(Quantum) Field Theory and the Electroweak Standard Model

Lecture II

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Outline

- Lecture I
  - What is the Standard Model?
  - Introducing Quantum Fields
  - Global Symmetries
- Lecture II
  - Introducing Interactions
  - Perturbation Theory
  - Renormalizable or Non-Renormalizable?
- Lecture III
  - Gauge Symmetries
  - Constructing the EW SM
  - Experimental tests of the EW SM
  - Issues and Prospects of the EW SM

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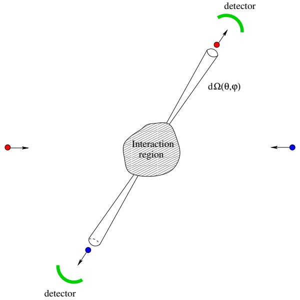

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- What about other particles?
- What about interactions?

Let us start with the first question ...

Fields, Fields, Fields ...

We can describe fields involving several DOFs by adding more (and more) indices $\phi(x) \to \Phi^i_\alpha(x)$. One can split indices into two groups:

space-time (α) and internal (i).

annihilation operator

guarantees $(\partial^2 + m^2)\Phi^i_\alpha = 0$

$$\Phi^i_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[u^s_\alpha(\mathbf{p})\,(a^-_{\mathbf{p}})^i_s\, e^{-ipx} + v^s_\alpha(\mathbf{p})(b^+_{\mathbf{p}})^i_\beta\, e^{+ipx} \right]$$

sum over all polarization states

polarization "vector" for a state s

$$\text{Lorentz transform } \Lambda : \Phi'^i_\alpha(\Lambda x) = S_{\alpha\beta}(\Lambda)\Phi^i_\beta(x)$$

$$\text{internal transform } a : \Phi'^i_\alpha(x) = U^{ij}(a)\Phi^j_\alpha(x)$$

- Quarks are color fermions Ψ^i_α and, e.g., $(a^-_{\mathbf{p}})^b_s$ annihilates blue quark in spin state s. The latter is characterized by spinor $u^s_\alpha(\mathbf{p})$.
- There are eight vector gluons G^a_μ. So $(a^-_{\mathbf{p}})^a_s$ annihilates gluon a in spin state s having polarization $u^s_\alpha(\mathbf{p}) \to \epsilon^s_\mu(\mathbf{p})$.

Massive Vector Fields

Charged Vector Field (e.g., W-bosons) can be written as

$$W_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1}^{3} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\epsilon_\mu^\lambda(\mathbf{p}) \left(a_\lambda^-(\mathbf{p}) e^{-ipx} + b_\lambda^+(\mathbf{p})\, e^{+ipx}\right)\right]$$

Massive spin-1 particle has 3 independent polarization vectors:

$$\epsilon_\mu^\lambda(\mathbf{p})\epsilon_\mu^{\lambda'}(\mathbf{p}) = -\delta^{\lambda\lambda'}, \qquad \sum_{\lambda=1}^{3} \epsilon_\mu^\lambda \epsilon_\nu^\lambda = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)$$

spin sum $\quad p_0 = \omega_p$

The Feynman propagator

$$\langle 0 | T(W_\mu(x) W_\nu^\dagger(y)) | 0 \rangle = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \left[\frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)}{p^2 - m^2 + i\epsilon}\right]$$

p_0 arbitrary

The Lagrangian

$$\mathcal{L} = -\frac{1}{2} W_{\mu\nu}^\dagger W_{\mu\nu} + m^2 W_\mu^\dagger W_\mu, \quad W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu$$

Ex: Find the explicit form of ϵ_μ^λ for $p_\mu = (E, 0, 0, p)$. Show that one of the vectors $\epsilon_\mu^L \simeq p_\mu / m + \mathcal{O}(m)$ diverges in the limit $p_\mu \to \infty$ ($m \to 0$).

Massless Vector Fields

Massless (say photon) Vector Field is usually represented by ($\omega_p = |\mathbf{p}|$)

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0}^{3} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\epsilon_\mu^\lambda(\mathbf{p}) \left(a_\lambda^-(\mathbf{p}) e^{-ipx} + a_\lambda^+(\mathbf{p})\, e^{+ipx}\right)\right].$$

with

$$\epsilon_\mu^\lambda(\mathbf{p})\epsilon_\mu^{\lambda'}(\mathbf{p}) = g^{\lambda\lambda'}, \quad \epsilon_\mu^\lambda(\mathbf{p})\epsilon_\nu^\lambda(\mathbf{p}) = g_{\mu\nu}, \quad [a_\lambda^-(\mathbf{p}), a_{\lambda'}^+(\mathbf{p}')] = -g_{\lambda\lambda'}\delta_{\mathbf{p},\mathbf{p}'}$$

The corresponding Feynman propagator is

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \frac{1}{(2\pi)^4}\int d^4p e^{-ip(x-y)}\left[\frac{-ig_{\mu\nu}}{p^2 + i\epsilon}\right]$$

But only 2 polarizations are physical!

