(Quantum) Field Theory and the Electroweak Standard Model

Lecture I

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Outline

- **Lecture I**
  - What is the Standard Model?
  - Introducing Quantum Fields
  - Global Symmetries

- **Lecture II**
  - Introducing Interactions
  - Perturbation Theory
  - Renormalizable or Non-Renormalizable?

- **Lecture III**
  - Gauge Symmetries
  - Constructing the EW SM
  - Experimental tests of the EW SM
  - Issues and Prospects of the EW SM
What is the Standard Model?

The Absolutely Amazing Theory of Almost Everything

- A Quantum Field Theory
- Based on (gauge) Symmetry principles
- Describes interactions between all known elementary particles
- Potentially can account for physics up to very high energies
- Experimentally established with rather high precision

But still it has several shortcomings...(see lectures by Ben Allanach)

What is the Standard Model?

A Quantum Field Theory

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Standard Model of Elementary Particles

three generations of matter
(fermions)

<table>
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<tr>
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<th>II</th>
<th>III</th>
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<td>c (charm)</td>
<td>t (top)</td>
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<td>W (W boson)</td>
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Courtesy to Wikipedia: "Standard Model of Elementary Particles", May 2018
Particle (Field) content of the SM

**Fermions ("Matter")**
- **Quarks (spin 1/2)**
  - 3 colors
    (lect. by F. Tramontano)
  - 6 flavours
    (lect. by J. Zupan)
- **Leptons (spin 1/2)**
  - 3 charged leptons
  - 3 neutrinos
    (lect. by S. Pascoli)

**Bosons ("Force Mediators")**
- **Vector (spin 1) bosons**
  - 8 gluons
    (lect. by F. Tramontano)
  - 4 electroweak bosons
    ($Z$, $W^{\pm}$, $\gamma$)
- **Scalar (spin 0) boson**
  (lect. by F. Maltoni)

**NB:** Gluons and photons ($\gamma$) are assumed to be **massless**. All other particles have mass (neutrino?).
The SM interactions (on a T-shirt)

All particle interactions can be read of the SM Lagrangian:

\[\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \psi \partial_\mu \phi + h.c. + \lambda_i y_{ij} \phi^i \phi^j + h.c. + |D_\mu \phi|^2 - V(\phi)\]

QFT allows one not only to understand why the short expression is unique in certain sense, but also to derive the long one!

- Symmetries (Lorentz, Gauge)
- Renormalizability

Moreover, given the Lagrangian one can obtain predictions for observables.
Units, Dimensions, etc.

We use natural units $\hbar = c = 1$. The reference unit is energy (mass):

$$[t] = [x] = -1, \quad [p] = [E] = [M] = 1$$

$$\hbar \simeq 6.6 \cdot 10^{-22} \text{ MeV} \cdot \text{s}, \quad \hbar c \simeq 2 \cdot 10^{-14} \text{ GeV} \cdot \text{cm}$$

Some useful formulas (check the dimension of both sides:)

- Commutation relation and uncertainty principle:
  $$[\hat{x}, \hat{p}] = i, \quad \Delta x \Delta p \geq 1$$

- Fourier transformation $f(x) \leftrightarrow f(p)$:
  $$f(x) = \frac{1}{2\pi} \int dp \ f(p) e^{-ipx}, \quad \partial_x f(x) = \frac{1}{2\pi} \int dp \ [−ip] f(p) e^{-ipx}$$

- Delta-function (distribution):
  $$\delta(x) = \frac{1}{2\pi} \int dp \ e^{-ipx}, \quad \int dx \delta(x) = 1$$
Lorentz symmetry and Index Summation Notation

- We consider Minkowski space in $d = 4$ dimensions. Greek letters are used to denote components of Lorentz 4-vectors

$$x_\mu = \{x_0, x\}, \text{ with time } t \equiv x_0,$$
$$p_\mu = \{p_0, p\}, \text{ with energy } E \equiv p_0,$$

while for 3-vectors we use **bold-face**: $x = \{x_1, x_2, x_3\}$, etc.

