(Quantum) Field Theory and the Electroweak Standard Model

Lecture I

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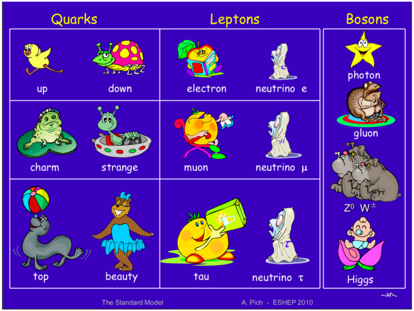
Outline

- Lecture I
  - What is the Standard Model?
  - Introducing Quantum Fields
  - Global Symmetries
- Lecture II
  - Introducing Interactions
  - Perturbation Theory
  - Renormalizable or Non-Renormalizable?
- Lecture III
  - Gauge Symmetries
  - Constructing the EW SM
  - Experimental tests of the EW SM
  - Issues and Prospects of the EW SM

What is the Standard Model?

The Absolutely Amazing Theory of Almost Everything

- A Quantum Field Theory
- Based on (gauge) Symmetry principles
- Describes interactions between all known elementary particles
- Potentially can account for physics up to very high energies
- Experimentally established with rather high precision



But still it has several shortcomings...(see lectures by Ben Allanach)

Robert Oerter, The Theory of Almost Everything: The Standard Model, the Unsung Triumph of Modern Physics, 2006

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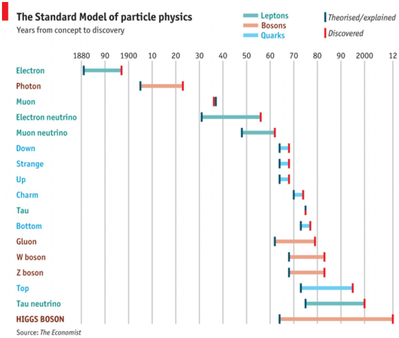

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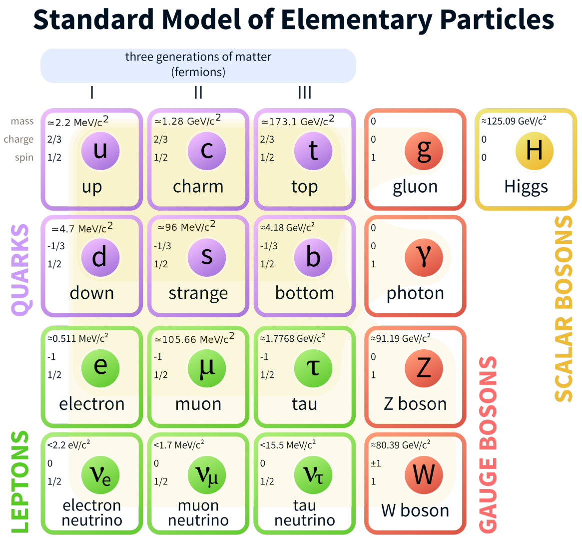


Courtesy to Wikipedia: "Standard Model of Elementary Particles", May 2018

Particle (Field) content of the SM

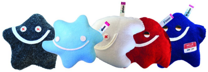
Fermions ("Matter")

- Quarks (spin 1/2)
  - 3 colors
    (lect. by F. Tramontano)
  - 6 flavours
    (lect. by J. Zupan)

- Leptons (spin 1/2)
  - 3 charged leptons
  - 3 neutrinos
    (lect. by S. Pascoli)

Bosons ("Force Mediators")

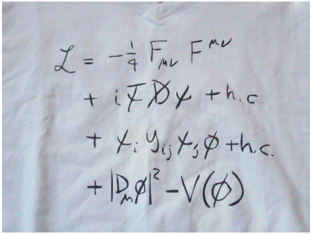
- Vector (spin 1) bosons
  - 8 gluons
    (lect. by F. Tramontano)
  - 4 electroweak bosons
    $(Z, W^{\pm}, \gamma)$
- Scalar (spin 0) boson
  (lect. by F. Maltoni)



NB: Gluons and photons (γ) are assumed to be massless. All other particles have mass (neutrino?).

The SM interactions (on a T-shirt)

All particle interactions can be read of the SM Lagrangian:



SU(3)x SU(2)x U(1)

QFT allows one not only to understand why the short expression is unique in certain sense, but also to derive the long one!

- Symmetries (Lorentz, Gauge)
- Renormalizability

Moreover, given the Lagrangian one can obtain predictions for observables.

Units, Dimensions, etc.