This reflects the fact that the vector-field Lagrangian for $m = 0$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ for arbitrary $a(x)$ (gauge symmetry).

Additional conditions are needed to get rid of unphysical d.o.f.!

Fermion Fields

Spin-1/2 fermion fields are given by .

$$\psi^{\alpha}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} \left[u_s^{\alpha}(\mathbf{p}) a_s^{-}(\mathbf{p}) e^{-ipx} + v_s^{\alpha}(\mathbf{p}) b_s^{+}(\mathbf{p}) e^{+ipx} \right],$$

Fermion Fields

Spin-1/2 fermion fields are given by ($\bar{\psi} = \psi^\dagger \gamma_0$).

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} \left[u_s(\mathbf{p}) a_s^-(\mathbf{p}) e^{-ipx} + v_s(\mathbf{p}) b_s^+(\mathbf{p}) e^{+ipx} \right],$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} \left[\bar{u}_s(\mathbf{p}) a_s^+(\mathbf{p}) e^{+ipx} + \bar{v}_s(\mathbf{p}) b_s^-(\mathbf{p}) e^{-ipx} \right].$$

Here the spinors u_s and v_s satisfy 4×4 matrix equations

$$(\hat{p} - m) u_s(\mathbf{p}) = 0, \quad (\hat{p} + m) v_s(\mathbf{p}) = 0, \quad \hat{p} \equiv \gamma_\mu p_\mu, \quad p_0 \equiv \omega_{\mathbf{p}}$$

and correspond to particles (u_s) or antiparticles (v_s) with two spin states ($s = 1, 2$). One can normalize the spinors as

$$\bar{u}_s(\mathbf{p}) u_r(\mathbf{p}) = 2m\delta_{rs}, \qquad \bar{v}_s(\mathbf{p}) v_r(\mathbf{p}) = -2m\delta_{rs},$$

NB: Gamma-matrix (Clifford) algebra involve anticommutators:

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu}\mathbf{1} \quad \Rightarrow \gamma_0^2 = \mathbf{1}, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\mathbf{1}.$$

Fermions & Pauli Principle

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} \left[u_{s,\mathbf{p}} a^-_{s,\mathbf{p}} e^{-ipx} + v_{s,\mathbf{p}} b^+_{s,\mathbf{p}} e^{+ipx} \right].$$

Contrary to the case of scalar (boson) fields, the creation/annihilation operators for fermions $a^\pm_{s,\mathbf{p}}$ and antifermions $b^\pm_{s,\mathbf{p}}$ anticommute.

$$\left[a^-_{r,\mathbf{p}}, a^+_{s,\mathbf{p}'}\right]_+ = \left[b^-_{r,\mathbf{p}}, b^+_{s,\mathbf{p}'}\right]_+ = \delta_{sr}\delta(\mathbf{p} - \mathbf{p}')$$

$$\left[a^\pm_{r,\mathbf{p}}, a^\pm_{s,\mathbf{p}'}\right]_+ = \left[b^\pm_{r,\mathbf{p}}, b^\pm_{s,\mathbf{p}'}\right]_+ = \left[a^\pm_{r,\mathbf{p}}, b^\pm_{s,\mathbf{p}'}\right]_+ = 0$$

Due to this, fermions obey Pauli principle: $a^+_{r,\mathbf{p}} a^+_{r,\mathbf{p}} = 0$.
NB: One can explicitly show that quantization of bosons (integer spin) with anticommutators and fermions (half-integer spin) with commutators leads to inconsistencies (violates Spin-Statistics theorem).

