

Perturbative solutions for relativistic hydrodynamics

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Zimányi School '17 - 2017.12.08. Budapest

Kurygis B., Csanád M., accepted at MDPI Universe arXiv:1711.05446 [nucl-th]

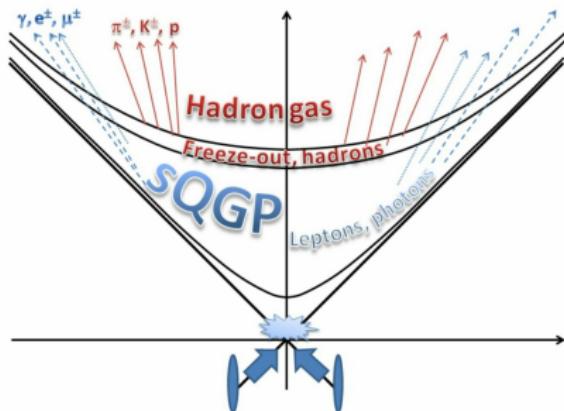
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Introduction

The strongly interacting quark-gluon plasma (sQGP)

- Discovered at the RHIC, created at LHC
- Hot, expanding, perfect quark-fluid
- Hadrons created at the freeze-out
- Photons and leptons "shine through"



Known solutions for relativistic hydrodynamics

- Many numerical solutions
- Exact, analytic solutions important: connect initial/final state
- Famous 1+1D solutions: Landau-Khalatnikov & Hwa-Bjorken

L. D. Landau, Izv. Akad. Nauk Ser. Fiz. **17**, 51 (1953)

I.M. Khalatnikov, Zhur. Eksp. Teor. Fiz. **27**, 529 (1954)

R. C. Hwa, Phys. Rev. **D 10**, 2260 (1974)

J. D. Bjorken, Phys. Rev. **D 27**, 140 (1983)

- Discovery of sQGP → Many new solutions
- First truly 3D relativistic solution: **Hubble-flow**

Csörgő, Csernai, Hama, Kodama, Heavy Ion Phys. **A21**, 73 (2004), nucl-th/0306004

- Describes well experimental data

Csanad, Vargyas, Eur. Phys. J. **A 44**, 473 (2010) nucl-th/09094842

Equations for relativistic hydrodynamics

Looking for (u^μ, p, ϵ, n or σ) fields

Assumptions:

- zero viscosity
- zero heat-conductivity
- local energy-momentum conservation

Properties:

- $u_\mu u^\mu = 1$
- $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$

Temperature: $T = (\epsilon + p)/\sigma$

Locally conserved entropy density
 $n \rightarrow$ conserved charge density

Energy, momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}$$

Continuity equation

$$\partial_\mu(nu^\mu) = 0 \text{ or } \partial_\mu(\sigma u^\mu) = 0$$

Equation of state (EoS)

$$\epsilon = \kappa p$$



Perturbative handling

Perturbative handling of the relativistic hydrodynamics

Perturbed fields:

- Start from a known solution: (u^μ, p, n)
- $u^\mu \rightarrow u^\mu + \delta u^\mu$
- $p \rightarrow p + \delta p$
- $n \rightarrow n + \delta n$
- works similarly for $n \rightarrow \sigma$
- Orthogonality:

$$u^\mu \delta u_\mu = 0 \quad (1)$$

Equations for perturbations:

- substitute perturbations into equations
- subtract 0th order equations
- neglect 2nd or higher order perturbations
- remainder: perturbed equation
- solution yields perturbations $\delta u^\mu, \delta n, \delta p$

Perturbations of Hubble-flow

Known solution: Hubble-flow

Hubble-flow:

Csörgő, Csernai, Hama, Kodama , Heavy Ion Phys. **A21**, 73 (2004), nucl-th/0306004

- $u^\mu = \frac{x^\mu}{\tau}$
- $n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$
- $p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$
- Scaling variable: $u_\mu \partial^\mu S = \partial_\tau S = 0$
- Describes well hadronic data and photons