We use natural units $\hbar = c = 1$. The reference unit is energy (mass):

$$[t] = [\mathbf{x}] = -1, \qquad [\mathbf{p}] = [E] = [M] = 1$$

$$\hbar \simeq 6.6 \cdot 10^{-22}\ \text{MeV} \cdot s, \qquad \hbar c \simeq 2 \cdot 10^{-14}\ \text{GeV} \cdot \text{cm}$$

Some useful formulas (check the dimension of both sides:)

- Commutation relation and uncertainty principle:

$$[\hat{x}, \hat{p}] = i, \qquad \Delta x \Delta p \geq 1$$

- Fourier transformation $f(x) \leftrightarrow f(p)$:

$$f(x) = \frac{1}{2\pi} \int dp\, f(p) e^{-ipx}, \quad \partial_x f(x) = \frac{1}{2\pi} \int dp\, [-ip] f(p) e^{-ipx}$$

- Delta-function (distribution):

$$\delta(x) = \frac{1}{2\pi} \int dp\, e^{-ipx}, \qquad \int dx \delta(x) = 1$$

Lorentz symmetry and Index Summation Notation

- We consider Minkowski space in $d = 4$ dimensions. Greek letters are used to denote components of Lorentz 4-vectors

$$x_\mu = \{x_0, \mathbf{x}\}, \text{ with time } t \equiv x_0,$$
$$p_\mu = \{p_0, \mathbf{p}\}, \text{ with energy } E \equiv p_0,$$

while for 3-vectors we use **bold-face**: $\mathbf{x} = \{x_1, x_2, x_3\}$, etc.
A scalar product of two 4-vectors in pseudo-euclidean space

$$px \equiv p_\mu x_\mu = g_{\mu\nu} p_\mu x_\nu = p_0 x_0 - \mathbf{p}\cdot\mathbf{x}, \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

is invariant under Lorentz transformations (rotations and boosts):

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu, \qquad x_\mu x_\mu = x'_\mu x'_\mu \Rightarrow \Lambda_{\mu\alpha}\Lambda_{\mu\beta} = g_{\alpha\beta}$$

- The 4-momentum p of a free particle with mass m satisfies

$$p^2 = E^2 - \mathbf{p}^2 = m^2 = \text{invariant}$$

Why do we need QFT?

Relativistic Quantum Mechanics (QM) describing fixed number of particles turns out to be inconsistent.

From the energy-momentum relation for a free relativistic particle

$$E^2 = \mathbf{p}^2 + m^2 \quad (\text{instead of } E = \frac{\mathbf{p}^2}{2m}),$$

and the correspondence principle

$$E \to i\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\nabla$$

we have the Klein-Gordon (KG) equation

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi(t,\mathbf{x}) = 0 \quad (\text{instead of } i\partial_t\psi = -\frac{\nabla^2}{2m}\psi)$$

for a wave-function $\phi(t,\mathbf{x}) \equiv \langle \mathbf{x}|\phi(t)\rangle$. For any $\mathbf{p}$

$$\phi_p(t,\mathbf{x}) = e^{-iEt+\mathbf{px}}, \quad \text{with } E = \pm\sqrt{\mathbf{p}^2+m^2}.$$

The spectrum is not bounded from below!

Wave-packets in Relativistic QM

General solution of the KG equation (a wave-packet)

$$\phi(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a(\mathbf{p})\, e^{-i\omega_p t + i\mathbf{p}\mathbf{x}} + b(\mathbf{p})\, e^{+i\omega_p t - i\mathbf{p}\mathbf{x}} \right]$$

with $\omega_p \equiv +\sqrt{\mathbf{p}^2 + m^2}$. Both $E = \omega_p$ and $E = -\omega_p$ contribute.

An attempt to introduce a positive-definite probability density ρ fails

$$\partial_\mu j_\mu = 0, \quad j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$$
$$\rho \equiv j_0 = i\left(\phi^* \partial_t \phi - \phi \partial_t \phi^*\right) \Rightarrow 2E \text{ for } \phi \propto e^{-iEt}.$$

Ex: Show that for the general solution we have

$$\int d\mathbf{x} \cdot \rho = \int d\mathbf{p} \left\{ |a(\mathbf{p})|^2 - |b(\mathbf{p})|^2 \right\},$$

which is not positively-defined (but time-independent).

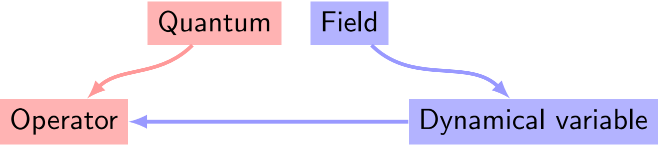
NB: Positive-energy condition $[b(\mathbf{p}) \equiv 0]$ is not stable under interactions!

From (R)QM to QFT

To get a relativistic quantum theory that treats space and time coordinates on the same footing, one re-interprets ϕ, satisfying

$$(\partial^2 + m^2)\phi(x) = 0$$

as a



$\phi(\mathbf{x}, t) \to \hat{\phi}(\mathbf{x}, t)$
(Heisenberg picture)

$x_i(t) \to \phi(\mathbf{x}, t)$
with infinite number
of degrees of freedom (DOFs)

- Particles in QFT are treated as field excitations.
- Single field accounts for infinite number of particles.