Fermions & Lorentz Symmetry

Fermion fields transform under Lorentz group $x' = \Lambda x$ as

$$\psi'(x') = \mathcal{S}_\Lambda \psi(x), \qquad \psi'(x')^\dagger = \psi(x)\mathcal{S}_\Lambda^\dagger.$$

The matrix $\mathcal{S}_\Lambda^\dagger \neq \mathcal{S}_\Lambda^{-1}$, but $\mathcal{S}^{-1} = \gamma_0 \mathcal{S}^\dagger \gamma_0$.
Dirac-conjugated spinor $\bar{\psi}(x) \equiv \psi^\dagger \gamma_0$ enters

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x), \qquad \text{Lorentz scalar}$$
$$\bar{\psi}'(x')\gamma_\mu\psi'(x') = \Lambda_{\mu\nu}\bar{\psi}(x)\gamma_\nu\psi(x), \quad \text{Lorentz vector.}$$

and allows one to prove that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\hat{\partial} - m\right)\psi$$

is also a Lorentz scalar = respects Lorentz symmetry.

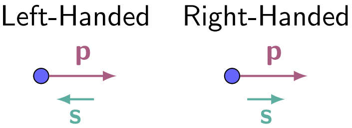
Ex: Find the expression for P_μ. Show that

$$P_0 \equiv \mathcal{H} = \int d\mathbf{p} \sum_s \omega_p \left(a_s^+ a_s^- + b_s^+ b_s^- - \delta(\mathbf{0})\right)$$

Left and Right Fermions: Chirality vs Helicity

Two independent solutions for particles ($u_{1,2}$) can be chosen to correspond to two different Helicities — projections of spin $\mathbf{s}$ onto direction of $\mathbf{p}$:

$$\mathcal{H} = \mathbf{s} \cdot \mathbf{n}, \quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|$$



In free motion it is conserved.
But not Lorentz-invariant!

Massive particles can be overtaken so that $\mathbf{n} \to -\mathbf{n}$ and $\mathcal{H} \to -\mathcal{H}$.

Helicity for a massless particle is the same for all inertial observers and coincides with Chirality, which is a Lorentz-invariant concept.

By definition Left (ψ_L) and Right (ψ_R) chiral spinors are eigenvectors of

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow [\gamma_\mu, \gamma_5]_+ = 0, \quad \gamma_5^2 = 1, \quad \gamma_5^\dagger = \gamma_5$$

$$\gamma_5\psi_L = -\psi_L, \qquad \gamma_5\psi_R = +\psi_R.$$

Chirality and Helicity are not the same for massive particles!

Fermions: Chirality and Mass

For any spinor ψ we can define

$$\psi = \psi_L + \psi_R, \qquad \psi_{L/R} = P_{L/R}\psi, \quad P_{L/R} = \frac{1 \mp \gamma_5}{2},$$

and rewrite the Lagrangian as

$$\mathcal{L} = i(\underbrace{\bar\psi_L \hat\partial \psi_L + \bar\psi_R \hat\partial \psi_R}_{\text{conserve chirality}}) - m(\underbrace{\bar\psi_L \psi_R + \bar\psi_R \psi_L}_{\text{break chirality}})$$

The mass term mixes left and right components.

Ex: Prove that the mass term violates chiral symmetry (independent rotation of left and right components):

$$\psi \to e^{i\gamma_5 \alpha}\psi.$$

NB: We considered the Dirac mass term. For neutral fermions (neutrino) there is another possibility — a Majorana mass. Charge-conjugation $\psi \to \psi^c$ flips chirality and we can use ψ_L^c in place of ψ_R to write

$$\mathcal{L} = \frac{1}{2}(i\bar\psi_L \hat\partial \psi_L - m\bar\psi_L \psi_L^c) \qquad \text{see lectures by Silvia Pascoli.}$$

Introducing Interactions

In HEP typical collision/scattering experiment deals with "free" initial and final states and considers transitions between these states.
In Quantum Theory one introduces S-matrix with matrix elements

$$\mathcal{M} = \langle\beta|S|\alpha\rangle, \qquad \mathcal{M} = \delta_{\alpha\beta} + (2\pi)^4\delta^4(p_\alpha - p_\beta) i M_{\alpha\beta}$$

giving amplitudes for possible transitions between in $|\alpha\rangle$ and out $|\beta\rangle$ states:

$$|\alpha\rangle = a^+_{\mathbf{p}_1}...a^+_{\mathbf{p}_r}|0\rangle, \quad |\beta\rangle = a^+_{\mathbf{k}_1}...a^+_{\mathbf{k}_s}|0\rangle,$$