Csanad, Vargyas, Eur. Phys. J. **A 44**, 473 (2010) nucl-th/09094842Csanad, Majer, Central Eur. J. Phys. **10** (2012)

- Multipole solutions also possible

Csanad, Szabo , Phys. Rev. **C 90**, 054911 (2014)

Perturbations of Hubble-flow

Finding solutions for perturbations

The way of solution:

- Choosing test functions: $\delta p, \delta u^\mu, \delta n$
- Fixing arbitrary functions
- Choosing scaling variable S
- Satisfying all the restrictions
- instead, one could look for sound waves on top of Hubble-flow

Perturbations:

$$u^\mu = \frac{x^\mu}{\tau} \quad \rightarrow \delta u^\mu = \delta \cdot F(\tau) g(x_\nu) \partial^\mu S \cdot \chi(S) \quad (2)$$

$$n = n_0 \left(\frac{\tau_0}{\tau} \right)^3 \mathcal{N}(S) \quad \rightarrow \delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau} \right)^3 h(x_\nu) \nu(S) \quad (3)$$

$$p = p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \quad \rightarrow \delta p = \delta \cdot p_0 \pi(S) \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \quad (4)$$

The new class of solutions

General form of solutions

Perturbations:

$$\delta u^\mu = \delta \cdot F(\tau) g(x_\nu) \partial^\mu S \cdot \chi(S) \quad (5)$$

$$\delta p = \delta \cdot p_0 \pi(S) \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \quad (6)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau} \right)^3 h(x_\nu) \nu(S) \quad (7)$$

- $\delta u^\mu \rightarrow$ defined by $g(x_\nu)$, $\chi(S)$ and $F(\tau)$ fixed by $g(x_\nu)$
- $\delta p \rightarrow \pi(S)$ determined by δu^μ
- $\delta n \rightarrow \nu(S)$ fixed by $h(x_\nu)$, $g(x_\nu)$, $h(x_\nu)$, S need to fulfill (8)-(10)

Equations for $\mathcal{N}(S)$, $\chi(S)$, $\nu(S)$, $\pi(S)$, $h(x_\mu)$, $g(x_\mu)$, S functions

$$\frac{\chi'(S)}{\chi(S)} = - \frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S} - \frac{\partial_\mu S \partial^\mu \ln g(x_\nu)}{\partial_\mu S \partial^\mu S} \quad (8)$$

$$\frac{\pi'(S)}{\chi(S)} = (\kappa + 1) \left[F(\tau) \left(u^\mu \partial_\mu g(x_\nu) - \frac{3g(x_\nu)}{\kappa \tau} \right) + F'(\tau) g(x_\nu) \right] \quad (9)$$

$$\frac{\nu(S)}{\chi(S) \mathcal{N}'(S)} = - \frac{F(\tau) g(x_\nu) \partial_\mu S \partial^\mu S}{u^\mu \partial_\mu h(x_\nu)} \quad (10)$$

The new class of solutions

Looking for concrete solution

To compute measurables \rightarrow fix the $g(x_\nu)$, $F(\tau)$, $h(x_\nu)$ functions:

$$g(x_\nu) = 1,$$

$$F(\tau) = \tau + c\tau_0 \left(\frac{\tau}{\tau_0} \right)^{\frac{3}{\kappa}}$$

$$h(x_\nu) = \begin{cases} \ln \left(\frac{\tau}{\tau_0} \right) + c \frac{\kappa}{3-\kappa} \left(\frac{\tau}{\tau_0} \right)^{\frac{3}{\kappa}-1} & \text{if } \kappa \neq 3 \\ (1+c) \ln \left(\frac{\tau}{\tau_0} \right) & \text{if } \kappa = 3 \end{cases}$$

Restrictions:

- $u_\mu \partial^\mu S = 0$
- $\frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S}$ is a function of the scaling variable
- $\tau^2 \partial_\mu S \partial^\mu S$ also a function of the scaling variable

Scaling variables found so far: $S = \frac{r^m}{t^m}$, $S = \frac{r^m}{\tau^m}$, $S = \frac{\tau^m}{t^m}$