NB: In the Heisenberg picture operators $\mathcal{O}_H(t)$ depend on time, while in the Schrödinger one the states evolve: $\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle = \langle\psi|\mathcal{O}_H(t)|\psi\rangle$.

Free Scalar Field

The solution of the KG equation* ($p_0 = \omega_p$)

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a^-_{\mathbf{p}} e^{-ipx} + b^+_{\mathbf{p}} e^{+ipx} \right],$$

is a linear combination of operators $a^\pm_{\mathbf{p}}$ and $b^\pm_{\mathbf{p}}$

$$\left[a^-_{\mathbf{p}}, a^+_{\mathbf{p}'} \right] = \delta^3(\mathbf{p} - \mathbf{p}'), \quad \left[b^-_{\mathbf{p}}, b^+_{\mathbf{p}'} \right] = \delta^3(\mathbf{p} - \mathbf{p}').$$

All other commutators are zero, e.g., $\left[a^\pm_{\mathbf{p}}, a^\pm_{\mathbf{p}'} \right] = 0.$

NB1: The operators also satisfy $a^\pm_{\mathbf{p}} = (a^\mp_{\mathbf{p}})^\dagger$ and $b^\pm_{\mathbf{p}} = (b^\mp_{\mathbf{p}})^\dagger$.

NB2: For $a^\pm_{\mathbf{p}} \equiv b^\pm_{\mathbf{p}}$ the field is hermitian $\phi^\dagger(x) = \phi(x)$.

*For brevity $\hat{\phi} \to \phi$.

Free Scalar Field: Fock Space

The operator

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a^-_{\mathbf{p}} e^{-ipx} + b^+_{\mathbf{p}} e^{+ipx} \right],$$

needs some space to act on. In QFT we consider Fock space. It consists of a vacuum $|0\rangle$, which is annihilated by $a^-_{\mathbf{p}}$ (and $b^-_{\mathbf{p}}$) for every $\mathbf{p}$

$$\langle 0|0\rangle = 1, \quad a^-_{\mathbf{p}}|0\rangle = 0, \quad \langle 0|a^+_{\mathbf{p}} = (a^-_{\mathbf{p}}|0\rangle)^\dagger = 0,$$

and states corresponding to field excitations:

$$|f_1\rangle = \int d\mathbf{k} \cdot f_1(\mathbf{k}) a^+_{\mathbf{k}}|0\rangle, \qquad \text{1- particle state}$$

$$|f_2\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 \cdot f_2(\mathbf{k}_1, \mathbf{k}_2) a^+_{\mathbf{k}_1} a^+_{\mathbf{k}_2}|0\rangle, \qquad \text{2- particle state}$$

NB1: Two sets of operators $a^{\pm}$ (particles) and $b^{\pm}$ (anti-particles).

NB2: Since $a^+_{\mathbf{p}} a^+_{\mathbf{k}} = a^+_{\mathbf{k}} a^+_{\mathbf{p}}$, particles are not distinguishable (bosons).

Free Scalar Field: Fock Space

The operator

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a^-_{\mathbf{p}} e^{-ipx} + b^+_{\mathbf{p}} e^{+ipx} \right],$$

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and states corresponding to field excitations:

$$\begin{aligned} |\mathbf{p}\rangle &= a^+_{\mathbf{p}}|0\rangle, & f_1(\mathbf{k}) &= \delta(\mathbf{p} - \mathbf{k}) \\ |\mathbf{p}_1, \mathbf{p}_2\rangle &= a^+_{\mathbf{p_1}} a^+_{\mathbf{p_2}}|0\rangle, & f_2(\mathbf{k}_1, \mathbf{k}_2) &= ? \end{aligned}$$

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Free Field and Harmonic Oscillators

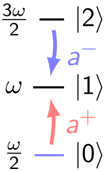
The commutation relations

$$\left[a^-_{\mathbf{p}}, a^+_{\mathbf{p}'}\right] = \delta^3(\mathbf{p} - \mathbf{p}')$$

should remind you about quantum harmonic oscillators with Hamiltonian

$$\hat{\mathcal{H}}_{osc} = \sum_j \frac{1}{2}(\hat{p}_j^2 + \omega_j^2 \hat{x}_j^2), \qquad [\hat{x}_j, \hat{p}_k] = i\delta_{ik},\ [x_j, x_k] = [p_j, p_k] = 0$$

expressed in terms of ladder operators $\sqrt{2\omega} a_j^\pm = (\omega \hat{x}_j \mp i \hat{p}_j)$



$$\hat{\mathcal{H}}_{osc} = \sum_j \frac{\omega_j}{2}\left(a_j^+ a_j^- + a_j^- a_j^+\right) \quad \text{after re-ordering}$$

$$= \sum_j \omega_j \left(\hat{n}_j + \tfrac{1}{2}\right),\ \hat{n}_j = a_j^+ a_j^-,\ \left[a_j^-, a_k^+\right] = \delta_{jk}.$$

Zero-point Energy

Here, $\hat{n}_j$ counts energy quanta for oscillator j: $\hat{n}_j |n_j\rangle = n_j |n_j\rangle$.

