\sqrt{\varepsilon x})}{2I_0(2\sqrt{\varepsilon})} \tag{29}$$

- And then the equilibrium of the third factorial moment has the form

$$\langle N(N-1)(N-2)\rangle_{eq.} = g'''_{eq.}(1) = \frac{3}{4}\sqrt{\varepsilon}\frac{I_1(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} - \frac{5}{4}\varepsilon\frac{I_2(2\sqrt{\varepsilon}) + I_0(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} + \frac{1}{4}\varepsilon^{3/2}\frac{I_3(2\sqrt{\varepsilon}) + 3I_1(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} \tag{30}$$

4th factorial moment

- The fourth derivative of the generating function is given by

$$
\begin{aligned}
g_{eq.}^{IV.}(x) = {} & \frac{3}{8}\varepsilon\frac{1}{x^3}\frac{I_2(2\sqrt{\varepsilon x}) + I_0(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} - \frac{15}{8}\sqrt{\varepsilon}x^{-7/2}\frac{I_1(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} \\
& + \frac{5}{2}\varepsilon\frac{1}{x^3}\frac{I_2(2\sqrt{\varepsilon x}) + I_0(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} - \frac{5}{8}\varepsilon^{3/2}\frac{1}{x^{5/2}}\frac{I_3(2\sqrt{\varepsilon x}) + I_1(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} \\
& - \frac{3}{8}\frac{1}{x^{5/2}}\frac{I_3(2\sqrt{\varepsilon x}) + 3I_1(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} \\
& + \frac{1}{8}\varepsilon^2\frac{1}{x^2}\frac{I_4(2\sqrt{\varepsilon x}) + 2I_2(2\sqrt{\varepsilon x}) + I_0(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})}
\end{aligned}
\tag{31}
$$

- And then the equilibrium value of the fourth factorial moment has the form

$$\langle N(N-1)(N-2)(N-3)\rangle_{eq.} = g_{eq.}^{IV.}(1) = \frac{23}{8}\varepsilon\frac{I_2(2\sqrt{\varepsilon}) + I_0(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})}$$
$$-\frac{15}{8}\sqrt{\varepsilon}\frac{I_1(2\sqrt{\varepsilon})}{I_0(2\sqrt{\varepsilon})} - \varepsilon^{3/2}\frac{4I_3(2\sqrt{\varepsilon}) + 7I_1(2\sqrt{\varepsilon})}{4I_0(2\sqrt{\varepsilon})}$$
$$+\frac{1}{8}\varepsilon^2\frac{I_4(2\sqrt{\varepsilon}) + 2I_2(2\sqrt{\varepsilon}) + I_0(2\sqrt{\varepsilon x})}{I_0(2\sqrt{\varepsilon})} \tag{32}$$

Binomial initial conditions

- On one hand, one can assume that initially there is at most one particle in given event
- Then the initial conditions are

$$P_0(\tau = 0) = 1 - N_0 \tag{33}$$

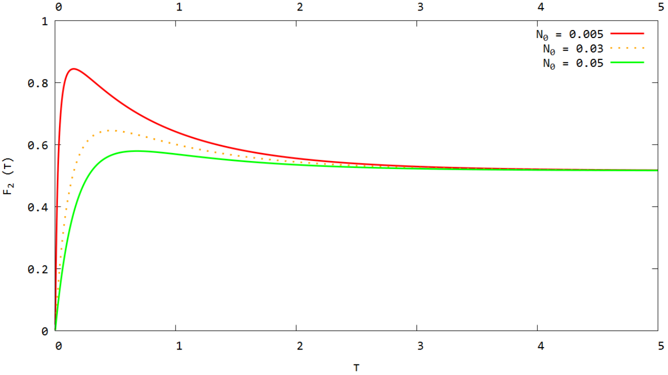
$$P_1(\tau = 0) = N_0 \tag{34}$$

$$P_n(\tau = 0) = 0, n > 1 \tag{35}$$

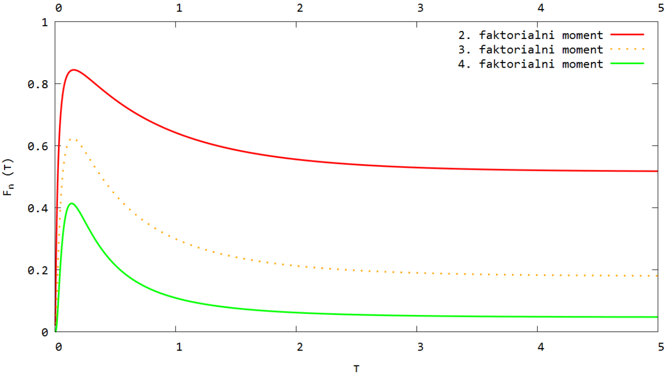
where N_0 is initial averaged number of particles