Scaling variable: $S = r^m/t^m$

Scaling variable: $S = r^m/t^m$

Scaling variable

$$S = \frac{r^m}{t^m} \quad (11)$$

The functions of the scaling variable:

Perturbations

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \pi(S)$$

$$\delta u^\mu = \delta \cdot \left(\tau + c \tau_0 \left(\frac{\tau}{\tau_0} \right)^{\frac{3}{\kappa}} \right) \partial^\mu S \chi(S)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau} \right)^3 \left(\ln \left(\frac{\tau}{\tau_0} \right) + c \frac{\kappa}{3-\kappa} \left(\frac{\tau}{\tau_0} \right)^{\frac{3}{\kappa}-1} \right) \nu(S)$$

$$\chi(S) = \left(\frac{r}{t} \right)^{-m-1} \quad (12)$$

$$\pi(S) = -\frac{(\kappa+1)(\kappa-3)}{\kappa} m \left(\frac{r}{t} \right)^{-1} \quad (13)$$

$$\nu(S) = m^2 \left(\frac{r}{t} \right)^{m-1} \left(\left(\frac{r}{t} \right)^2 - 1 \right) \left(1 - \left(\frac{r}{t} \right)^{-2} \right) \mathcal{N}' \left(\frac{r^m}{t^m} \right) \quad (14)$$

Scaling variable: $S = t/r$

Concrete solution with $S = t/r$, $\mathcal{N}(S) = \exp(-S^{-2})$

Particular case: $m = -1$

$$S = \frac{t}{r} \quad (15)$$

Let us choose Gaussian $\mathcal{N}(S)$: $\mathcal{N}(S) = e^{-\frac{r^2}{t^2}} = e^{-S^{-2}}$

The functions of the scaling variable:

$$\chi(S) = 1 \quad (16)$$

$$\pi(S) = \frac{(\kappa + 1)(\kappa - 3)}{\kappa} \left(\frac{t}{r} \right) \quad (17)$$

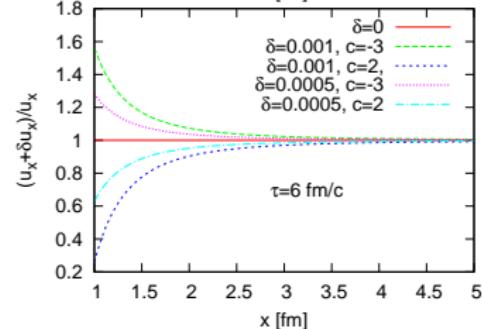
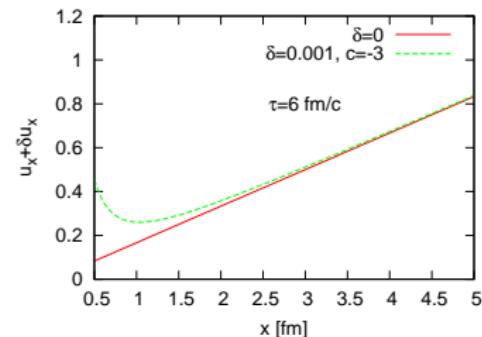
$$\nu(S) = 2 \left(\frac{t}{r} \right)^{-3} \left(1 - \left(\frac{t}{r} \right)^2 \right)^2 \mathcal{N} \left(\frac{t}{r} \right) \quad (18)$$

Scaling variable: $S=t/r$

Four-velocity perturbation

$$\delta u^\mu = \delta \cdot \left(\tau + c\tau_0 \left(\frac{\tau}{\tau_0} \right)^{\frac{3}{\kappa}} \right) \partial^\mu S$$

