QCD.1

Francesco Tramontano
Università di Napoli Federico II
and INFN, sezione di Napoli

Basics

ESHEP 2018, Maratea, Italy

Plan of lecture 1

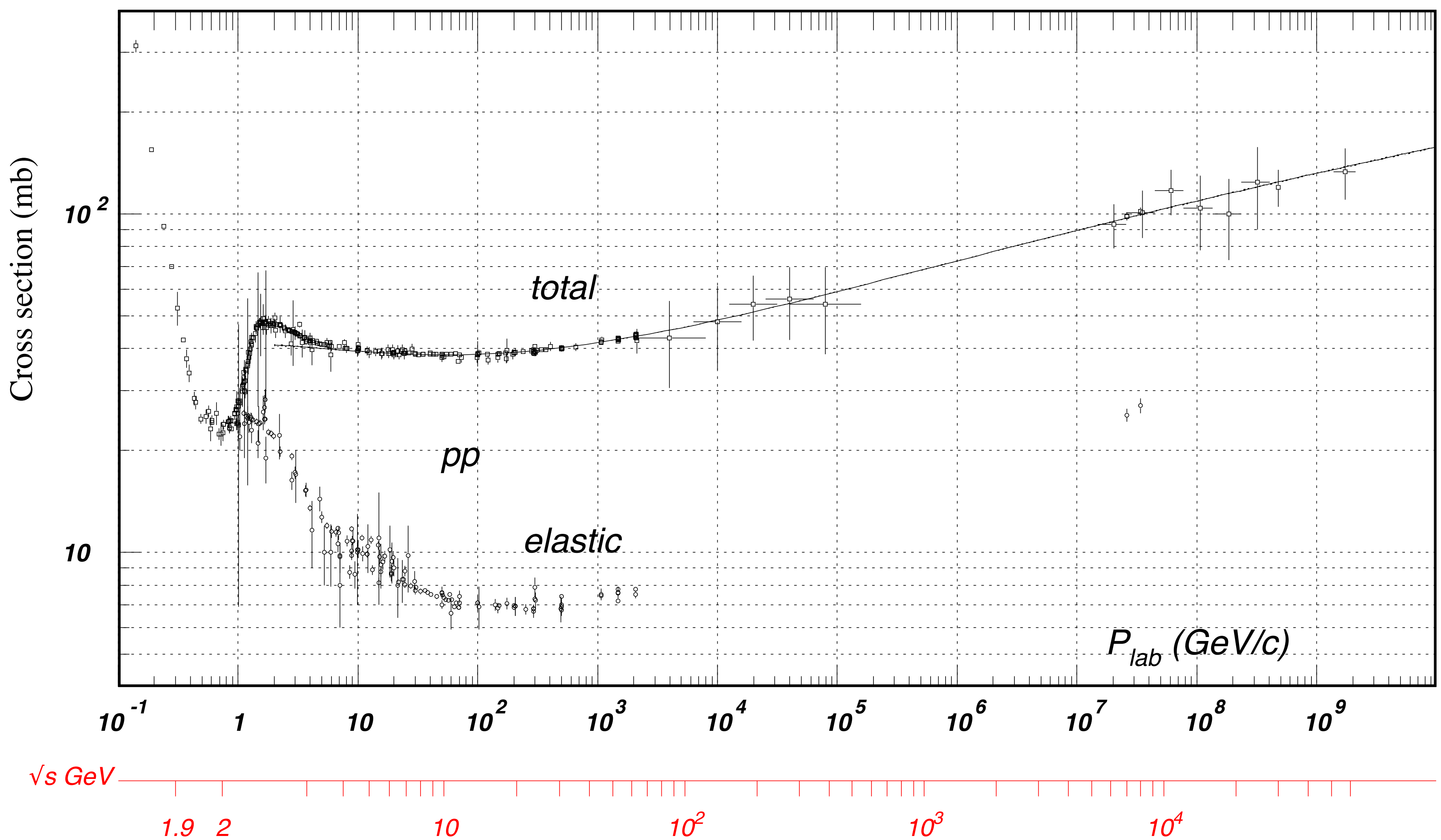
- Motivations for QCD

- QCD Lagrangian and Feynman rules

- QCD and phenomenology (e^+e^-)
  - Renormalization and running coupling
  - Infrared and collinear safety
  - Differential predictions

Basics

- protons, neutrons and mesons mainly interact via a strong force
- the strong force has an higher level of symmetry than weak and electromagnetic interactions, conserve parity and isospin (almost)
- characteristic scale of about 200 ÷ 300 MeV
  - ➡ Typical lifetime of excitations are ≈ O(10^{-24}s) ≈ O(1/(300MeV))
  - ➡ Typical cross sections are ≈ O(10mb) ≈ O(300MeV)^2



- No evidence of a small parameter (like in QED)
- Very much harder then EW interactions: we don't "see" the vertices

Motivations for QCD

1. Spectrum of hadrons. The whole hadron spectrum can be classified by assuming that:

   - hadrons are made up of quarks:

quark flavour	spin	charge (e)
d=down	1/2	-1/3
u=up	1/2	2/3
s=strange	1/2	-1/3
....		

   - Each quark flavour q^f comes in three different colours (index i): q^f_i

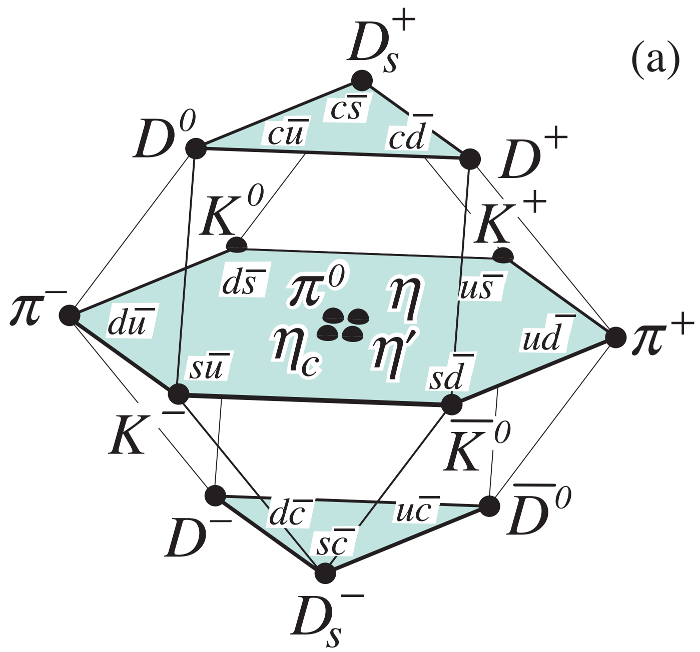
   - Observable hadrons are colour singlets under $SU(3)_{COLOURS}$