Free Field: Hamiltonian

Indeed, if we put our field in a box of size L, $\mathbf{p}$ and ω_p will be quantized

$$\mathbf{p} \to \mathbf{p_j} = (2\pi/L)\mathbf{j}, \quad \mathbf{j} = (j_1, j_2, j_3),\, j_i \in \mathbb{Z},$$

$$\omega_p \to \omega_{\mathbf{j}} = \sqrt{(2\pi/L)^2\mathbf{j}^2 + m^2}$$

The QFT Hamiltonian is obtained by taking the limit $L \to \infty$ in $\hat{\mathcal{H}}_{osc}$:

$$\hat{\mathcal{H}}_{part} = \underbrace{\lim_{L\to\infty}\left[\left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{j}}\right]}_{\int d\mathbf{p}} \omega_{\mathbf{j}} \left[\underbrace{\left(\frac{L}{2\pi}\right)^{\frac{3}{2}} a^+_{\mathbf{j}}}_{a^+_{\mathbf{p}}} \underbrace{\left(\frac{L}{2\pi}\right)^{\frac{3}{2}} a^-_{\mathbf{j}}}_{a^-_{\mathbf{p}}} + \frac{1}{2} \underbrace{\left(\frac{L}{2\pi}\right)^3}_{\delta(\mathbf{0})} \right]$$

We had two kind of operators, so

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{part} + \hat{\mathcal{H}}_{antipart} = \int d\mathbf{p}\, \omega_p \left\{ \left[n_{\mathbf{p}} + \frac{1}{2}\delta(\mathbf{0}) \right] + \left[\bar{n}_{\mathbf{p}} + \frac{1}{2}\delta(\mathbf{0}) \right] \right\},$$

with $(\bar{n}_{\mathbf{p}} \equiv b^+_{\mathbf{p}} b^-_{\mathbf{p}})$ $n_{\mathbf{p}} \equiv a^+_{\mathbf{p}} a^-_{\mathbf{p}}$ counting (anti-)particles with momentum $\mathbf{p}$.

Free Field and Vacuum Energy

There is a disturbing problem in

$$\hat{\mathcal{H}} = \int d\mathbf{p}\,\omega_p\,[n_\mathbf{p} + \bar{n}_\mathbf{p}] + \int d\mathbf{p}\,\omega_p \delta(\mathbf{0}), \quad n_\mathbf{p} \equiv a^+_\mathbf{p} a^-_\mathbf{p},\ \bar{n}_\mathbf{p} \equiv b^+_\mathbf{p} b^-_\mathbf{p}.$$

The additive "constant", associated with vacuum (no particles):

$$E_0 = \langle 0|\hat{\mathcal{H}}|0\rangle = \int d\mathbf{p}\,\omega_p \delta(\mathbf{0})$$

is infinite. There are two kind of infinities:

- InfraRed (large distances, $L \to \infty$) due to $L^3 \to (2\pi)^3 \delta(\mathbf{0})$.
- UltraViolet (small distances, $\mathbf{p}/\omega_p \to \infty$).

To "solve" the problem, let's measure all energies w.r.t the vacuum:

$$\hat{\mathcal{H}} \to\ :\hat{\mathcal{H}}:= \hat{\mathcal{H}} - \langle 0|\hat{\mathcal{H}}|0\rangle$$

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Equivalently, we can say that by definition the operators in $:\hat{\mathcal{H}}:$ are normal-ordered, e.g.,

$$:\hat{\mathcal{H}}_{osc}:= \frac{\omega_j}{2} \left(:a^+_j a^-_j + a^-_j a^+_j: \right) = \omega_j \ :a^+_j a^-_j := \omega_j a^+_j a^-_j$$

Free Field: Momentum and Charge

Now we have

$$\hat{\mathcal{H}} = \int d\mathbf{p}\, \omega_p \left(n_\mathbf{p} + \bar{n}_\mathbf{p}\right), \quad \bar{n}_\mathbf{p} \equiv b^+_\mathbf{p} b^-_\mathbf{p}, \quad n_\mathbf{p} \equiv a^+_\mathbf{p} a^-_\mathbf{p}, \qquad \hat{\mathcal{H}}|0\rangle = 0.$$

It is easy to check that (no negative energies)

$$\hat{\mathcal{H}}|\mathbf{p}\rangle = \omega_p|\mathbf{p}\rangle, \qquad |\mathbf{p}\rangle = a^+_\mathbf{p}|0\rangle, \qquad \hat{\mathcal{H}}|\bar{\mathbf{p}}\rangle = \omega_p|\bar{\mathbf{p}}\rangle, \qquad |\bar{\mathbf{p}}\rangle = b^+_\mathbf{p}|0\rangle.$$