Differential probability to evolve from $|\alpha\rangle$ to $|\beta\rangle$ is

$$dw = \frac{n_1...n_r}{(2\omega_{p_1})...(2\omega_{p_r})}|M_{\alpha\beta}|^2 d\Phi_s, \qquad n_i - \text{particle densities}$$

$$d\Phi_s = (2\pi)^4\delta^4\left(p_{in} - k_{out}\right)\frac{d\mathbf{k}_1}{(2\pi)^3(2\omega_{k_1})}\cdots\frac{d\mathbf{k}_i}{(2\pi)^3(2\omega_{k_i})}$$

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Differential decay width $d\Gamma$ of particle with mass m, and cross-section $d\sigma$ for $2 \to s$ process can be calculated via:

$$d\Gamma = \Phi_\Gamma |M|^2 d\Phi_s, \qquad \Phi_\Gamma = \frac{1}{2m},$$

$$d\sigma = \Phi_\sigma |M|^2 d\Phi_s, \qquad \Phi_\sigma = \frac{1}{4\sqrt{(p_1 p_2)^2 - p_1^2 p_2^2}}$$

Scattering Matrix

- In QFT the S-matrix is given by

$$S = T e^{-i\int dx \mathcal{H}_I(x)} = T e^{i\int dx \mathcal{L}_I(x)}.$$

  Interaction Hamiltonian $\mathcal{H}_I$ (Lagrangian $\mathcal{L}_I$) is built from free* field operators (certain combinations of $a^\pm$ and $b^\pm$).

- $\mathcal{L}_I = \mathcal{L}_{full} - \mathcal{L}_0$ is a sum of Lorentz-invariant terms involving more than two fields and more ∂_μ than in the free $\mathcal{L}_0$.
- Time-ordering operation

$$T\Phi_1(x_1)...\Phi_n(x_n) = (-1)^k \Phi_{i_1}(x_{i_1})...\Phi_{i_n}(x_{i_n}), \qquad x^0_{i_1} > ... > x^0_{i_n},$$

  where $(-1)^k$ appears due to k permutations of fermions fields.

NB: Higgs self-interactions in the SM is described by $\mathcal{L}_I = -\lambda\phi^4/4!$

*More precisely, operators in the interaction picture.

Interaction Lagrangians

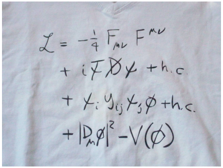
Interaction Lagrangian should be hermitian and can include any scalar combination of quantum fields, e.g.,

$$\mathcal{L}_I: \quad g\phi^3(x), \qquad \lambda\phi^4(x), \qquad y\bar{\psi}(x)\psi(x)\phi(x)$$
$$e\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x), \qquad G\left[(\bar{\psi}_1\gamma_\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) + \mathrm{h.c.}\right]$$

The parameters (couplings) g, λ, e, y, and G sets the strength of interactions. Usually, we assume that couplings are small and we can treat $\mathcal{L}_I$ as a perturbation to $\mathcal{L}_0$.

The T-shirt Lagrangian is unique, since all the couplings there are dimensionless!

Ex: Show that $[\phi] = [A_\mu] = 1$, $[\psi] = 3/2$. Find the (mass) dimensions of g, λ, e, y, and G.



Hint: $[\mathcal{L}] = 4$

Well, $V(\phi)$ is not specified, but I am pretty sure that only 2 terms were implied. Which ones?

Perturbation Theory

Given $\mathcal{L}_I$ we can calculate $\langle\alpha|S|\beta\rangle$. In practise, one uses perturbative expansion of the T-exponent and evaluates terms like ($\mathcal{L}_I = -\lambda\phi^4/4!$):

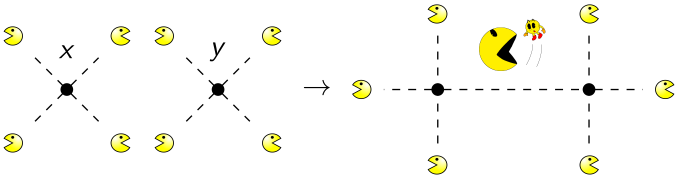
$$\frac{i^n}{n!}\left[\frac{\lambda}{4!}\right]^n \langle 0|a^-_{\mathbf{k}_1}...a^-_{\mathbf{k}_s}\int dx_1...dx_n T\left[\phi(x_1)^4...\phi(x_n)^4\right] a^+_{\mathbf{p}_1}...a^+_{\mathbf{p}_r}|0\rangle,$$