$$u^\mu = \frac{x^\mu}{\tau}$$

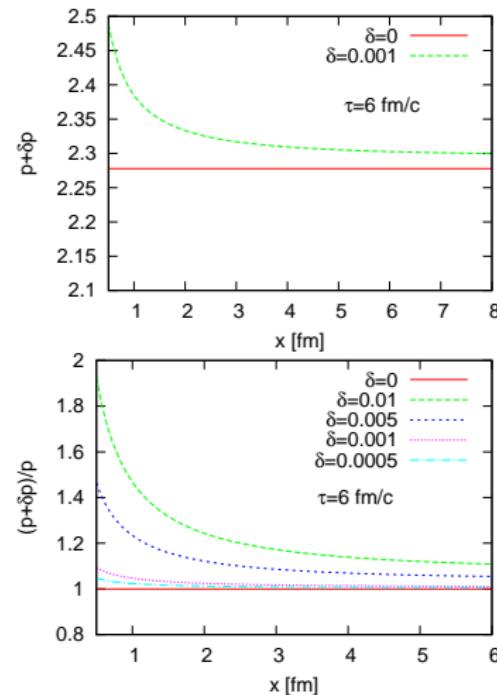
Used parameters (describes v_2 , $N(p_T)$, R_{HBT}):M. Csand, M. Vargyas, Eur. Phys. J. A **44**, 473

Scaling variable: $S=t/r$

Pressure perturbation

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \frac{(\kappa+1)(\kappa-3)}{\kappa} S$$

$$p = p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}}$$

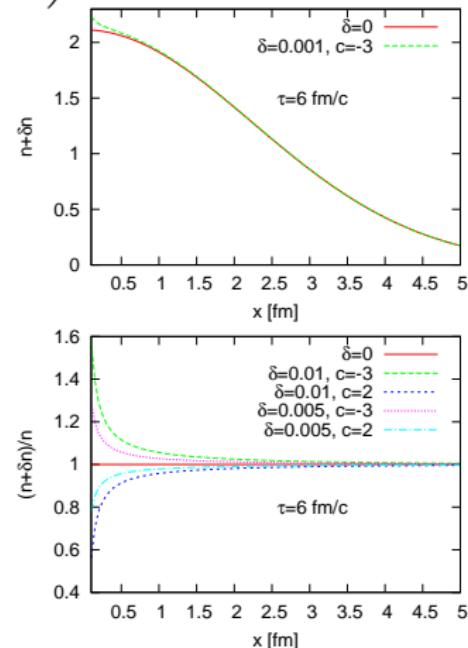


Scaling variable: $S=t/r$

Particle density perturbation

$$\delta n = \delta \cdot 2bn_0 \left(\frac{\tau_0}{\tau}\right)^3 \left(\ln\left(\frac{\tau}{\tau_0}\right) + c \frac{\kappa}{3-\kappa} \left(\frac{\tau}{\tau_0}\right)^{\frac{3}{\kappa}-1} \right) S^{-3} (1-S^2)^2 \mathcal{N}(S)$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$$



Single-particle distribution

Source function \rightarrow Jüttner-distribution:

$$S(x, p) d^4x = N n \exp\left(-\frac{p_\mu u^\mu}{T}\right) H(\tau) p_\mu d^3 \Sigma^\mu(x^\mu) d\tau \quad (19)$$

The Cooper–Frye factor: $p_\mu d^3 \Sigma^\mu(x^\mu) = \frac{p_\mu u^\mu}{u^0} d^3x$

Freeze out at const. proper time $(\tau_0) \rightarrow H(\tau) = \delta(\tau - \tau_0)$

With the perturbations:

$$S(x, p) = N n \exp\left(-\frac{p_\mu u^\mu}{T}\right) \delta(\tau - \tau_0) \frac{p_\mu u^\mu}{u^0} \cdot (1 + \Delta) d\tau dx^3$$

$$\Delta = \left[\frac{\delta u^0}{u^0} + \frac{p_\mu \delta u^\mu}{p_\nu u^\nu} - \frac{p_\mu \delta u^\mu}{T} + \frac{p_\mu u^\mu \delta T}{T^2} + \frac{\delta n}{n} \right]$$