We can also "cook up" the 3-momentum operator (Ex:)

$$\hat{\mathbf{P}}|0\rangle = 0|0\rangle, \qquad \hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \qquad \hat{\mathbf{P}}|\bar{\mathbf{p}}\rangle = \mathbf{p}|\bar{\mathbf{p}}\rangle$$

and the charge operator that distinguishes particles from antiparticles

$$\hat{Q}|0\rangle = 0|0\rangle, \quad \hat{Q}|\mathbf{p}\rangle = +\,|\mathbf{p}\rangle \qquad \hat{Q}|\bar{\mathbf{p}}\rangle = -\,|\bar{\mathbf{p}}\rangle.$$

NB: The operators $\hat{\mathbf{P}}$ and $\hat{Q}$ do not depend on time and $\left[\hat{\mathbf{P}}, \hat{Q}\right] = 0$.

Ex: Show that multiparticle states $|\mathbf{p_1}...\mathbf{p_n}\rangle$ are eigenvectors of $\hat{\mathcal{H}}$, $\hat{\mathbf{P}}$, $\hat{Q}$.

Free Scalar Field Propagator

The field $\phi^\dagger$ (ϕ) increases (decreases) charge of a state

$$\left[\hat{Q}, \phi^\dagger(x)\right] = +\phi^\dagger(x), \quad \left[\hat{Q}, \phi(x)\right] = -\phi(x)$$

Consider the following amplitudes

$$t_2 > t_1 : \quad \langle 0| \underbrace{\phi(x_2)}_{a^-} \underbrace{\phi^\dagger(x_1)}_{a^+} |0\rangle \qquad\qquad t_1 > t_2 : \quad \langle 0| \underbrace{\phi^\dagger(x_1)}_{b^-} \underbrace{\phi(x_2)}_{b^+} |0\rangle$$

Particle (charge $+1$) propagates from x_1 to x_2

Antiparticle (charge -1) propagates from x_2 to x_1

Both possibilities can be taken into account in one function:

$$\begin{aligned}\langle 0| T[\phi(x_2)\phi^\dagger(x_1)]|0\rangle \equiv \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi^\dagger(x_1)|0\rangle \\ + \theta(t_1 - t_2)\langle 0|\phi^\dagger(x_1)\phi(x_2)|0\rangle,\end{aligned}$$

with T being time-ordering operation.

Free Scalar Field Propagator

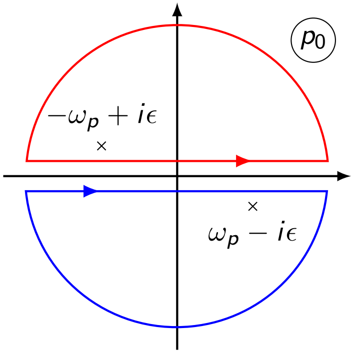
This is Feynman Propagator:

vanishes for $t_2 < t_1$

$$\underbrace{\langle 0|T[\phi(x_2)\phi^\dagger(x_1)]|0\rangle}_{-iD_c(x-y)} \equiv \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi^\dagger(x_1)|0\rangle + \theta(t_1 - t_2)\langle 0|\phi^\dagger(x_1)\phi(x_2)|0\rangle,$$

Fourier transform

$$D_c(x-y) = \frac{-1}{(2\pi)^4}\int d^4p \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$



The $i\epsilon$-prescription ($\epsilon \to 0$) picks up certain poles in the p_0 complex plane.

The propagator is a Green-function:

$$\underbrace{\left(\partial_x^2 + m^2\right)}_{\text{KG equation}} D_c(x-y) = \delta(x-y)$$

NB: Feynman propagator is a Lorentz-invariant function (distribution)!

From Field to Particle to Force

The propagator of particles can be connected to the force between two static classical sources $J_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$ located at $\mathbf{x}_i = (\mathbf{x}_1, \mathbf{x}_2)$. Presence of the sources disturbs the vacuum $|0\rangle \to |\Omega\rangle$, since Hamiltonian $\mathcal{H} \to \mathcal{H}_0 + J \cdot \phi$. Assuming for simplicity that $\phi = \phi^\dagger$ we can find

$$\langle\Omega|e^{-i\mathcal{H}T}|\Omega\rangle \equiv e^{-iE_0(J)T} \Rightarrow \text{in the limit } T \to \infty$$

$$= e^{\frac{i^2}{2!}\int dxdy J(x)\langle 0|T(\phi(x)\phi(y))|0\rangle J(y)} = e^{+\frac{i}{2}\int dxdy J(x) D_c(x-y) J(y)}$$

Evaluating the integral for $J(x) = J_1(x) + J_2(x)$ we get the contribution δE_0 to $E_0(J)$ due to interactions between two sources

$$\lim_{T\to\infty} \delta E_0 T = -\int dxdy J_1(x) D_c(x-y) J_2(y)$$

$$\delta E_0 = -\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{e^{+i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{p}^2 + m^2} = -\frac{1}{4\pi r} e^{-mr}, \qquad r = |\mathbf{x_1} - \mathbf{x_2}|$$

This is nothing else, but Yukawa potential due to scalar massive field. It is attractive and fall off exponentially over the distance scale $1/m$.