The calculation is carried out by means of Wick theorem:

$$T\Phi_1...\Phi_n = \sum(-1)^\sigma\langle 0|T(\Phi_{i_1}\Phi_{i_2})|0\rangle...\langle 0|T(\Phi_{i_{m-1}}\Phi_{i_m})|0\rangle :\Phi_{i_{m+1}}...\Phi_{i_n}:,$$

where sum goes over all possible ways to pair the fields.
Remember normal ordering? Now it cares about fermions.

$$:a^-_1 a^+_2 a^-_3 a^-_4 a^+_5 a^-_6 := (-1)^\sigma a^+_2 a^+_5 a^-_1 a^-_3 a^-_4 a^-_6$$



Feynman Rules: External States

We have to evaluate

$$\langle 0| a^-_{\mathbf{k}_1} ... a^-_{\mathbf{k}_s} :\Phi_{i_{m+1}} ... \Phi_{i_n}: a^+_{\mathbf{p}_1} ... a^+_{\mathbf{p}_r} |0\rangle.$$

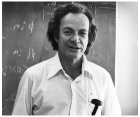
To get a non-zero matrix element all $a^-(a^+)$ in the normal product of fields from the Lagrangian have to be "killed" by (commuted with) $a^+(a^-)$ from the initial (final) states.
For our generalized field

$$\left[\Phi^i_\alpha(x), (a^+_{\mathbf{p}})^i_s\right] = \underbrace{\frac{e^{-ipx}}{(2\pi)^{3/2}\sqrt{2\omega_p}}}_{\text{common to all fields}} u^s_\alpha(\mathbf{p}), \qquad \text{initial state polarization}$$

$$\left[(b^-_{\mathbf{p}})^i_s, \Phi^i_\alpha(x)\right] = \frac{e^{+ipx}}{(2\pi)^{3/2}\sqrt{2\omega_p}} v^s_\alpha(\mathbf{p}), \qquad \text{final state polarization}$$

All this machinery can be implemented in a set of Feynman rules, which are used to draw (and evaluate) Feynman diagrams for the amplitudes.

Feynman Rules: Propagators



- Propagators (from $\mathcal{L}_0$)

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \begin{cases} 1 & \phi \\ \hat{p} + m & \psi \\ -g_{\mu\nu} + p_\mu p_\nu / m^2 & W_\mu \end{cases}$$

(p; p; μ p ν)

- External states (from $\mathcal{L}_0$)

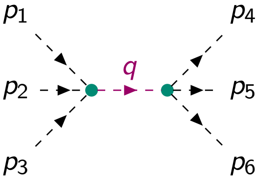
scalar	1	p	incoming fermion	$u_s(\mathbf{p})$	p
	1	p	outgoing fermion	$\bar{u}_s(\mathbf{p})$	p
vector	$\epsilon^\lambda_\mu(\mathbf{p})$	p μ	outgoing antifermion	$v_s(\mathbf{p})$	p
	$\epsilon^{*\lambda}_\mu(\mathbf{p})$	μ p	incoming antifermion	$\bar{v}_s(\mathbf{p})$	p

Feynman Rules: Vertices

- Interaction Vertices (from $\mathcal{L}_I$ or $S_I = \int dx \mathcal{L}_I$)

$$i \frac{\delta^4 S_I[\phi]}{\delta\phi(p_1)\delta\phi(p_2)\delta\phi(p_3)\delta\phi(p_4)} \Rightarrow \underbrace{(2\pi)^4 \delta^4(p_1+p_2+p_3+p_4)}_{\text{conservation of energy-momentum}} \times [-i\lambda]$$

In a typical diagram all $(2\pi)^4\delta(...)$ factors (but one) are removed by the momentum integration originating from propagators.



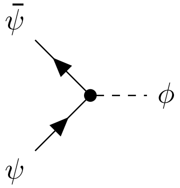
$$(2\pi)^4\delta^4\left(\sum_{i=1}^{3} p_i - \sum_{i=4}^{6} p_i\right)(-i\lambda)^2 \frac{i}{q^2 - m^2 + i\epsilon}$$

NB: All integrations (from propagators) are "killed" by δ-functions (from vertices) only in tree diagram (w/o loops)!