Motivations for QCD

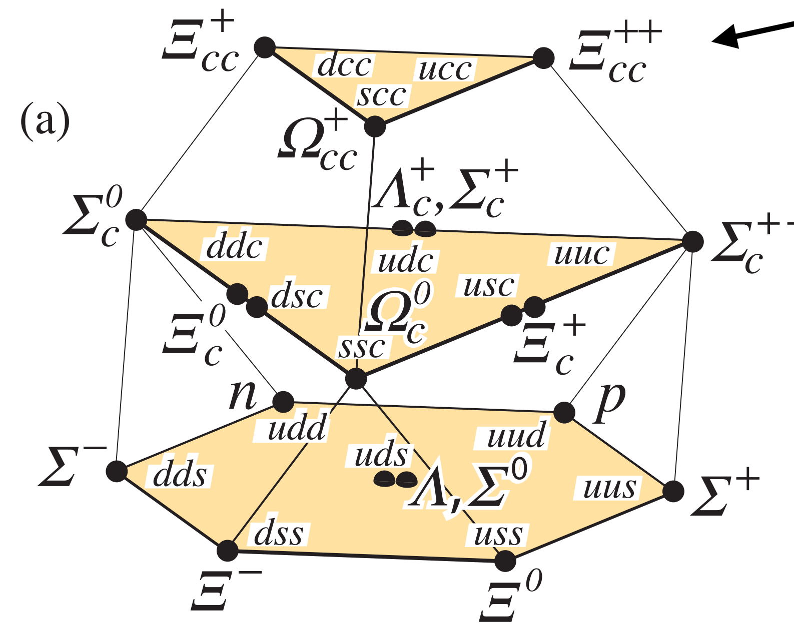
- singlet under $SU(3)_C$ are easily built for a quark-antiquark or a three quark system
- example: $SU(4)_F$ lowest mass multiplets

[LHCb 2017]

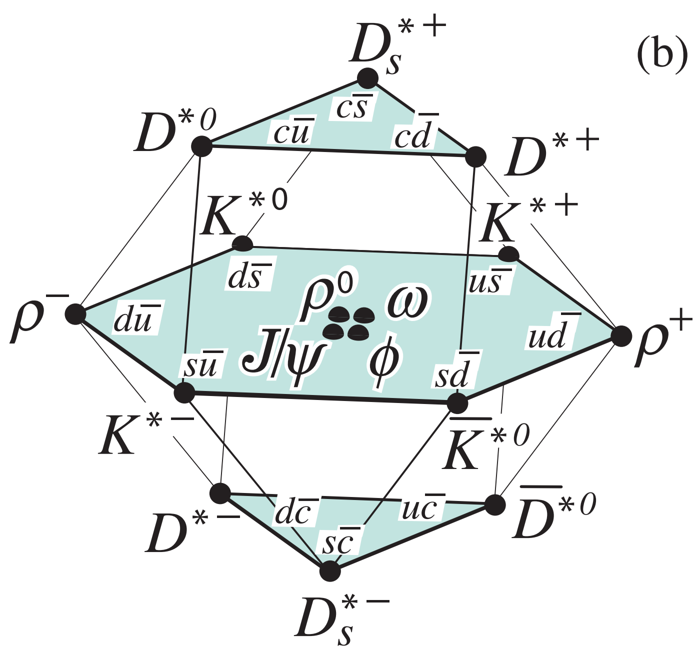
spin 0



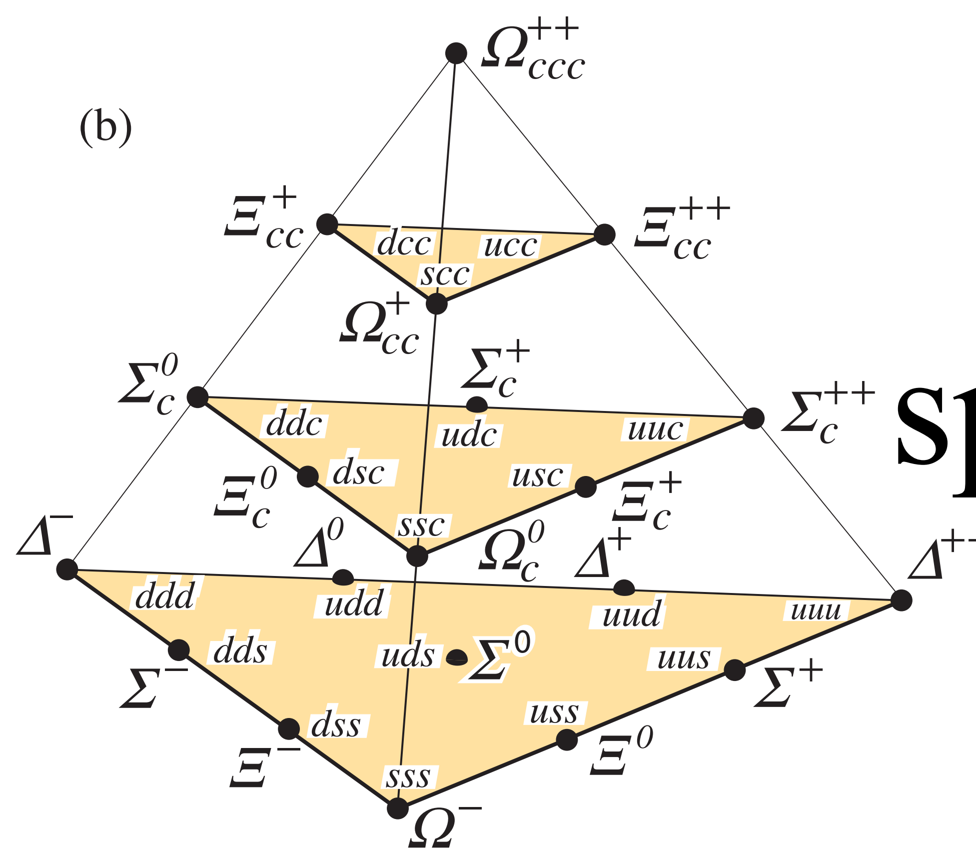
spin 1/2



spin 1



spin 3/2



- Nice symmetry in each group:
  differences well explained by quark masses, difficult to explain without colour (Δ^{++})

Motivations for QCD

- However the simple vision of just qqbar and qqq state does not represent the whole story. There is now evidence of states that do not fit the spectrum dictated by these two combinations only, for both the meson and baryon sector and for light-light, heavy-light and heavy-heavy particles.