Free Scalar Field: Lagrangian

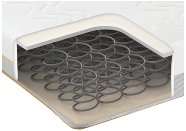
A convenient way to deal with (quantum) fields is to consider the following Action functional ("function → number"):

$$\mathcal{S}[\phi(x)] = \int d^4x \underbrace{\mathcal{L}(\phi(x), \partial_\mu\phi)}_{\text{Lagrangian (density)}} = \int d^4x \underbrace{\left(\partial_\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi\right)}_{\phi^\dagger\cdot K\cdot\phi}.$$

To have an analogy with Classical Mechanics one can rewrite the Action as

$$\mathcal{S}[\phi(x)] = \int dt\, L(t), \qquad L = T - U, \quad H = T + U$$

$$T = \int d\mathbf{x}|\partial_t\phi|^2, \quad U = \int d\mathbf{x}(|\partial_{\mathbf{x}}\phi|^2 + m^2|\phi|^2)$$



A system of coupled oscillators with kinetic energy T and potential U.

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We can derive the equations of motions (EOM) via the Action Principle:

$$\underbrace{\mathcal{S}[\phi'(x)] - \mathcal{S}[\phi(x)]}_{\delta\mathcal{S}[\phi(x)]} = \int d^4x \left[\underbrace{\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi}\right)}_{(\partial_\mu^2 + m^2)\phi} \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi\right)}_{\text{surface term}}\right].$$

$$\phi'(x) = \phi(x) + \delta\phi(x), \quad \delta\phi(x) \text{ is infinitesimal ("tiny")}$$

Free Scalar Field: Lagrangian

A convenient way to deal with (quantum) fields is to consider the following Action functional ("function $\to$ number"):

$$\mathcal{S}[\phi(x)] = \int d^4x \underbrace{\mathcal{L}(\phi(x), \partial_\mu\phi)}_{\text{Lagrangian (density)}} = \int d^4x \underbrace{\left(\partial_\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi\right)}_{\phi^\dagger\cdot K\cdot\phi}.$$

We can derive the equations of motions (EOM) via the Action Principle:

$$\underbrace{\mathcal{S}[\phi'(x)] - \mathcal{S}[\phi(x)]}_{\delta\mathcal{S}[\phi(x)]=0} = \int d^4x \left[\underbrace{\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} - \frac{\partial\mathcal{L}}{\partial\phi}\right)}_{(\partial_\mu^2+m^2)\phi=0}\delta\phi + \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)}_{\text{surface term}=0}\right].$$

We look for specific $\phi(x)$ that gives $\delta S[\phi(x)] = 0$ for any variation[a] $\delta\phi(x)$.

NB1: Fields satisfying EOMs are said to be "on-mass-shell".

NB2: $\langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle$ can be found by inverting the quadratic form K.

[a]Satisfying boundary conditions.

About Symmetries (a Relaxing Slide)

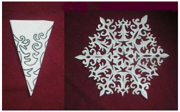
With Action you can study Symmetries...

The latter are intimately connected with transformations, which leaves something invariant...

Symmetries are not only beautiful but also very useful:

An architect can design only half of the building (parity $x \to -x$)

And winter decoration will take much less time (rotation by a finite angle)

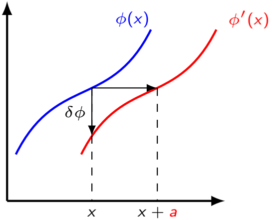


Field Theory: Symmetries

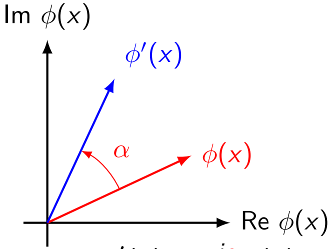
- Transformations can be discrete, e.g.,

$$\text{Parity}: \phi'(\mathbf{x},t) = P\phi(\mathbf{x},t) = \phi(-\mathbf{x},t),$$
$$\text{Time-reversal}: \phi'(\mathbf{x},t) = T\phi(\mathbf{x},t) = \phi(\mathbf{x},-t),$$
$$\text{Charge-conjugation}: \phi'(\mathbf{x},t) = C\phi(\mathbf{x},t) = \phi^\dagger(\mathbf{x},t),$$

or depend on continuous parameters, e.g.,



$$\phi'(x+a) = \phi(x)$$



$$\phi'(x) = e^{i\alpha}\phi(x)$$

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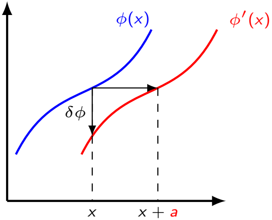

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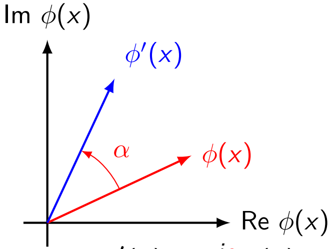

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- We can distinguish space-time and internal symmetries.