Feynman Rules: Vertices

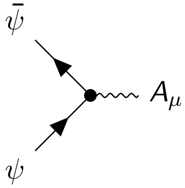
More Examples:

$$\mathcal{L}_I = -y\bar{\psi}\psi\phi$$



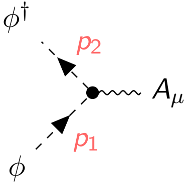
$$-iy$$

$$\mathcal{L}_I = e\bar{\psi}\gamma_\mu\psi A_\mu$$



$$ie\gamma_\mu$$

$$\mathcal{L}_I = ieA_\mu\left(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger\right)$$



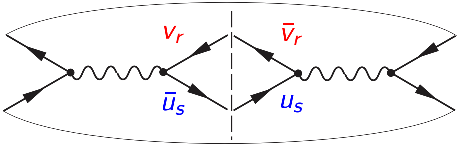
$$ie(p_1 + p_2)_\mu$$

Now you can do tree-level calculations of amplitudes...

From Amplitudes To Probabilities

To get probabilities we have to square matrix elements:

$$|M|^2 = MM^\dagger \Rightarrow$$



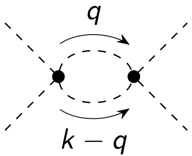
Sometimes we do not care about polarization states of initial or final particles so we have to sum over final polarization and average over initial ones. That is where spin-sum formulas become handy, e.g.

$$\sum_s u_s(\mathbf{p_1})\bar{u}_s(\mathbf{p_1}) = \hat{p}_1 + m, \qquad \sum_s v_s(\mathbf{p_2})\bar{v}_s(\mathbf{p_2}) = \hat{p}_2 - m$$

$$MM^\dagger \to \sum_{s,r} (\bar{u}_s A v_r)(\bar{v}_r A^\dagger u_s) = \mathrm{Tr}\left[(\hat{p}_1 + m)A(\hat{p}_2 - m)A^\dagger\right]$$

Loops and (UV) Divergences

In QFT particle propagators can form loops and we have integrals over unconstrained momenta



$$I_2(k) \equiv \int \frac{d^4 q}{[q^2 + i\epsilon][(k-q)^2 + i\epsilon]}$$
$$\sim \int^{\infty} \frac{|q|^3 d|q|}{|q|^4} \sim \ln \infty$$

which can lead to divergent (meaningless?) results. This is again UV divergence, due to large momenta ("small distances").

Q: Do we have to abandon QFT?

A: Nope, and there are reasons...

- (Phys:) We do not know physics up to infinitely small scales.
- (Math:) We are dealing with distributions, not functions.

NB: Overall degree of divergence can be deduced by power counting

Parametrizing our Ignorance: Regularization

To make sense of formally divergent integrals we introduce some regularization, e.g., a "cut-off" $|q| < \Lambda$

$$I_2^\Lambda(k) = i\pi^2 \left[\ln\frac{\Lambda^2}{k^2} + 1\right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) = i\pi^2\left[\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{k^2}{\mu^2} + 1\right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right)$$

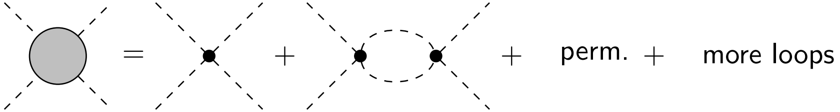
Another convenient possibility is dimensional regularization
$d = 4 \to d = 4 - 2\varepsilon$

$$I_2^{4-2\varepsilon}(k) = \mu^{2\varepsilon}\int\frac{d^{4-2\varepsilon}q}{q^2(k-q)^2} = i\pi^2\left(\frac{1}{\varepsilon} - \ln\frac{k^2}{\mu^2} + 2\right) + \mathcal{O}(\varepsilon)$$

The crucial property of the divergent ($\Lambda \to \infty$ or $\varepsilon \to 0$) terms is that they either do not depend on external momenta (k) at all or depend on k polynomially*. This means that we can cancel them by adding local (yet divergent) terms to $\mathcal{L}_I$ (CounterTerms).

Parametrizing our Ignorance: Renormalization

Indeed, we are interested in scattering amplitude in perturbation theory:



$$\lambda \qquad -\frac{\lambda^2}{2}\left(\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{k^2}{\mu^2} + \ldots\right)$$

Parametrizing our Ignorance: Renormalization

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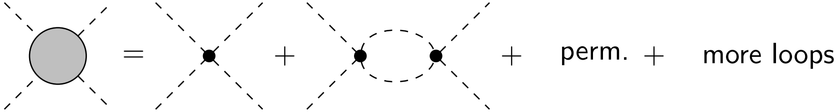


$$\lambda - \frac{\lambda^2}{2} \ln \frac{\Lambda^2}{\mu^2} \qquad -\frac{\lambda^2}{2}\left(\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + \ldots\right)$$

$$\lambda_B(\Lambda)$$

- We can defined bare coupling $\lambda_B(\Lambda)$ that depends on Λ (or ε):

$$\lambda_B(\Lambda) = \lambda(\mu)\left(1 + \frac{3}{2}\lambda(\mu) \ln \frac{\Lambda^2}{\mu^2} + \ldots\right),$$