A scalar product of two 4-vectors in pseudo-euclidean space

$$px \equiv p_\mu x_\mu = g_{\mu\nu} p_\mu x_\nu = p_0 x_0 - p \cdot x, \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

is invariant under Lorentz transformations (rotations and boosts):

$$x_\mu \rightarrow x'_\mu = \Lambda_{\mu\nu} x_\nu, \quad x_\mu x_\mu = x'_\mu x'_\mu \Rightarrow \Lambda_{\mu\alpha} \Lambda_{\mu\beta} = g_{\alpha\beta}$$

- The 4-momentum $p$ of a free particle with mass $m$ satisfies

$$p^2 = E^2 - p^2 = m^2 = \text{invariant}$$
Why do we need QFT?

Relativistic Quantum Mechanics (QM) describing fixed number of particles turns out to be inconsistent.

From the energy-momentum relation for a free relativistic particle

\[ E^2 = p^2 + m^2 \]  \hspace{1cm} (instead of \( E = \frac{p^2}{2m} \)),

and the correspondence principle

\[ E \rightarrow i \frac{\partial}{\partial t}, \quad p \rightarrow -i \nabla \]

we have the Klein-Gordon (KG) equation

\[ \left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(t, \mathbf{x}) = 0 \]  \hspace{1cm} (instead of \( i \partial_t \psi = -\frac{\nabla^2}{2m} \psi \))

for a wave-function \( \phi(t, \mathbf{x}) \equiv \langle \mathbf{x} | \phi(t) \rangle \). For any \( p \)

\[ \phi_p(t, \mathbf{x}) = e^{-iEt+px}, \quad \text{with } E = \pm \sqrt{p^2 + m^2}. \]

The spectrum is not bounded from below!
Wave-packets in Relativistic QM

General solution of the KG equation (a wave-packet)

\[ \phi(t, x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[ a(\mathbf{p}) e^{-i\omega_p t + ipx} + b(\mathbf{p}) e^{+i\omega_p t - ipx} \right] \]

with \( \omega_p \equiv +\sqrt{p^2 + m^2} \). Both \( E = \omega_p \) and \( E = -\omega_p \) contribute.

An attempt to introduce a positive-definite probability density \( \rho \) fails

\[ \partial_\mu j_\mu = 0, \quad j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \]

\[ \rho \equiv j_0 = i (\phi^* \partial_t \phi - \phi \partial_t \phi^*) \Rightarrow 2E \text{ for } \phi \propto e^{-iEt} \]

Ex: Show that for the general solution we have

\[ \int d\mathbf{x} \cdot \rho = \int d\mathbf{p} \left\{ |a(\mathbf{p})|^2 - |b(\mathbf{p})|^2 \right\}, \]

which is not positively-defined (but time-independent).

NB: Positive-energy condition \( [b(\mathbf{p}) \equiv 0] \) is not stable under interactions!
From (R)QM to QFT

To get a relativistic quantum theory that treats space and time coordinates on the same footing, one re-interprets $\phi$, satisfying

$$(\partial^2 + m^2)\phi(x) = 0$$

as a

Quantum Field Operator

$\phi(x, t) \rightarrow \hat{\phi}(x, t)$

(Heisenberg picture)

Dynamical variable

$x_i(t) \rightarrow \phi(x, t)$

with infinite number of degrees of freedom (DOFs)

- Particles in QFT are treated as field excitations.
- Single field accounts for infinite number of particles.

NB: In the Heisenberg picture operators $\mathcal{O}_H(t)$ depend on time, while in the Schrödinger one the states evolve: $\langle \psi(t)|\mathcal{O}_S|\psi(t)\rangle = \langle \psi|\mathcal{O}_H(t)|\psi\rangle$. 
Free Scalar Field

The solution of the KG equation* \((p_0 = \omega_p)\)

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dp}{\sqrt{2\omega_p}} \left[ a_p^- e^{-ipx} + b_p^+ e^{+ipx} \right],
\]

is a linear combination of operators \(a_p^\pm\) and \(b_p^\pm\)

\[
\left[ a_p^-, a_p^+ \right] = \delta^3(p - p'), \quad \left[ b_p^-, b_p^+ \right] = \delta^3(p - p').
\]

All other commutators are zero, e.g., \(\left[ a_p^\pm, a_{p'}^\pm \right] = 0\).