- for the light-light sector there is observation of $\pi\pi$ resonances with isospin 2

- for the hidden charm and beauty sector, there is evidence of many states that for their mass, decay modes and width do not fit the quarkonium spectrum

- they have to be necessarily multi-quark systems!

  ➡ although quarks have fractional charges, to build a colour singlet necessarily:

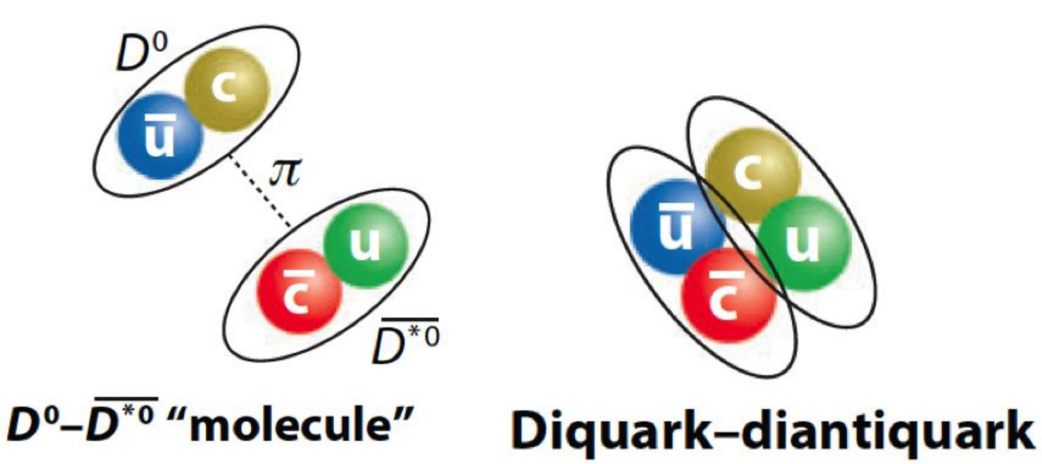
$$n_q - n_{\bar{q}} = \text{multiple of } 3$$

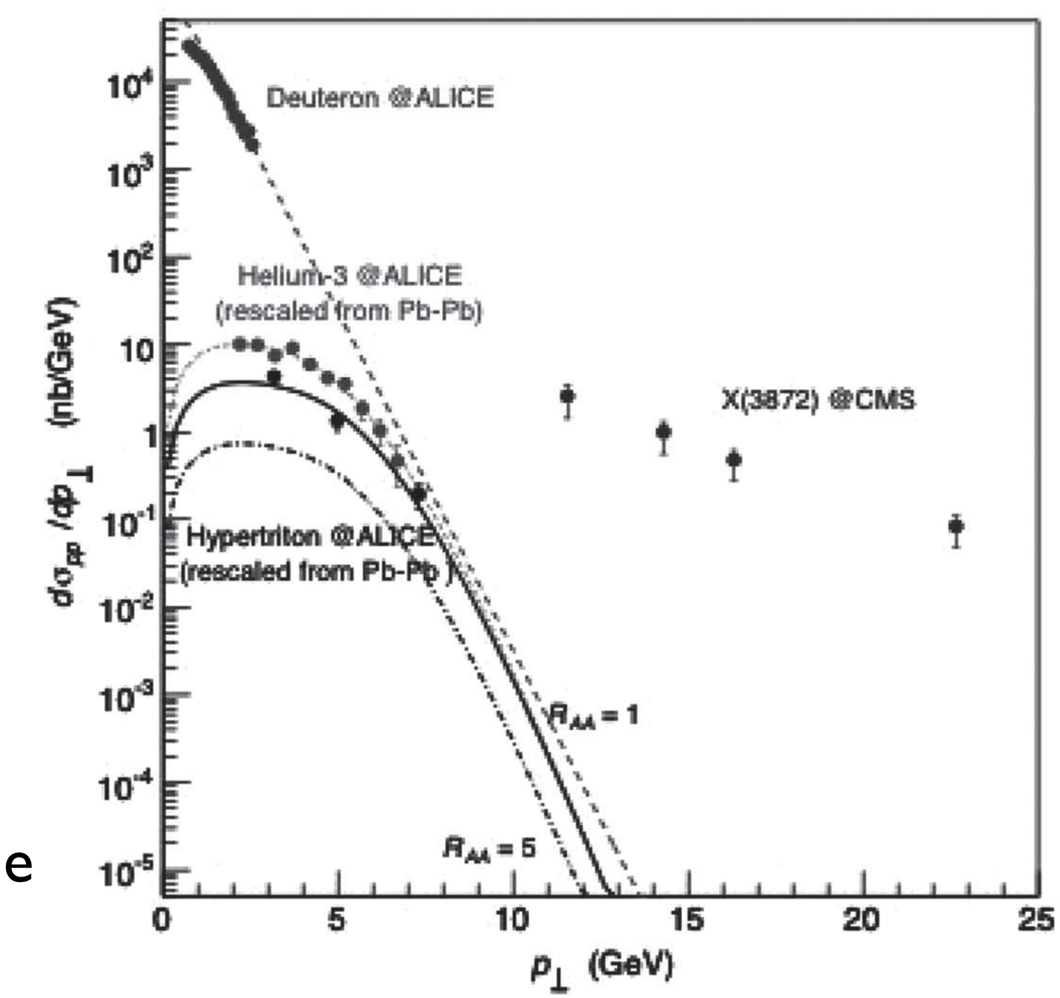
  ➡ and so there cannot be hadrons with fractional charge