Single-particle distribution:

$$N_1(p) = \int S(x, p) d^4x \quad (20)$$

Single-particle transverse momentum distribution, $S = t/r$

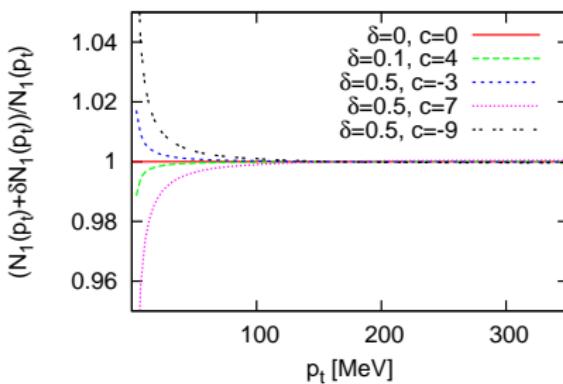
Two component Gaussian:

$$N(p) = Nn_0 \mathcal{E}_1 \mathcal{V}_1 (1 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) + Nn_0 \mathcal{E}_2 \mathcal{V}_2 (\mathcal{P}_4 + \mathcal{P}_5)$$

$$\mathcal{E}_1 = \exp \left[-\frac{E^2 + m^2}{2ET_0} - \frac{p^2}{2ET_{\text{eff}}} \right] \quad \mathcal{E}_2 = \exp \left[-\frac{E^2 + m^2}{2ET_0} - \frac{p^2}{2ET_{\text{eff},\delta}} \right]$$

Used parameters: describes hadronic & photonic data (v_2 , R_{HBT} , $N(p_T)$)

M. Csand, M. Vargyas, Eur. Phys. J. A **44**, 473 (2010)



Effective temperatures:

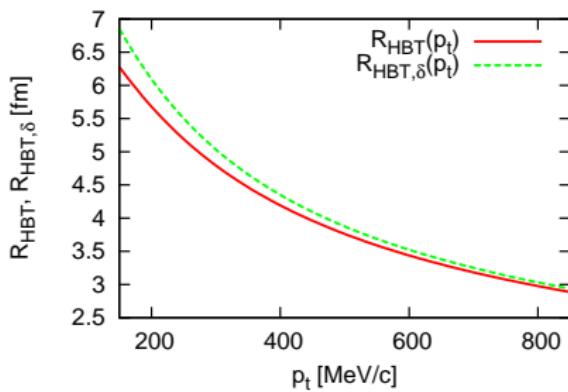
$$T_{\text{eff}} = T_0 + \frac{T_0 E \dot{R}_0^2}{2b(T_0 - E)}$$

$$T_{\text{eff},\delta} = T_0 + \frac{T_0 E \dot{R}_0^2}{2b(2T_0 - E)}$$

Measurables

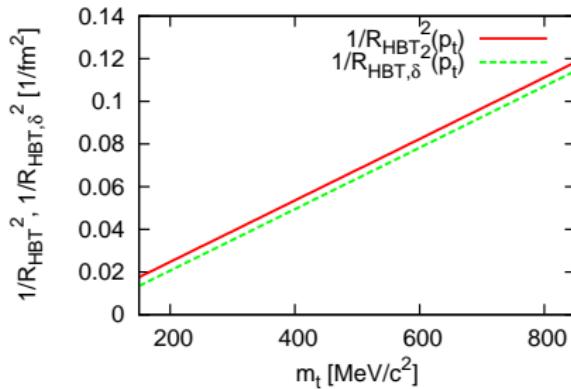
HBT-radii for $S = t/r$ Size of the source \rightarrow HBT-radii

- $R^{-2} \propto m_t$ scaling

Parameters: $\delta = 0.5$, $c = -3$

$$R_{\text{HBT}}^2 = \frac{T_0 \tau_0^2 (T_{\text{eff}} - T_0)}{E T_{\text{eff}}}$$

$$R_{\text{HBT},\delta}^2 = \frac{T_0 \tau_0^2 (T_{\text{eff},\delta} - T_0)}{E T_{\text{eff},\delta}}$$



Summary

Hubble-flow

$$u^\mu = \frac{x^\mu}{\tau}$$

$$p = p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}}$$

$$n = n_0 \left(\frac{\tau_0}{\tau} \right)^3 \mathcal{N}(S)$$

Perturbations

$$\delta u^\mu = \delta \cdot F(\tau) g(x_\nu) \partial^\mu S \cdot \chi(S)$$

$$\delta p = \delta \cdot p_0 \pi(S) \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}}$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau} \right)^3 h(x_\nu) \nu(S)$$

Accomplishments:

- New relativistic, accelerating, perturbative family of solutions
- Many possible particular solutions
- Measurables can be computed

Outlook:

- Non-spherical symmetry
- Other S , $g(x_\nu)$ and $h(x_\nu)$ functions
- Another known solution

Thank you for your attention!