**NB1:** The operators also satisfy \(a_p^\pm = (a_p^{\mp})^\dagger\) and \(b_p^\pm = (b_{p'}^{\mp})^\dagger\).

**NB2:** For \(a_p^\pm \equiv b_p^\pm\) the field is hermitian \(\phi^\dagger(x) = \phi(x)\).

*For brevity \(\hat{\phi} \rightarrow \phi\).
Free Scalar Field: Fock Space

The operator

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[ a_p^- e^{-ipx} + b_p^+ e^{+ipx} \right],$$

needs some space to act on. In QFT we consider Fock space. It consists of a vacuum $|0\rangle$, which is annihilated by $a_p^-$ (and $b_p^-$) for every $p$

$$\langle 0|0 \rangle = 1, \quad a_p^- |0\rangle = 0, \quad \langle 0|a_p^+ = (a_p^- |0\rangle)^\dagger = 0,$$

and states corresponding to field excitations:

$$|f_1\rangle = \int d\mathbf{k} \cdot f_1(\mathbf{k})a_{\mathbf{k}}^+ |0\rangle, \quad \text{1-particle state}$$
$$|f_2\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 \cdot f_2(\mathbf{k}_1, \mathbf{k}_2)a_{\mathbf{k}_1}^+ a_{\mathbf{k}_2}^+ |0\rangle, \quad \text{2-particle state}$$

NB1: Two sets of operators $a^\pm$ (particles) and $b^\pm$ (anti-particles).
NB2: Since $a_{\mathbf{p}}^+ a_{\mathbf{k}}^+ = a_{\mathbf{k}}^+ a_{\mathbf{p}}^+$, particles are not distinguishable (bosons).
Free Scalar Field: Fock Space

The operator

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{dp}{\sqrt{2\omega_p}} \left[ a_p^- e^{-ipx} + b_p^+ e^{+ipx} \right],$$

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and states corresponding to field excitations:

$$|p\rangle = a_p^+ |0\rangle, \quad f_1(k) = \delta(p - k)$$

$$|p_1, p_2\rangle = a_{p_1}^+ a_{p_2}^+ |0\rangle, \quad f_2(k_1, k_2) = ?$$

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Free Field and Harmonic Oscillators

The commutation relations

\[
[a_p^-, a_{p'}^+] = \delta^3(p - p')
\]

should remind you about quantum harmonic oscillators with Hamiltonian

\[
\hat{\mathcal{H}}_{osc} = \sum_j \frac{1}{2}(\hat{p}_j^2 + \omega_j^2 \hat{x}_j^2), \quad [\hat{x}_j, \hat{p}_k] = i\delta_{jk}, \quad [x_j, x_k] = [p_j, p_k] = 0
\]

expressed in terms of ladder operators

\[
\sqrt{2\omega}a_j^\pm = (\omega \hat{x}_j \mp i\hat{p}_j)
\]

expressed in terms of ladder operators

\[
\hat{\mathcal{H}}_{osc} = \sum_j \frac{\omega_j}{2} \left( a_j^+ a_j^- + a_j^- a_j^+ \right) \quad \text{after re-ordering}
\]

Here, \(\hat{n}_j\) counts energy quanta for oscillator \(j\): \(\hat{n}_j |n_j\rangle = n_j |n_j\rangle\).
Free Field: Hamiltonian