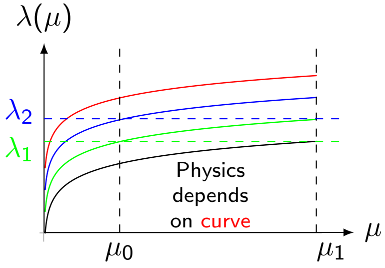
- The bare parameters can not be measured. But, we can (and do) express them in terms of renormalized quantities $\lambda(\mu)$ to get finite results, which can be confronted with experiment.

Renormalization Group

- The scale μ inevitably appears in any renormalization scheme.
- Changing μ corresponds to finite renormalization (rescaling) of the coupling, the latter becomes running

$$\lambda(\mu_0) \to \lambda(\mu), \quad \frac{d}{d\ln\mu}\lambda = \beta_\lambda(\lambda), \quad \beta_\lambda = \frac{3}{2}\lambda^2 + ..., \quad \text{RGE}$$

- The crucial point is that observables (if all orders of PT are taken into account) should not depend on the choice of μ.
- The dependence on μ is given by Renormalization Group Equations that theory can predict (β_λ).
- The value of $\lambda(\mu_0)$ is not predicted, but extracted from experiment!

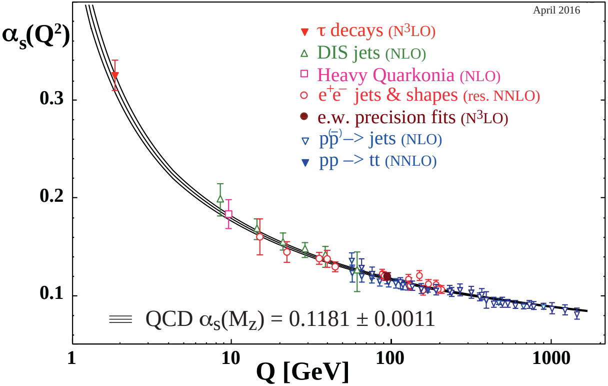


Two values λ_1 and λ_2 correspond to

- different Physics, if both measured at μ_0
- same Physics, if measured at μ_0 and μ_1, respectively.

Renormalization Group: QCD Example

$$\beta_{\alpha_s} = -\frac{\alpha_s^2}{4\pi}\left(11 - \frac{2}{3}n_f\right) + \ldots + \mathcal{O}(\alpha_s^7), \quad n_f - \text{number of flavours}$$



Experiments prove that Quantum ChromoDynamics is a consistent theory of strong interactions for a wide range of scales...

Renormalizable vs Non-Renormalizable

- We were able to cancel UV-divergencies by counter-terms that have the same structure as our initial Lagrangian. In general, each term in $\mathcal{L}_{full}$ gets a renormalization constant Z to subtract relevant divergence:

$$\mathcal{L}_{full} = \frac{Z_2}{2}(\partial\phi)^2 - \frac{Z_m m^2}{2}\phi^2 + Z_4\frac{\lambda\phi^4}{4!} = \frac{1}{2}(\partial\phi_B)^2 - \frac{m_B^2}{2} + \frac{\lambda_B\phi_B^4}{4!},$$

where $\phi_B = \sqrt{Z_2}\phi$, $m_B^2 = m^2 Z_m Z_2^{-1}$, $\lambda_B = \lambda Z_4 Z_2^{-2}$ are bare field, mass and coupling.

- (Divergent) Z-factors are chosen in such a way that the diverences in amplitudes are removed order by order in perturbation theory.

> If all divergences can be canceled by such a procedure
> the model is called renormalizable!

- One can determine whether a model is renormalizable by checking the dimension of the couplings. Remember the T-shirt Lagrangian?

Renormalizable vs Non-Renormalizable

- What happens if we have a divergent amplitude but the structure of the required subtraction does not have a counter-part in $\mathcal{L}$?
- We can add the required structure to the Lagrangian...