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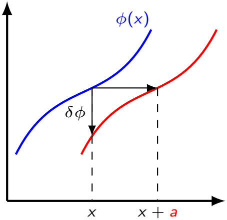

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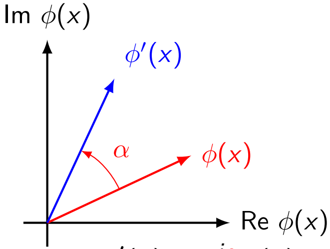

$$\phi'(x) = e^{i\alpha}\phi(x)$$

- We can distinguish space-time and internal symmetries.
- For x-dependent parameters we have local (gauge) transformations.

Quantum Field Theory: Symmetries

In Classical Physics symmetry transformations allows one to find

- new solutions to EOMs from the given one, keeping some features of the solutions (invariants) intact.
- how a solution in one coordinate system (as seen by one observer) looks in another coordinates (as seen by another observer).

In Quantum World a symmetry $\mathcal{S}$ guarantees that transition probabilities $\mathcal{P}$ between states do not change upon transformation:

$$|A_i\rangle \stackrel{\mathcal{S}}{\to} |A_i'\rangle, \qquad \mathcal{P}(A_i \to A_j) = \mathcal{P}(A_i' \to A_j'), \qquad |\langle A_i|A_j\rangle|^2 = |\langle A_i'|A_j'\rangle|^2$$

Symmetries are represented by unitary[†] operators U:

$$|A_i'\rangle = U|A_j\rangle, \quad \langle A_i'|A_j'\rangle = \langle A_i|\underbrace{U^\dagger U}_{1}|A_j\rangle$$

[†]or anti-unitary (time-reversal).

Quantum Field Theory: Symmetries

A transformation of states can be reformulated as a change of operators:

$$\langle A_i|\mathcal{O}_k(x)|A_j\rangle \xrightarrow{\mathcal{S}} \langle A_i'|\mathcal{O}_k(x)|A_j'\rangle = \langle A_i|U^\dagger\mathcal{O}_k(x)U|A_j\rangle$$

$$\langle A_i|\mathcal{O}_k(x)|A_j\rangle \xrightarrow{\mathcal{S}} \langle A_i|\mathcal{O}_k'(x)|A_j\rangle, \quad O_k'(x) \equiv U^\dagger\mathcal{O}_k(x)U$$

Symmetry relates these quantities

For example, translational invariance leads to

$$\langle A_i|\phi(x)|A_j\rangle = \langle A_i|\phi'(x+a)|A_j\rangle = \langle A_i|U^\dagger(a)\phi(x+a)U(a)|A_j\rangle$$

Quantum Field Theory: Symmetries

A transformation of states can be reformulated as a change of operators:

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Symmetry relates these quantities

For example, translational invariance leads to

$$\phi(x) = \phi'(x+a) = U^\dagger(a)\phi(x+a)U(a)$$

so quantum field should satisfy

$$\phi(x+a) = U(a)\phi(x)U^\dagger(a)$$

We can have non-trivial (realizations of) symmetries mixing different fields:

$$\phi_i'(x') = S_{ij}(a)\phi_j(x) \Rightarrow \phi_i(x') = S_{ij}(a)U(a)\phi_j(x)U^\dagger(a), \quad x' = x'(x,a)$$

Examples will be provided later...For the moment, let us find a connection between Symmetries of Action, Conserved Quantities and Unitary Operators that realize the symmetries at the quantum level.

Global Continuous Symmetries: Noether Theorem

Given $\mathcal{S}[\phi]$ one can find its symmetries, i.e., particular infinitesimal variations $\delta\phi(x)$ that for any ϕ leave $\mathcal{S}[\phi]$ invariant up to a surface term

$$\mathcal{S}[\phi'(x)] - \mathcal{S}[\phi(x)] = \int d^4x\, \partial_\mu \mathcal{K}_\mu, \quad \phi'(x) \equiv \phi(x) + \delta\phi(x).$$

We compare this with

$$\mathcal{S}[\phi'(x)] - \mathcal{S}[\phi(x)] = \int d^4x \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \right].$$

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and require $\phi(x)$ to satisfy EOMs. This results in a local conservation law:

$$\partial_\mu J_\mu = 0, \quad J_\mu \equiv \mathcal{K}_\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi$$

Integration over space leads to the conserved charge

$$\frac{d}{dt} Q = 0, \qquad Q = \int d\mathbf{x} J_0$$

NB: If $\delta\phi = \rho_i \delta_i \phi$ depends on parameters ρ_i, we have a conservation law for every ρ_i. For Global symmetries ρ_i do not depend on coordinates.