- In this case, the factorial moments then start at 0

Time evolution of the 2nd factorial moment for the binomial initial conditions. The 2nd factorial moment for different values of the averaged initial number of particles N_0 and for $\epsilon = 0.1$



2nd, 3rd and 4th factorial moment for the binomial initial conditions for $\epsilon = 0.1$ and $N_0 = 0.005$



Temperature dependent master equation

- Because of averaging over relative velocities, we will assume that the momenta are distributed according to Boltzmann distribution

$$n_i(p) \propto exp\left(-\frac{\sqrt{m_i^2+p^2}}{T}\right). \tag{36}$$

- The averaged cross section is then obtained as

$$\left\langle v_{ij}\sigma_{ij}^X \right\rangle = \frac{\int_{\sqrt{s_0}}^{\infty} dx \sigma_{ij}^X(x) K_1(\frac{x}{T}) \left[x^2-(m_i+m_j)^2\right]\left[x^2-(m_i-m_j)^2\right]}{4m_i^2 m_j^2 T K_2(m_i/T) K_2(m_j/T)} \tag{37}$$

where K_i's are the modified Bessel functions and $\sqrt{s_0} = max(m_i+m_j, \Sigma_{final} m_a)$ is the reaction threshold.

- If we know cross section for the reactions $a_1 a_2 \to b_1 b_2$, the cross section for the inverse reactions follows from phase-space considerations as

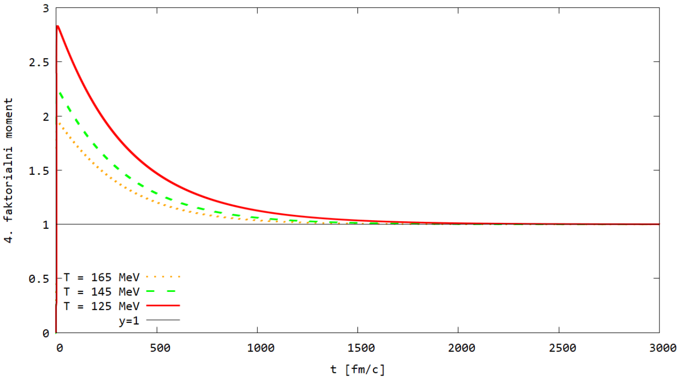
$$\sigma_{34 \longrightarrow 12}(\sqrt{s}) = \frac{(2J_3+1)(2J_4+1)}{(2J_1+1)(2J_2+1)} \frac{p_{cm}^2(s, m_1, m_2)}{p_{cm}^2(s, m_3, m_4)} \times \sigma_{12 \longrightarrow 34}(\sqrt{s}) \tag{38}$$

where J_i and m_i are spins and masses of the participating species, and p_{cm} is the center-of-mass momentum defined as

$$p_{cm}^2(s, m_1, m_2) = \frac{\left[s - (m_1^2 + m_2^2)\right]^2 - 4m_1^2 m_2^2}{4s}. \tag{39}$$

Temperature dependent master equation - constant temperature

- 4th factorial moment divided by its equilibrium value for different temperatures T = 165 MeV, T = 145 MeV and T = 125 MeV for 15 pions a 10 neutrons.



Other ratios of central moments

- While the 2nd, 3rd and 4th central moment are decreasing, the coefficient of skewness and kurtosis increases $\rightarrow$ it is dependent on the ratio we choose.
- Volume independent ratios $\rightarrow$ useful for comparison with experimental data, eg.

$$R_{32} = \frac{\mu_3}{\mu_2} = S\sigma \tag{40}$$

or

$$R_{12} = \frac{\mu_1}{\mu_2} = M/\sigma^2 \tag{41}$$

where S is coefficient of skewness, σ is standard deviation and M is number of particles $\langle N \rangle$.

Coefficient R_{32} (left) and R_{12} (right) for gradual change of temperature

- Decrease from 165 MeV to 100 MeV for different cross sections, for 15 pions and 10 neutrons