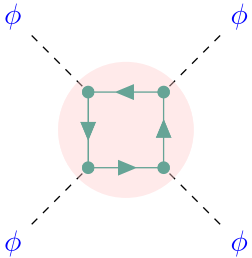
Imagine that we couple a scalar (ϕ) and a fermion (ψ) via

$$\mathcal{L}_I \ni \delta\mathcal{L}_Y = -y\bar{\psi}\psi\phi$$

but forgot to consider

$$\delta\mathcal{L}_4 = -\lambda\phi^4/4!$$

But it is required to cancel divergences due to fermion loops!



This divergent diagram will force us to add $\delta\mathcal{L}_4$ to $\mathcal{L}_I$.

- New terms in $\mathcal{L}_I$ will generate new diagrams, which can require new interactions to be added to $\mathcal{L}_I$. Will this process terminate?

NB: Every fermion loop produces an additional minus sign! Why?

Renormalizable vs Non-Renormalizable

- If one has to add more and more terms to $\mathcal{L}_I$, this is a signal of Non-Renormalizable model. Have to measure infinite number of couplings to predict something!

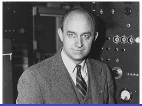
  Q: Is it BAD? Does it make sense?

  A: It is not satisfactory...but still we can get something!

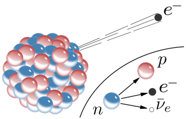
- Non-Renormalizable models contrary to the Renormalizable ones involves couplings G_i with negative mass dimension $[G_i] < 0$! Not all of them are important for predictions at low energies E

$$G_i E^{-[G_i]} \ll 1.$$

- This explains the success of the Fermi model of the β-decay $n \to p + e^- + \bar{\nu}_e$:



$$\mathcal{L}_I = G\bar{\Psi}_p\gamma_\rho\Psi_n \cdot \bar{\Psi}_e\gamma_\rho\Psi_\nu + \text{h.c.}$$



Fermi Model: Harbinger of the EW theory

In 1957 R. Marshak & G.Sudarshan, R. Feynman & M. Gell-Mann modified the original Fermi theory of beta-decay to incorporate 100 % violation of Parity discovered by C.S. Wu in 1956

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}}(J^+_\mu J^-_\mu + \mathrm{h.c.}),$$

$$J^-_\rho = (V - A)^{\text{nucleons}}_\rho + \overline{\Psi}_e \gamma_\rho (1 - \gamma_5) \Psi_{\nu_e} + \overline{\Psi}_\mu \gamma_\rho (1 - \gamma_5) \Psi_{\nu_\mu} + \ldots$$

This is current-current interactions with $G_F \simeq 10^{-5}$ GeV^{-1}.
From dimensional grounds we can estimate

$$\sigma(\nu_e e \to \nu_e e) \propto G_F^2 s, \qquad s = (p_e + p_\nu)^2.$$

× Non-Renormalizable theories eventually violate unitarity!

From Fermi Model to EW theory

- The modern view on the Fermi model treats it as an effective theory with certain limits of applicability.
- The value of the dimensionful coupling constant G_F tells us something about more fundamental theory (SM?):

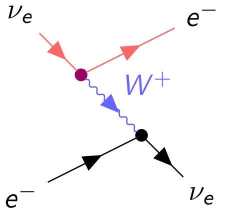
Warning!

around $G_F^{-1/2} \sim 10^2 - 10^3$ GeV there should be some "New Physics".

- QED is renormalizable. By analogy we introduce mediators - electrically charged vector fields $W_\mu^\pm$:

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}}(J_\mu^+ J_\mu^- + \text{h.c.})$$

$$\to \mathcal{L}_{\text{int}} = -\frac{g}{2\sqrt{2}}(W_\mu^+ J_\mu^- + \text{h.c.}).$$



From Fermi Model to EW theory

- The field W_μ in

$$\mathcal{L}_{\text{int}} = -\frac{g}{2\sqrt{2}}(J^+_\mu W^-_\mu + \text{h.c.}).$$

should be massive to account for short-range weak interactions.

- The scattering amplitude

$$T = i(2\pi)^4 \frac{g^2}{8} J^+_\alpha \left[\frac{g_{\alpha\beta} - p_\alpha p_\beta / M_W^2}{p^2 - M_W^2} \right] J^-_\beta$$

reproduces the result due to the current-current interaction in the limit $|p| \ll M_W$ if we identify ("match")

$$\text{(effective theory)} \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W{}^2} \quad \text{(more fundamental theory)}$$

- However, we have to be more clever, since the behavior of the amplitude in the opposite limit ($p \gg M_W$) is still the same.

The solution to this problem is to utilize gauge symmetry...