Indeed, if we put our field in a box of size $L$, $\mathbf{p}$ and $\omega_p$ will be quantized

$$
\mathbf{p} \rightarrow \mathbf{p}_j = \left( \frac{2\pi}{L} \right) \mathbf{j}, \quad \mathbf{j} = (j_1, j_2, j_3), \quad j_i \in \mathbb{Z},
$$

$$
\omega_p \rightarrow \omega_j = \sqrt{\left( \frac{2\pi}{L} \right)^2 j^2 + m^2},
$$

The QFT Hamiltonian is obtained by taking the limit $L \rightarrow \infty$ in $\hat{\mathcal{H}}_{osc}$:

$$
\hat{\mathcal{H}}_{part} = \lim_{L \rightarrow \infty} \left\{ \int d\mathbf{p} \omega_j \left[ \left( \frac{L}{2\pi} \right)^3 \sum_j \mathbf{a}_j^+ \mathbf{a}_j^- + \frac{1}{2} \left( \frac{L}{2\pi} \right)^3 \delta(0) \right] \right\}
$$

We had two kind of operators, so

$$
\hat{\mathcal{H}} = \hat{\mathcal{H}}_{part} + \hat{\mathcal{H}}_{antipart} = \int d\mathbf{p} \omega_p \left\{ \left[ \mathbf{n}_p + \frac{1}{2} \delta(0) \right] + \left[ \mathbf{\bar{n}}_p + \frac{1}{2} \delta(0) \right] \right\},
$$

with $(\mathbf{\bar{n}}_p \equiv b^+_p b^-_p)$ $\mathbf{n}_p \equiv \mathbf{a}_p^+ \mathbf{a}_p^-$ counting (anti-)particles with momentum $\mathbf{p}$. 
Free Field and Vacuum Energy

There is a disturbing problem in

\[ \hat{\mathcal{H}} = \int dp \, \omega_p \left[ n_p + \bar{n}_p \right] + \int dp \, \omega_p \delta(0), \quad n_p \equiv a^+_p a^-, \quad \bar{n}_p \equiv b^+_p b^- . \]

The additive “constant”, associated with vacuum (no particles):

\[ E_0 = \langle 0 | \hat{\mathcal{H}} | 0 \rangle = \int dp \, \omega_p \delta(0) \]

is infinite. There are two kind of infinities:

- **InfraRed** (large distances, \( L \to \infty \)) due to \( L^3 \to (2\pi)^3 \delta(0) \).
- **UltraViolet** (small distances, \( p/\omega_p \to \infty \)).

To “solve” the problem, let’s measure all energies w.r.t the vacuum:

\[ \hat{\mathcal{H}} \to :\hat{\mathcal{H}}:= \hat{\mathcal{H}} - \langle 0 | \hat{\mathcal{H}} | 0 \rangle \]
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\[ \hat{\mathcal{H}} \to :\hat{\mathcal{H}}:= \hat{\mathcal{H}} - \langle 0 | \hat{\mathcal{H}} | 0 \rangle \]

Equivalently, we can say that by definition the operators in \( :\hat{\mathcal{H}}: \) are normal-ordered, e.g.,

\[ :\hat{\mathcal{H}}_{osc}: = \frac{\omega j}{2} \left( :a_j^+ a_j^- + a_j^- a_j^+ : \right) = \omega_j :a_j^+ a_j^- := \omega_j a_j^+ a_j^- \]
Free Field: Momentum and Charge

Now we have
\[ \hat{\mathcal{H}} = \int dp \omega_p (n_p + \bar{n}_p), \quad \bar{n}_p \equiv b_p^+ b_p^-, \quad n_p \equiv a_p^+ a_p^-, \quad \hat{\mathcal{H}} |0\rangle = 0. \]

It is easy to check that (no negative energies)
\[ \hat{\mathcal{H}} |p\rangle = \omega_p |p\rangle, \quad |p\rangle = a_p^+ |0\rangle, \quad \hat{\mathcal{H}} |\bar{p}\rangle = \omega_p |\bar{p}\rangle, \quad |\bar{p}\rangle = b_p^+ |0\rangle. \]