Perturbations on a standing fluid: waves

Known solution: Standing fluid

- $u^\mu = (1, 0, 0, 0)$
- $p = \text{const.}$
- $n = \text{const.}$

Exploit these fields:

- $\partial_\mu u^\mu = 0$
- $\partial_\mu p = 0$
- $u^\mu \partial_\mu = \partial_0$
- $Q^{\mu\nu} = (u^\mu u^\nu - g^{\mu\nu})$
- $Q^{\mu\nu} \partial_\mu = (0, \nabla)$

Perturbed Energy equation

$$\kappa \partial_0 \delta p + (\kappa + 1) p \partial_\mu \delta u^\mu = 0$$

Perturbed Euler equation

$$(\kappa + 1) p \partial_0 \delta u^\nu - Q^{\mu\nu} \partial_\mu \delta p = 0$$

Wave solution for pressure

$$\partial_0^2 \delta p = \frac{1}{\kappa} \Delta \delta p$$

Decomposition of energy-momentum tensor

Two equations:

- Lorentz-orthogonal to u^μ
- Lorentz-perpendicular to u^μ

Euler equation

$$(\kappa + 1) p u^\nu \partial_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu) \partial_\nu p$$

Energy equation

$$\kappa u^\mu \partial_\mu p + (\kappa + 1) p \partial_\mu u^\mu = 0$$

Perturbative equations

Euler equation

$$(\kappa + 1)\delta p u^\mu \partial_\mu u^\nu + (\kappa + 1)p \delta u^\mu \partial_\mu u^\nu + (\kappa + 1)p u^\mu \partial_\mu \delta u^\nu = (g^{\mu\nu} - u^\mu u^\nu) \partial_\mu \delta p - \delta u^\mu u^\nu \partial_\mu p - u^\mu \delta u^\nu \partial_\mu p \quad (21)$$

Energy equation

$$\kappa \delta u^\mu \partial_\mu p + \kappa u^\mu \partial_\mu \delta p + (\kappa + 1) \delta p \partial_\mu u^\mu + (\kappa + 1) p \partial_\mu \delta u^\mu = 0 \quad (22)$$

Continuity equation

$$u^\mu \partial_\mu \delta n + \delta n \partial_\mu u^\mu + \delta u^\mu \partial_\mu n + n \partial_\mu \delta u^\mu = 0 \quad (23)$$

Perturbed equations

Euler equation:

$$\frac{\partial_\mu \delta p}{(\kappa + 1)p} [g^{\mu\nu} - u^\mu u^\nu] = \frac{\kappa - 3}{\tau \kappa} \delta u^\nu + u^\mu \partial_\mu \delta u^\nu \quad (24)$$

Energy equation:

$$\kappa u^\mu \partial_\mu \delta p + \frac{3(\kappa + 1)}{\tau} \delta p = -(\kappa + 1) p \partial_\mu \delta u^\mu \quad (25)$$

Continuity:

$$\delta u^\mu n \frac{\mathcal{N}'(S)}{\mathcal{N}(S)} \partial_\mu S + u^\mu \partial_\mu \delta n + \frac{3\delta n}{\tau} + n \partial_\mu \delta u^\mu = 0 \quad (26)$$

Solving the energy equation

Pressure perturbation

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau} \right)^{3+\frac{3}{\kappa}} \pi(S). \quad (27)$$

Four-velocity perturbation

$$\delta u^\mu = \delta \cdot F(\tau) g(x_\mu) \partial^\mu S \cdot \chi(S) \quad (28)$$

- Orthogonality satisfied ($\delta u_\mu u^\mu = 0$)

Energy equation

$$\frac{\chi'(S)}{\chi(S)} = -\frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S} - \frac{\partial_\mu S \partial^\mu \ln g(x_\mu)}{\partial_\mu S \partial^\mu S} \quad (29)$$

Right side is a function of S !