Motivations for QCD

- QCD has indeed more then one solution to the problem!
  - molecule of color singlet states keep together by kind of nuclear forces: that would explain the fact that these states have masses typically close threshold
  - made of color parts confined by long-range color forces, similarly to what happens in normal mesons (tetraquark, pentaquark, hexaquark…) as indicated by the pt spectrum of the X(3872) that is a 1^{++} meson, measured in CMS that is much harder than (rescaled) deuteron and tritium spectrum measured in Pb-Pb collisions

$$B^{\pm} \to K^{\pm} X \;\; (X \to J/\psi\, \pi^+ \pi^-)$$

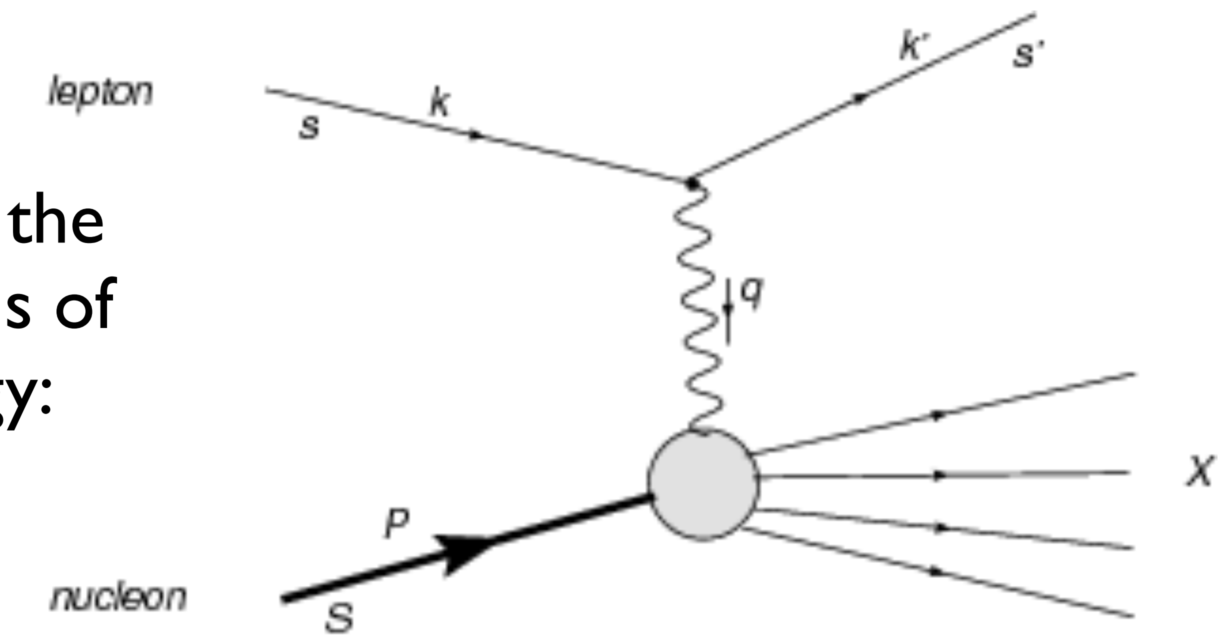




- These studies might in principle provide new clues to the understanding of QCD in the fully non-perturbative regime

Motivations for QCD

2. Scaling in Deep Inelastic Scattering (DIS)



- for inclusive inelastic large angle scattering, the differential cross section expressed in terms of dimensionless variables, "scales" with energy:

$$\frac{d\sigma}{dxdy} \sim \frac{1}{s} \qquad (s = E_{CM}^2)$$

- strong interaction at high energy resembles a weakly interacting theory with dimensionless coupling!

3. Theoretical discovery of asymptotically free theories

- Non abelian gauge theories are weakly coupled at high energies (short distances)
- Good candidates to be the theory of strong interactions

Motivations for QCD

All the three ingredients together

- A non abelian gauge theory coupled to $SU(3)_C$

  ➡ the only asymptotically free field theories

  ➡ it becomes strong at low energy: it may bind hadrons into color neutral systems (confinement) with a single scale at low energy

  ➡ assuming strong interaction group to be completely independent from the weak interaction group (commute) parity violating terms remain of order α/M_W^2

  ➡ same type of theory for weak, strong and electromagnetic interactions!

QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F_{\mu\nu A} + \sum_{f=1}^{N_f} \bar{q}^i_f (i\gamma^\mu \partial_\mu \delta_{ij} - g_S \gamma^\mu t^A_{ij} A^C_\mu - m_f \delta_{ij}) q^i_f$$

$i, j = 1, 2, 3 = N_c$ $\qquad$ $A, B, C = 1, 2, ..., 8 = N_c^2 - 1$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_S f_{ABC} A^B_\mu A^C_\mu$$

$$[t^A, t^B] = i f_{ABC} t^C$$

$$t^A = \frac{1}{2} \lambda^A$$

- gluon field strength term generates 3g and 4g couplings: gluons bring colour charge
- gluon radiation from a quark or a gluon changes its colour charge
- explicit colour matrices not important for most practical purposes

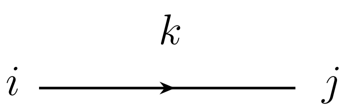
➡ normalisation: $Tr(t^A t^B) = T_R \delta^{AB}$ $\qquad$ $T_R = \frac{1}{2}$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

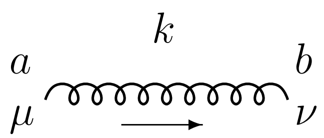
$$\vdots$$

QCD Feynman rules

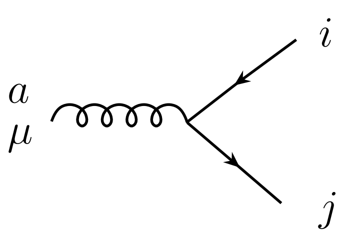


$i\,\delta_{ij}\dfrac{(\not{k}+m)}{k^2-m^2+i\epsilon}$

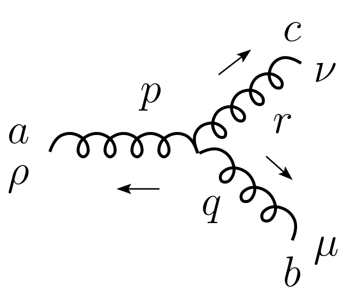
To compute amplitudes is needed to specify the gluon pol. tensor



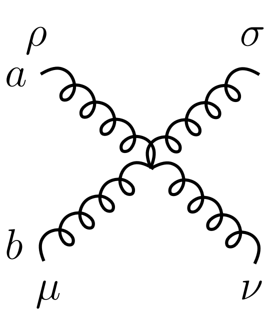
$\dfrac{-i\,\delta_{ab}}{k^2+i\epsilon}\;d_{\mu\nu}(k) \quad\Longleftrightarrow\quad d_{\mu\nu}(k)=\sum_\lambda \varepsilon^\mu_\lambda(k)\varepsilon^{\nu *}_\lambda(k) = \begin{cases} -g^{\mu\nu}+(1-\lambda)\frac{k^\mu k^\nu}{k^2+i\epsilon} & \text{covariant} \\ -g^{\mu\nu}+\frac{n_\mu k_\nu + k_\mu n_\nu}{n\cdot k} - \frac{(n^2-\lambda k^2)k^\mu k^\nu}{(n\cdot k)^2} & \text{axial} \end{cases}$