We can also “cook up” the 3-momentum operator (Ex: )
\[ \hat{P} |0\rangle = 0 |0\rangle, \quad \hat{P} |p\rangle = p |p\rangle \quad \hat{P} |\bar{p}\rangle = p |\bar{p}\rangle \]
and the charge operator that distinguishes particles from antiparticles
\[ \hat{Q} |0\rangle = 0 |0\rangle, \quad \hat{Q} |p\rangle = + |p\rangle \quad \hat{Q} |\bar{p}\rangle = - |\bar{p}\rangle. \]

NB: The operators \( \hat{P} \) and \( \hat{Q} \) do not depend on time and \( [\hat{P}, \hat{Q}] = 0 \).

Ex: Show that multiparticle states \( |p_1...p_n\rangle \) are eigenvectors of \( \hat{\mathcal{H}}, \hat{P}, \hat{Q} \).
Free Scalar Field Propagator

The field $\phi^\dagger (\phi)$ increases (decreases) charge of a state

\[ \left[ \hat{Q}, \phi^\dagger (x) \right] = +\phi^\dagger (x), \quad \left[ \hat{Q}, \phi (x) \right] = -\phi (x) \]

Consider the following amplitudes

\[ \langle 0 | \phi(x_2) \phi^\dagger (x_1) | 0 \rangle \quad \text{for} \quad t_2 > t_1 \]
\[ \langle 0 | \phi^\dagger (x_1) \phi(x_2) | 0 \rangle \quad \text{for} \quad t_1 > t_2 \]

Particle (charge +1) propagates from $x_1$ to $x_2$

Antiparticle (charge $-1$) propagates from $x_2$ to $x_1$

Both possibilities can be taken into account in one function:

\[ \langle 0 | T[\phi(x_2)\phi^\dagger (x_1)]| 0 \rangle \equiv \theta(t_2 - t_1) \langle 0 | \phi(x_2) \phi^\dagger (x_1) | 0 \rangle 
\quad + \theta(t_1 - t_2) \langle 0 | \phi^\dagger (x_1) \phi(x_2) | 0 \rangle, \]

with $T$ being time-ordering operation.
Free Scalar Field Propagator

This is Feynman Propagator:

\[
\langle 0 | T [\phi(x_2)\phi^\dagger(x_1)] | 0 \rangle \equiv \theta(t_2 - t_1)\langle 0 | \phi(x_2)\phi^\dagger(x_1) | 0 \rangle - i D_c(x - y)
\]

\[
+ \theta(t_1 - t_2)\langle 0 | \phi^\dagger(x_1)\phi(x_2) | 0 \rangle,
\]

Fourier transform

\[
D_c(x - y) = \frac{-1}{(2\pi)^4} \int d^4 p \frac{e^{-ip(x - y)}}{p^2 - m^2 + i\epsilon}
\]

The \(i\epsilon\)-prescription (\(\epsilon \rightarrow 0\)) picks up certain poles in the \(p_0\) complex plane.

The propagator is a Green-function:

\[
\left( \partial^2_x + m^2 \right) D_c(x - y) = \delta(x - y)
\]

KG equation

NB: Feynman propagator is a Lorentz-invariant function (distribution)!

vanishes for \(t_2 < t_1\)
From Field to Particle to Force

The propagator of particles can be connected to the force between two static classical sources $J_i(x) = \delta(x - x_i)$ located at $x_i = (x_1, x_2)$. Presence of the sources disturbs the vacuum $|0\rangle \to |\Omega\rangle$, since Hamiltonian $\mathcal{H} \to \mathcal{H}_0 + J \cdot \phi$. Assuming for simplicity that $\phi = \phi^\dagger$ we can find

$$\langle \Omega | e^{-i\mathcal{H}T} | \Omega \rangle \equiv e^{-iE_0(J)T} \Rightarrow \text{in the limit } T \to \infty$$

$$= e^{\frac{i^2}{2!} \int dx dy J(x) \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle J(y) = e^{\frac{i}{2} \int dx dy J(x) D_c(x-y) J(y)}$$