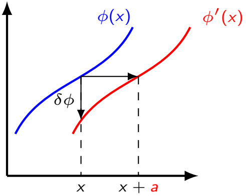
The Noether Theorem: Space-time symmetries

Consider space-time translations

$$\phi'(x+a) = \phi(x)$$

$$\text{expand in } a \Rightarrow \delta\phi(x) = -a_\nu \partial_\nu \phi(x),$$

$$\delta\mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \partial_\nu\left(-a_\nu \mathcal{L}\right)$$



A conserved Energy-Momentum Tensor $T_{\mu\nu}$:

$$J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}, \qquad \partial_\mu T_{\mu\nu} = 0$$

leads to time-independent "charges"

$$P_\nu = \int d\mathbf{x} T_{0\nu}$$

Ex1: Consider $\mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2$ and find the expression for P_μ.
Ex2: Substitute $\phi(x)$ by its expansion in terms of operators $a_p^\pm$ and $b_p^\pm$ and prove that modulo operator ordering ambiguities $P_\mu \to (\hat{\mathcal{H}}, \hat{\mathbf{P}})$.

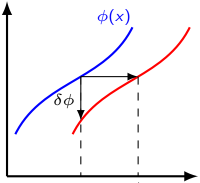
The Noether Theorem: Space-time symmetries

Consider space-time translations

$$\phi'(x + a) = \phi(x)$$

$$\text{expand in } a \Rightarrow \delta\phi(x) = -a_\nu \partial_\nu \phi(x),$$

$$\delta\mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \partial_\nu\left(-a_\nu \mathcal{L}\right)$$



A conserved Energy-Momentum Tensor $T_{\mu\nu}$:

$$J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}, \qquad \partial_\mu T_{\mu\nu} = 0$$

Lorentz transform for a **scalar** field $\phi'(\Lambda x) = \phi(x)$

leads to time-independent "charges"

$$P_\nu = \int d\mathbf{x} T_{0\nu}$$

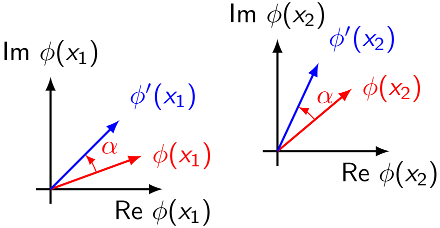
Ex1: Consider $\mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2$ and find the expression for P_μ.
Ex2: Substitute $\phi(x)$ by its expansion in terms of operators $a^\pm_p$ and $b^\pm_p$ and prove that normal-ordered expression $P_\mu \to (\hat{\mathcal{H}}, \hat{\mathbf{P}})$.

The Noether Theorem: Internal symmetries

There is an additional symmetry of

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi$$

$$\phi'(x) = e^{i\alpha}\phi(x)$$
$$\delta\phi(x) = i\alpha\phi(x)$$
$$J_\mu = i(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger)$$



It is a $U(1)$ symmetry:

- It acts in internal space ("rotates" complex number $\phi(x)$ at every x)
- It is a global symmetry (rotation angle α does not depend on x).

Ex: Check again that we will obtain the expression for the operator $\hat{Q}$.

Lagrange Approach to Quantum Fields: Mini Summary

The approach based on Lagrangians allows one to (given $\mathcal{L}$)

- Derive EOMs (via Action Principle).
- Find Symmetries of the Action.
- Find Conserved quantities (via the Noether Theorem)

After quantisation the operators of conserved quantities

- can be used to define a convenient basis of states, e.g.,:

$$|\mathbf{p}\rangle \equiv |\mathbf{p}, +1\rangle, \; |\bar{\mathbf{p}}\rangle \equiv |\mathbf{p}, -1\rangle \Rightarrow \hat{Q}|\mathbf{p}, q\rangle = q|\mathbf{p}, q\rangle, \hat{\mathbf{P}}|\mathbf{p}, q\rangle = \mathbf{p}|\mathbf{p}, q\rangle$$

- act as generators of symmetries, e.g. for space-time translations:

$$U(a) = \exp\left(i\hat{P}_\mu a_\mu\right), \qquad \hat{\phi}(x+a) = U(a)\hat{\phi}(x)U^\dagger(a)$$

NB: For $a_\mu = (t, \mathbf{0})$ we obtain the connection between Schrödinger and Heisenberg pictures:

$$O_H(t) = e^{i\hat{\mathcal{H}}t} O_S e^{-i\hat{\mathcal{H}}t}.$$

Lagrange Approach to Quantum Fields: Mini Summary

In QFT we usually start building our models by postulating symmetries (and other good properties) of the Action/Lagrangian!

We assume that general $\mathcal{L}$ is

- Lorentz (Poincare) invariant* (a sum of Lorentz scalars),
- Local (involve finite number of partial derivatives),
- Real (hermitian) (respects unitarity=conservation of probability)

In addition, we can impose other symmetries and get further restrictions on the model...

*Lorentz invariance is crucial for proving the $\mathcal{CPT}$-theorem.