$i g_s \gamma_\mu T^a_{ji}$



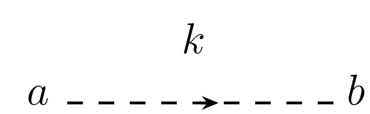
$-g_s f^{abc}\,[(p-q)_\nu\, g_{\rho\mu}$
$+(q-r)_\rho\, g_{\mu\nu}$
$+(r-p)_\mu\, g_{\nu\rho}]$



$-ig_s^2 f^{abe}f^{cde}\,(g_{\rho\nu}g_{\mu\sigma}-g_{\rho\sigma}g_{\mu\nu})$
$-ig_s^2 f^{ace}f^{bde}\,(g_{\rho\mu}g_{\nu\sigma}-g_{\rho\sigma}g_{\mu\nu})$
$-ig_s^2 f^{ade}f^{cbe}\,(g_{\rho\nu}g_{\mu\sigma}-g_{\rho\mu}g_{\sigma\nu})$

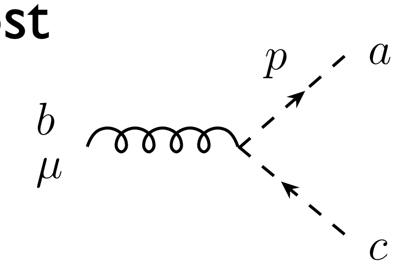
- the easiest choice seems to be the covariant version with $\lambda=1$
- it propagates unphysical (non transverse) degrees of freedom
- they have to be cancelled introducing other non physical degrees of freedom: the ghosts

➡ coloured scalars



$\dfrac{-i\,\delta_{ab}}{k^2+i\epsilon}$

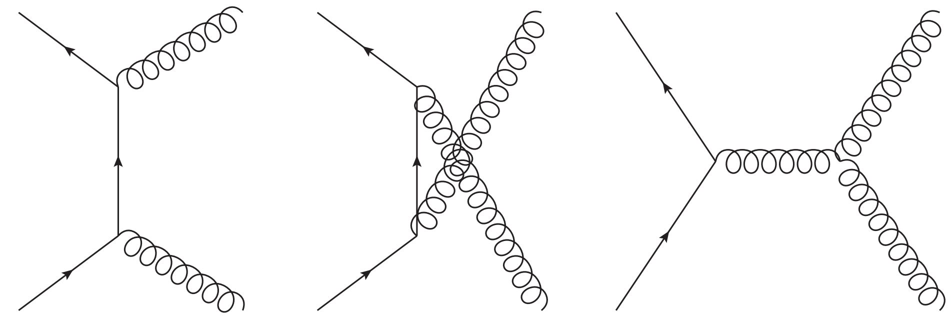
➡ (-1) factor for each ghost loop, as for fermions



$g_s\, f^{abc} p_\mu\;(p_\mu \text{ outgoing})$

QCD Feynman rules

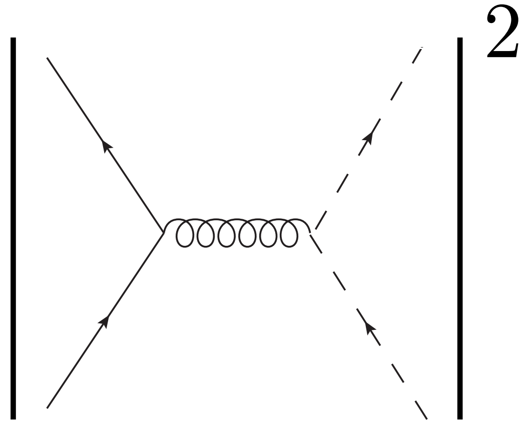
- Example: $q\bar{q} \to gg$

$A_{\mu\nu} =$ 

$$\sigma \propto |A|^2 = A_{\mu\nu} A_{\mu'\nu'} d^{\mu\mu'} d^{\nu\nu'}$$

- To build the squared amplitude using simply: $d^{\alpha\beta} = -g^{\alpha\beta}$

➡ one has to include also the ghost contribution:

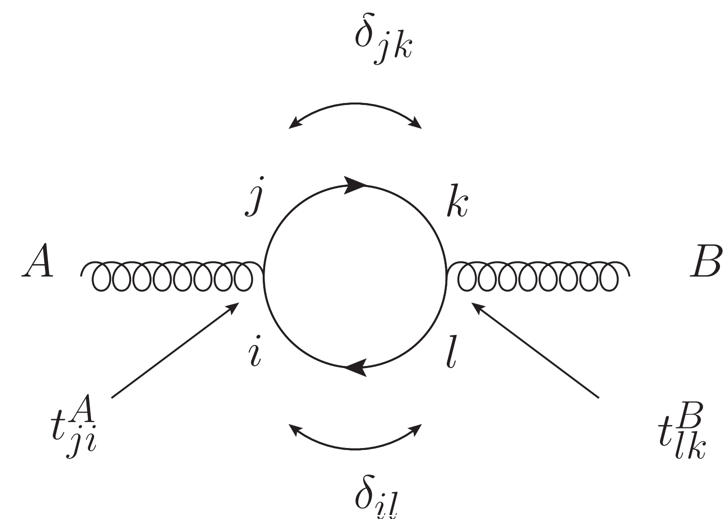


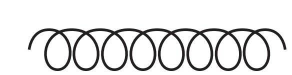
➡ or one can work out numerics at amplitude level, with helicity amplitudes, using only physical external gluon polarisations and having ghosts only in closed loops (no ghosts in tree level amps)

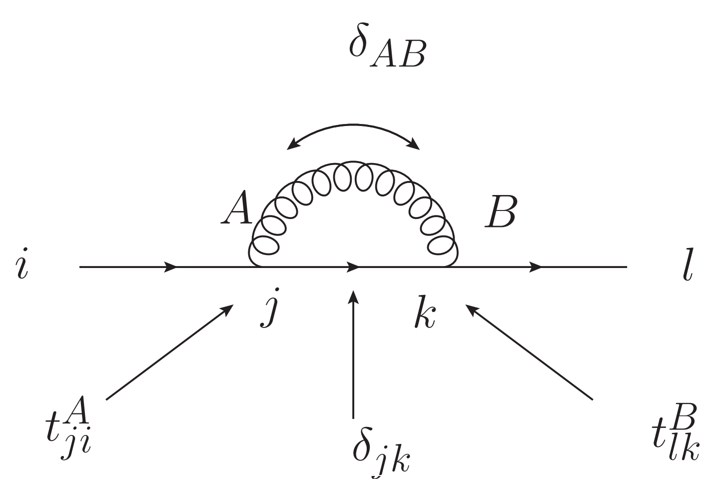
$$\sigma = \sum_{\substack{helicity \\ configurations}} |A_{h.c.}|^2$$