Solution of the Euler equation

Using (27) and (28) perturbations:

Euler equation:

$$\frac{\pi'(S)}{\chi(S)} = (\kappa + 1) \left[F(\tau) \left(u^\mu \partial_\mu g(x_\mu) - \frac{3g(x_\mu)}{\kappa\tau} \right) + F'(\tau)g(x_\mu) \right] \quad (30)$$

- Right side is a function of S
- Restriction for S , $g(x_\mu)$, $F(\tau)$

Solving the continuity equation

The particle density perturbation

Using (28) form of perturbation

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau} \right)^3 h(x_\mu) \nu(S) \quad (31)$$

Continuity equation

$$\frac{\nu(S)}{\chi(S)\mathcal{N}'(S)} = -\frac{F(\tau)g(x_\mu)\partial_\mu S\partial^\mu S}{u^\mu\partial_\mu h(x_\mu)} \quad (32)$$

Right side is a function of S

- Restriction for S , $h(x_\mu)$, $F(\tau)$

Scaling variable $S = r^m/\tau^m$ Scaling variable $S = r^m/\tau^m$

$$\chi(S) = \frac{S^{-\frac{m+1}{m}}}{\sqrt{S^{\frac{2}{m}} + 1}} \quad (33)$$

$$\pi(S) = \frac{(\kappa + 1)(\kappa - 3)}{\kappa} \left[\pi_0 - m\sqrt{1 + S^{-\frac{2}{m}}} \right], \quad (34)$$

$$\nu(S) = m^2 S^2 \left[S^{-\frac{2}{m}} + 1 \right] \frac{S^{-\frac{m+1}{m}}}{\sqrt{S^{\frac{2}{m}} + 1}} \mathcal{N}'(S) \quad (35)$$

Scaling variable $S = \tau^m/t^m$ Scaling variable $S = \tau^m/t^m$

$$\chi(S) = \frac{S^{\frac{2}{m}-1}}{\left(1 - S^{\frac{2}{m}}\right)^{\frac{3}{2}}} \quad (36)$$

$$\pi(S) = \frac{(\kappa+1)(\kappa-3)}{\kappa} \left(\pi_0 + \frac{m}{\sqrt{1 - S^{\frac{2}{m}}}} \right) \quad (37)$$

$$\nu(S) = m^2 S^2 \frac{S^{\frac{2}{m}-1}}{1 - S^{\frac{2}{m}}} \mathcal{N}'(S) \quad (38)$$

Equations of non-relativistic hydrodynamics

Looking for (u, p, ρ) fields

Assumptions:

- zero viscosity
- zero heat conductivity

Euler-equation

$$\frac{\partial u}{\partial t} + (u \nabla) u = -\frac{1}{\rho} \nabla p$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla(\rho u) = 0$$

Equation of state

$p - \rho$ relation

Perturbative equations

Perturbed fields

- $u \rightarrow u + \delta u$
- $p \rightarrow p + \delta p$
- $\rho \rightarrow \rho + \delta \rho$

Perturbed equations

- first order perturbation
- using another solution

Wave solution

Known solution: Standing fluid

- $u = 0$
- $p = \text{const.}$
- $\rho = \text{const.}$

Sound speed from equation of state:

$$\frac{\delta p}{\delta \rho} = c^2$$

Perturbed Euler-equation

$$\frac{\partial \delta u}{\partial t} = -\frac{1}{\rho} \nabla \delta p$$

Perturbed continuity equation

$$\frac{\partial \delta \rho}{\partial t} + \rho \nabla \delta u = 0$$

Wave solution for pressure

$$\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$$

Source function

$$\begin{aligned} S(x, p) = & N \delta(\tau - \tau_0) d\tau d^3x n_0 \left(\frac{\tau_0}{\tau} \right)^3 \mathcal{N}(S) \\ & \exp \left[-\frac{Et - xp_x - yp_y - zp_z}{\tau T_0 \left(\frac{\tau_0}{\tau} \right)^{\frac{3}{\kappa}}} \mathcal{N}(S) \right] \left(E - \frac{xp_x + yp_y + zp_z}{t} \right) . \\ & \cdot \left[1 + \delta \left(-\frac{(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}) \partial^0 S \chi(S) \tau}{t} + \right. \right. \\ & + \frac{(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}) \chi(S) t}{Et - xp_x - yp_y - zp_z} p_\mu \partial^\mu S + \\ & + \frac{(Et - xp_x - yp_y - zp_z)(\mathcal{N}(S)\pi(S) - h(x, y, z, t)\nu(S))}{\tau T_0 \left(\frac{\tau_0}{\tau} \right)^{\frac{3}{\kappa}}} + \\ & \left. \left. + \frac{h(x, y, z, t)\nu(S)}{\mathcal{N}(S)} \right) \right] \end{aligned}$$

Single-particle distribution

$$N(p) = Nn_0 \mathcal{E}_1 \mathcal{V}_1 (1 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) + Nn_0 \mathcal{E}_2 \mathcal{V}_2 (\mathcal{P}_4 + \mathcal{P}_5) \quad (39)$$

The newly introduced functions:

$$\mathcal{E}_1 = \exp \left[-\frac{E^2 + m^2}{2ET_0} - \frac{p^2}{2ET_{\text{eff}}} \right], \quad \mathcal{V}_1 = \sqrt{\frac{2\pi T_0 \tau_0^2}{E}} \left(1 - \frac{T_0}{T_{\text{eff}}} \right)^3 \left(E - \frac{p^2}{E} \left(1 - \frac{T_0}{T_{\text{eff}}} \right) \right), \quad (40)$$

$$\mathcal{E}_2 = \exp \left[-\frac{E^2 + m^2}{2ET_0} - \frac{p^2}{2ET_{\text{eff},\delta}} \right], \quad \mathcal{V}_2 = \sqrt{\frac{2\pi T_0 \tau_0^2}{E}} \left(1 - \frac{T_0}{T_{\text{eff},\delta}} \right)^3 \left(E - \frac{p^2}{E} \left(1 - \frac{T_0}{T_{\text{eff},\delta}} \right) \right). \quad (41)$$

The perturbative terms are:

$$\mathcal{P}_1 = -\frac{\delta(1+c)\tau_0^2}{r_1 \sqrt{\tau_0^2 + r_1^2}}, \quad \mathcal{P}_2 = \frac{\delta(1+c)\tau_0}{E - \frac{p^2 \rho_1^2}{\sqrt{\tau_0^2 + r_1^2}}} \left(\frac{E}{r_1} - (p^2 \rho_1^2) \frac{\sqrt{\tau_0^2 + r_1^2}}{r_1^3} \right), \quad (42)$$

$$\mathcal{P}_3 = \frac{\delta 2bc\kappa}{(3-\kappa)R_0^2} \left(\frac{r_1}{\sqrt{\tau_0^2 + r_1^2}} \right)^3 \left(\frac{\tau_0}{r_1} \right)^4, \quad \mathcal{P}_5 = -\frac{\delta(\tau_0 + c\tau_0)}{T_0} \left(\frac{E}{r_2} - (p^2 \rho_2^2) \frac{\sqrt{\tau_0^2 + r_2^2}}{r_2^3} \right), \quad (43)$$

$$\mathcal{P}_4 = \frac{\delta 2bE \sqrt{\tau_0^2 + r_2^2} - p^2 \rho_2^2}{\dot{R}_0^2 \tau_0 T_0} \left(\frac{(\kappa+1)(\kappa-3)}{\kappa} \frac{\tau_0^2 + r_2^2}{r_2} - \frac{c\kappa}{3-\kappa} \tau_0 \right) \left(\frac{r_2}{\sqrt{\tau_0^2 + r_2^2}} \right)^3 \left(\frac{\tau_0}{r_2} \right)^4. \quad (44)$$

The way of finding a concrete solution