Evaluating the integral for $J(x) = J_1(x) + J_2(x)$ we get the contribution $\delta E_0$ to $E_0(J)$ due to interactions between two sources

$$\lim_{T \to \infty} \delta E_0 T = - \int dx dy J_1(x) D_c(x-y) J_2(y)$$

$$\delta E_0 = - \int \frac{dp}{(2\pi)^3} \frac{e^{+ip(x_1-x_2)}}{p^2 + m^2} = - \frac{1}{4\pi r} e^{-mr}, \quad r = |x_1 - x_2|$$

This is nothing else, but Yukawa potential due to scalar massive field. It is attractive and fall off exponentially over the distance scale $1/m$. 

A. Bednyakov (JINR)
Free Scalar Field: Lagrangian

A convenient way to deal with (quantum) fields is to consider the following Action functional ("function → number"): \[
S[\phi(x)] = \int d^4x \ \mathcal{L}(\phi(x), \partial_{\mu} \phi) = \int d^4x \left( \partial_{\mu} \phi^\dagger \partial_{\mu} \phi - m^2 \phi^\dagger \phi \right).
\]

To have an analogy with Classical Mechanics one can rewrite the Action as

\[
S[\phi(x)] = \int dt L(t), \quad L = T - U, \quad H = T + U
\]

\[
T = \int dx |\partial_t \phi|^2, \quad U = \int dx (|\partial_x \phi|^2 + m^2 |\phi|^2)
\]

A system of coupled oscillators with kinetic energy \(T\) and potential \(U\).
Free Scalar Field: Lagrangian

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$$ S[\phi(x)] = \int d^4 x \ L(\phi(x), \partial_\mu \phi) = \int d^4 x \left( \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \right). $$

We can derive the equations of motions (EOM) via the Action Principle:

$$ S[\phi'(x)] - S[\phi(x)] = \int d^4 x \left[ \left( \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} - \frac{\partial L}{\partial \phi} \right) \delta \phi + \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi \right) \right]. $$

$$ \phi'(x) = \phi(x) + \delta \phi(x), \quad \delta \phi(x) \text{ is infinitesimal ("tiny") } $
Free Scalar Field: Lagrangian

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\[ S[\phi(x)] = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi) = \int d^4x \left( \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \right). \]

We can derive the equations of motions (EOM) via the Action Principle:

\[ \delta S[\phi(x)] = 0 \]

\[ \int d^4x \left[ \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta \phi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right) \right] = 0. \]

We look for specific \( \phi(x) \) that gives \( \delta S[\phi(x)] = 0 \) for any variation \( \delta \phi(x) \).

**NB1:** Fields satisfying EOMs are said to be "on-mass-shell".

**NB2:** \( \langle 0 | T[\phi(x)\phi^\dagger(y)]|0 \rangle \) can be found by inverting the quadratic form \( K \).

\[ ^a \text{Satisfying boundary conditions.} \]
With Action you can study Symmetries...

The latter are intimately connected with transformations, which leaves something invariant...

Symmetries are not only beautiful but also very useful:

An architect can design only half of the building (parity $x \rightarrow -x$)

And winter decoration will take much less time (rotation by a finite angle)
Field Theory: Symmetries

- Transformations can be **discrete**, e.g.,

  Parity: \( \phi'(x, t) = P\phi(x, t) = \phi(-x, t) \),

  Time-reversal: \( \phi'(x, t) = T\phi(x, t) = \phi(x, -t) \),

  Charge-conjugation: \( \phi'(x, t) = C\phi(x, t) = \phi^\dagger(x, t) \),

- or depend on **continuous** parameters, e.g.,

\[
\phi'(x + a) = \phi(x)
\]
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- We can distinguish space-time and internal symmetries.
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  \]
  or depend on continuous parameters, e.g.,

\[
\begin{align*}
\Re\phi(x) &\quad \phi'(x) \\
\Im\phi(x) &\quad \phi'(x)
\end{align*}
\]

- We can distinguish space-time and internal symmetries.
- For \(x\)-dependent parameters we have local (gauge) transformations.
Quantum Field Theory: Symmetries

In Classical Physics symmetry transformations allows one to find

- new solutions to EOMs from the given one, keeping some features of the solutions (invariants) intact.
- how a solution in one coordinate system (as seen by one observer) looks in another coordinates (as seen by another observer).