QCD Feynman rules

- beyond Dirac algebra we also need to compute colour factors
- we list the most relevant ones:

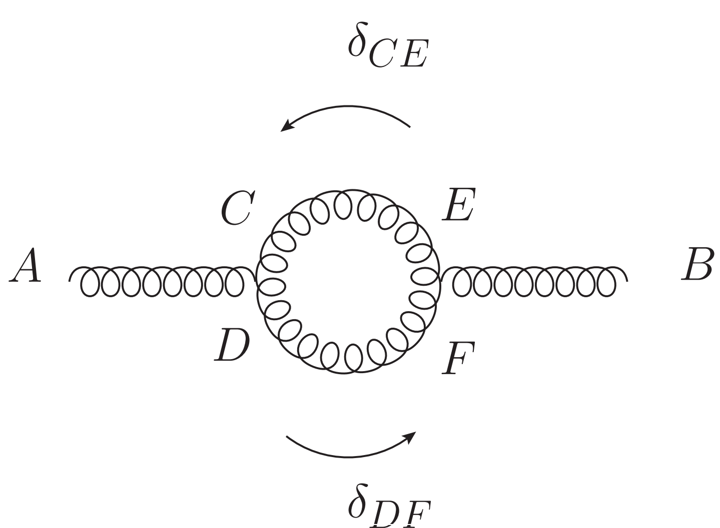


$Tr(t^A t^B) = T_R \delta^{AB} \quad \rightarrow \quad T_R$ A  B

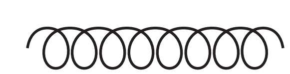


4/3 fermion colour charge

$(t^A t^A)_{li} = C_F \delta^{li} \quad \rightarrow \quad C_F$ i  l



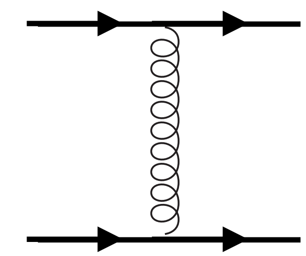
3 gluon colour charge

$f_{ACD} f_{BCD} = C_A \delta^{CD} \quad \rightarrow \quad C_A$ A  B

$C_A > C_F \qquad \dfrac{C_A}{C_F} \sim 2$ gluon radiates more than quark

QCD Feynman rules

- Fierz transformation $t^A_{ij} t^A_{kl} = \frac{1}{2}\delta_{kj}\delta_{il} - \frac{1}{N_C}\delta_{ij}\delta_{kl}$ useful to deal with colour flowing from a fermion line to another:



- In principle, starting from the diagrams one has that the full amplitude can be written as a combination of the colour tensor structures (C_i) that are just strings of colour matrices, and fully contracted Lorentz structures (numbers $K_{i,j}$) for each colour and helicity configurations (j)

$$A_j = \sum_i C_i \cdot K_{i,j}$$

- The same numerical construction can be done also at one (and higher) loop
- so, beyond Dirac spinors, what is needed is just the colour matrix built interfering (contracting) all the present colour structures
- full automation of tree level and one loop amplitude computation has been reached in the recent years
  - ➡ NLO revolution: based on a large number of optimisations in the computation of the kinematical factors ($K_{i,j}$) and the availability of faster computer and larger memories with respect to the past (although it has a prequel as we will see later on)

QCD phenomenology

- Low energy:
  - hadron spectrum as a consequence of the symmetry
  - lattice computations

- effective field theory

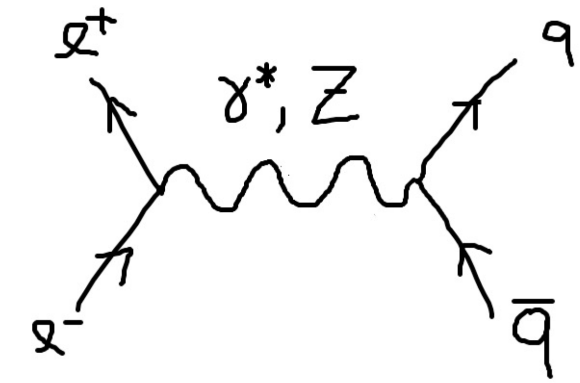
- Finite temperature (see lectures by E. Fraga)

- High energy:
  - perturbation theory with quarks and gluons, although only hadrons appear in the initial and final states
  - the assumption is that the processes of quark extraction from an initial hadron and that of hadron formation do not spoil the predictions
  - this has implications on the definition of observables that can be predicted in perturbation theory

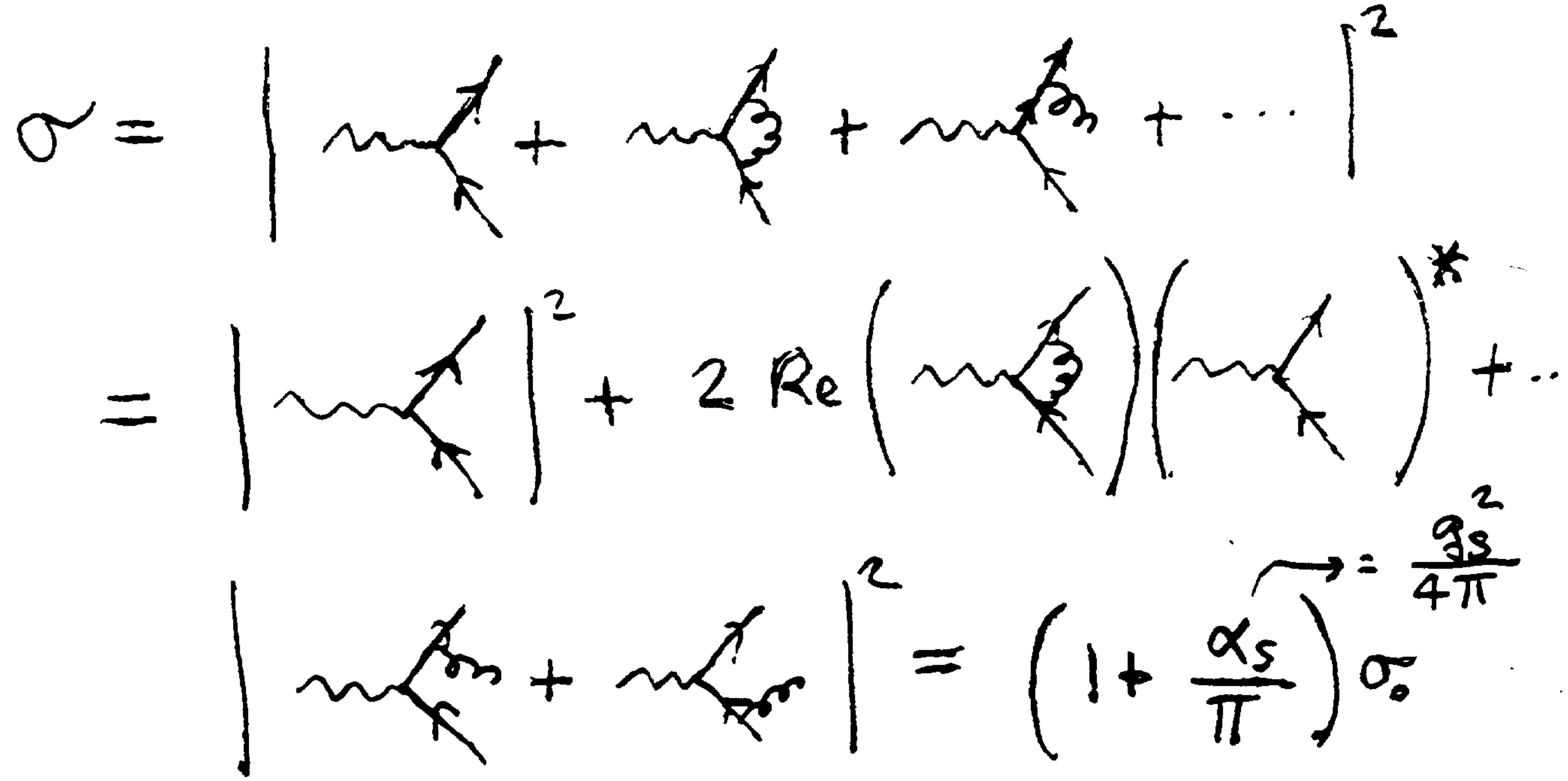
Let's start from the easiest situation: $e^+e^- \rightarrow$ hadrons

- QCD at order α_S^0 (neglecting masses):

$$\frac{\sigma(\gamma^* \to hadrons)}{\sigma(\gamma^* \to \mu^+\mu^-)} = 3 \cdot \left(\overset{u}{\frac{4}{9}} + \overset{d}{\frac{1}{9}} + \overset{s}{\frac{1}{9}} + \overset{c}{\frac{4}{9}} + \overset{b}{\frac{1}{9}} + \ldots \right)$$



- QCD at order α_S :



- same as in QED except for a colour factor $C_F = \frac{4}{3}$

- At order α_S^2:

$$\sigma = \sigma_0 \left(1 + \frac{\alpha_S}{\pi} + \left(c + \pi b_0 \log \frac{M^2}{Q^2}\right) \frac{\alpha_S^2}{\pi^2}\right)$$

- At order α_S^2:

Logarithmically Divergent!
M is an ultraviolet cutoff

$$\sigma = \sigma_0 \left(1 + \frac{\alpha_S}{\pi} + \left(c + \pi b_0 \log \frac{M^2}{Q^2} \right) \frac{\alpha_S^2}{\pi^2} \right)$$

Q is the invariant mass of the system, the scale of the process

$$c = 1.986 - 0.115 n_f \qquad b_0 = \frac{33 - 2n_f}{12\pi}$$

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$$c = 1.986 - 0.115 n_f \qquad b_0 = \frac{33 - 2n_f}{12\pi}$$

- One can show that for any physical quantity G computed as:

$$G = G_0 \alpha_S^n + (\ldots)\alpha_S^{n+1} + \ldots$$

the expansion has the form:

$$G = G_0 \alpha_S^n + \left(G_1 + n G_0 b_0 \log\left(\frac{M^2}{Q^2}\right)\right) \alpha_S^{n+1} + \ldots$$

- Indeed we have:

$$\frac{\sigma}{\sigma_0} - 1 = \frac{1}{\pi}\alpha_S + \left(\frac{c}{\pi^2} + \frac{1}{\pi} b_0 \log\left(\frac{M^2}{Q^2}\right)\right) \alpha_S^2$$

- now define:

$$\widetilde{\alpha}_S(\mu) = \alpha_S + b_0 \log\left(\frac{M^2}{\mu^2}\right)\alpha_S^2$$

- and find that our physical quantity

$$G = G_0\alpha_S^n + \left(G_1 + nG_0 b_0 \log\left(\frac{M^2}{Q^2}\right)\right)\alpha_S^{n+1} + \ldots$$

gets the form (neglecting terms of order α_S^{n+2} and higher):

$$G = G_0\widetilde{\alpha}_S^n(\mu) + \left(G_1 + nb_0 \log\left(\frac{\mu^2}{Q^2}\right)\right)\widetilde{\alpha}_S^{n+1}(\mu) + \ldots$$

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This is renormalization

Renormalisation

- For any physical quantity:

$$G\left(\alpha_S, M, ...\right)$$

coupling UV cutoff physical variables (masses, momenta…)

- We can always define a charge containing the whole dependence on the UV cutoff M:

$$\widetilde{\alpha}_S(\mu, M, \alpha_S) = \alpha_S + c_1(\mu, M)\alpha_S^2 + ...$$

in such a way that:

$$G\left(\alpha_S, M, ...\right) = G'\left(\widetilde{\alpha}_S(\mu, M, \alpha_S), \mu, ...\right)$$

➡ i.e. the physical quantity has a finite expression in terms of $\widetilde{\alpha}_S, \mu$ and the physical variables

➡ we can measure $\widetilde{\alpha}_S$ in one process and use it to make predictions for another process!