In Quantum World a symmetry $\mathcal{S}$ guarantees that transition probabilities $\mathcal{P}$ between states do not change upon transformation:

$$|A_i\rangle \xrightarrow{\mathcal{S}} |A'_i\rangle, \quad \mathcal{P}(A_i \rightarrow A_j) = \mathcal{P}(A'_i \rightarrow A'_j), \quad |\langle A_i|A_j\rangle|^2 = |\langle A'_i|A'_j\rangle|^2$$

Symmetries are represented by unitary\(^\dagger\) operators $U$:

$$|A'_i\rangle = U|A_j\rangle, \quad \langle A'_i|A'_j\rangle = \langle A_i|U^\dagger U \rangle |A_j\rangle$$

\(^\dagger\)or anti-unitary (time-reversal).
Quantum Field Theory: Symmetries

A transformation of states can be reformulated as a change of operators:

\[ \langle A_i | \mathcal{O}_k(x) | A_j \rangle \xrightarrow{S} \langle A'_i | \mathcal{O}'_k(x) | A'_j \rangle = \langle A_i | U^\dagger \mathcal{O}_k(x) U | A_j \rangle \]

\[ \langle A_i | \mathcal{O}_k(x) | A_j \rangle \xrightarrow{S} \langle A_i | \mathcal{O}'_k(x) | A_j \rangle, \quad \mathcal{O}'_k(x) \equiv U^\dagger \mathcal{O}_k(x) U \]

Symmetry relates these quantities

For example, translational invariance leads to

\[ \langle A_i | \phi(x) | A_j \rangle = \langle A_i | \phi'(x + a) | A_j \rangle = \langle A_i | U^\dagger(a) \phi(x + a) U(a) | A_j \rangle \]
Quantum Field Theory: Symmetries

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For example, translational invariance leads to

\[ \phi(x) = \phi'(x + a) = U^\dagger(a) \phi(x + a) U(a) \]

so quantum field should satisfy

\[ \phi(x + a) = U(a) \phi(x) U^\dagger(a) \]

We can have non-trivial (realizations of) symmetries mixing different fields:

\[ \phi'_i(x') = S_{ij}(a) \phi_j(x) \Rightarrow \phi_i(x') = S_{ij}(a) U(a) \phi_j(x) U^\dagger(a) , \quad x' = x'(x, a) \]

Examples will be provided later...For the moment, let us find a connection between Symmetries of Action, Conserved Quantities and Unitary Operators that realize the symmetries at the quantum level.
Global Continuous Symmetries: Noether Theorem

Given $S[\phi]$ one can find its symmetries, i.e., particular infinitesimal variations $\delta \phi(x)$ that for any $\phi$ leave $S[\phi]$ invariant up to a surface term

$$S[\phi'(x)] - S[\phi(x)] = \int d^4x \partial_\mu K_\mu, \quad \phi'(x) \equiv \phi(x) + \delta \phi(x).$$

We compare this with

$$S[\phi'(x)] - S[\phi(x)] = \int d^4x \left[ \left( \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} - \frac{\partial L}{\partial \phi} \right) \delta \phi + \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi \right) \right].$$
Global Continuous Symmetries: Noether Theorem

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and require $\phi(x)$ to satisfy EOMs. This results in a local conservation law:

$$\partial_\mu J_\mu = 0, \quad J_\mu \equiv K_\mu - \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi.$$

Integration over space leads to the conserved charge

$$\frac{d}{dt} Q = 0, \quad Q = \int dx J_0.$$

NB: If $\delta \phi = \rho_i \delta_i \phi$ depends on parameters $\rho_i$, we have a conservation law for every $\rho_i$. For Global symmetries $\rho_i$ do not depend on coordinates.
The Noether Theorem: Space-time symmetries

Consider space-time translations
\[
\phi'(x + a) = \phi(x)
\]
expand in \( a \) \( \Rightarrow \delta \phi(x) = -a_\nu \partial_\nu \phi(x), \)
\[
\delta \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \partial_\nu (-a_\nu \mathcal{L})
\]

A conserved Energy-Momentum Tensor \( T_{\mu\nu} \):
\[
J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}, \quad \partial_\mu T_{\mu\nu} = 0
\]
leads to time-independent “charges”
\[
P_\nu = \int d\mathbf{x} T_{0\nu}
\]

Ex1: Consider \( \mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2 \) and find the expression for \( P_\mu \).
Ex2: Substitute \( \phi(x) \) by its expansion in terms of operators \( a_\pm \) and \( b_\pm \)
and prove that modulo operator ordering ambiguities \( P_\mu \rightarrow (\hat{\mathcal{H}}, \hat{\mathbf{P}}) \).
The Noether Theorem: Space-time symmetries

Consider space-time translations
\[ \phi'(x + a) = \phi(x) \]
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A conserved Energy-Momentum Tensor \( T_{\mu\nu} \):
\[ J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}, \]
leads to time-independent "charges"
\[ P_\nu = \int d\mathbf{x} T_{0\nu} \]

Ex1: Consider \( \mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2 \) and find the expression for \( P_\mu \).
Ex2: Substitute \( \phi(x) \) by its expansion in terms of operators \( a_\pm p \) and \( b_\pm p \)
and prove that normal-ordered expression \( P_\mu \rightarrow (\hat{\mathcal{H}}, \hat{\mathbf{P}}) \).
The Noether Theorem: Internal symmetries

There is an additional symmetry of

\[ \mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \]

\[ \phi'(x) = e^{i\alpha} \phi(x) \]

\[ \delta \phi(x) = i\alpha \phi(x) \]

\[ J_\mu = i(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) \]

It is a \textit{U(1)} symmetry:

- It acts in \textit{internal} space ("rotates" complex number \( \phi(x) \) at every \( x \))
- It is a \textit{global} symmetry (rotation angle \( \alpha \) does not depend on \( x \)).

\textbf{Ex:} Check again that we will obtain the expression for the operator \( \hat{Q} \).
Lagrange Approach to Quantum Fields: Mini Summary

The approach based on Lagrangians allows one to (given $\mathcal{L}$)

- Derive EOMs (via Action Principle).
- Find Symmetries of the Action.
- Find Conserved quantities (via the Noether Theorem)

After quantisation the operators of conserved quantities can be used to define a convenient basis of states, e.g.,:

$$|p\rangle \equiv |p, +1\rangle, \quad |\bar{p}\rangle \equiv |p, -1\rangle \Rightarrow \hat{Q}|p, q\rangle = q|p, q\rangle, \quad \hat{P}|p, q\rangle = p|p, q\rangle$$

- act as generators of symmetries, e.g. for space-time translations:

$$U(a) = \exp \left( i\hat{P}_\mu a_\mu \right), \quad \hat{\phi}(x + a) = U(a)\hat{\phi}(x)U^\dagger(a)$$

NB: For $a_\mu = (t, 0)$ we obtain the connection between Schrödinger and Heisenberg pictures:

$$O_H(t) = e^{i\hat{H}t} O_S e^{-i\hat{H}t}.$$
In QFT we usually **start** building our models by **postulating** symmetries (and other good properties) of the Action/Lagrangian!

We assume that general $\mathcal{L}$ is

- Lorentz (Poincare) invariant* (a sum of Lorentz scalars),
- Local (involve finite number of partial derivatives),
- Real (hermitian) (respects unitarity=conservation of probability)

In addition, we can impose other symmetries and get further restrictions on the model...

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* Lorentz invariance is crucial for proving the $CPT$-theorem.