Consequences

$$G\left(\alpha_S, M, ...\right) = G'\left(\widetilde{\alpha}_S(\mu, M, \alpha_S), \mu, ...\right)$$

- take derivative on both sides wrt $\log(\mu^2)$

$$0 = \frac{\partial G'\left(\widetilde{\alpha}_S, \mu, ...\right)}{\partial \widetilde{\alpha}_S} \frac{\partial \widetilde{\alpha}_S}{\partial \log(\mu^2)} + \frac{\partial G'\left(\widetilde{\alpha}_S, \mu, ...\right)}{\partial \log(\mu^2)}$$

$$\frac{\partial \widetilde{\alpha}_S}{\partial \log(\mu^2)} = -\frac{\frac{\partial G'(\widetilde{\alpha}_S, \mu, ...)}{\partial \log(\mu^2)}}{\frac{\partial G'(\widetilde{\alpha}_S, \mu, ...)}{\partial \widetilde{\alpha}_S}} = \beta\left(\widetilde{\alpha}_S, \mu\right)$$

- But β is dimensionless, so if it does not depend explicitly on M it cannot depend on explicitly on μ

$$\beta \equiv \beta\left(\widetilde{\alpha}_S\right) = \frac{\partial \widetilde{\alpha}_S}{\partial \log(\mu^2)}$$

- In fact for e+e- we have:

$$\widetilde{\alpha}_S(\mu) = \alpha_S + b_0 \log\left(\frac{M^2}{\mu^2}\right)\alpha_S^2 \quad \Longrightarrow \quad \frac{\partial \widetilde{\alpha}_S}{\partial \log \mu^2} = -b_0 \widetilde{\alpha}_S^2 + \mathcal{O}\left(\widetilde{\alpha}_S^3\right)$$

Consequences (summary)

- any physical quantity can be given as an expansion in $\widetilde{\alpha}_S$:

$$G'\left(\widetilde{\alpha}_S, \mu, ...\right) = \sum_i G_i(\mu, ...)\widetilde{\alpha}_S^i$$

- if we change μ and α_S in such a way that:

$$\delta\widetilde{\alpha}_S = \beta(\widetilde{\alpha}_S)\delta\log\mu^2$$

the prediction of physical quantities do not change (Renormalisation Group Invariance)

Back to e+e- cross section

(drop the tilde from now on)

$$\sigma = \sigma_0 \left(1 + \frac{\alpha_S(\mu)}{\pi} + \left(c + \pi b_0 \log \frac{\mu^2}{Q^2} \right) \frac{\alpha_S^2(\mu)}{\pi^2} \right)$$

- choosing $\mu = Q$ (or taking the two in a fixed ratio), the energy dependence of the cross section is entirely described by the running of α_S

- better to choose $\mu \sim Q$ otherwise second order may become larger then the first (and subsequent orders even larger!)

- up to now just a rewriting

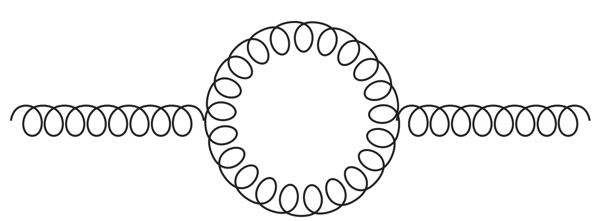
Running coupling

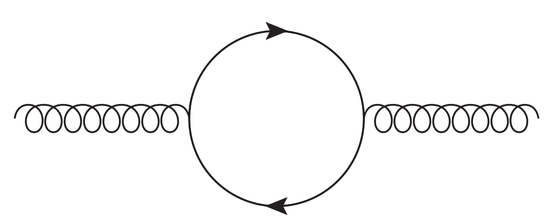
Equation to solve: $$\frac{\partial \alpha_S}{\partial \log \mu^2} = -b_0 \alpha_S^2$$

$$\frac{\partial}{\partial \log \mu^2} \frac{1}{\alpha_S} = b_0 \xrightarrow{\text{integrating}} \frac{1}{\alpha_S} = b_0 \log \mu^2 + cost.$$

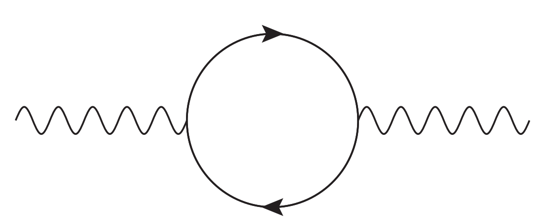
that is: $$\alpha_S = \frac{1}{b_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)}$$

- the logarithmic divergences come mainly from bubble graphs





QCD: $b_0 = \dfrac{33 - 2n_f}{12\pi} > 0$



QED: $b_0 = -\dfrac{4n_f}{12\pi} < 0$

Running coupling:

- In QED we have $\alpha(m_e) = \alpha_{QED}$, so:

$$\alpha_{QED} = \frac{1}{\left(-\frac{4n_f}{12\pi}\right)\log\left(\frac{m_e^2}{\Lambda_{QED}^2}\right)} \quad \Rightarrow \quad \Lambda_{QED}^2 = m_e^2\, exp\left(\frac{3\pi}{n_f \alpha_{QED}}\right)$$

- that is a huge scale, that's why we never talk about Λ_{QED}

- In QCD, Λ has to be the typical hadronic scale (~ 500 MeV), the scale at which the hadronic systems become strongly coupled:

  ➡ let's predict the value of α_S at higher scales with $\alpha_S = \frac{1}{b_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)}$

- assuming $\Lambda \sim 100 \div 500 MeV$ we have:

$\alpha_S(M_Z) = 0.1 \div 0.13 \ (\pm 13\%)$ and $\alpha_S(10^7 GeV) = 0.040 \div 0.044 \ (\pm 5\%)$

Status of $e^+e^- \to$ hadrons total cross section

$$\frac{\sigma}{\sigma_0} = 1 + \frac{\alpha_S}{\pi}\left(1 + 0.448\alpha_S - 1.30\alpha_S^2 - 2.59\alpha_S^3\right) + \ldots$$

[Baikov et al 2012]

- Coefficients for nf=5 (general expression very complex)
- $\alpha_S = \alpha_S^{\overline{MS}}(Q)$
- corrections are well behaved: on the Z peak $\alpha_S \sim .12$ so:

$$\mathcal{O}(\alpha_S^2) \sim 5\%\,, \quad \mathcal{O}(\alpha_S^3) \sim 2\% \quad \text{and } \mathcal{O}(\alpha_S^4) \sim 4‰$$

➡ In principle we could determine α_S with an accuracy better then 1%

- β function: $$\frac{\partial \alpha_S}{\partial \log \mu^2} = -b_0\alpha_S^2 - b_1\alpha_S^3 - b_2\alpha_S^4 - b_3\alpha_S^5$$ [Czakon 2005]

$$b_0 = \frac{33 - 2n_f}{12\pi}, \quad b_1 = \frac{153 - 19n_f}{24\pi^2}, \quad b_2 = \frac{2857 - \frac{5033n_f}{18} + \frac{325n_f^2}{54}}{(4\pi)^4}$$

➡ generally used: $\alpha_S(\mu) = \dfrac{1}{b_0 \log \frac{\mu^2}{\Lambda^2}}\left(1 - \dfrac{b_1 \log\log \frac{\mu^2}{\Lambda^2}}{b_0^2 \log \frac{\mu^2}{\Lambda^2}}\right)$

α_S determination: $R = \dfrac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ below the Z peak

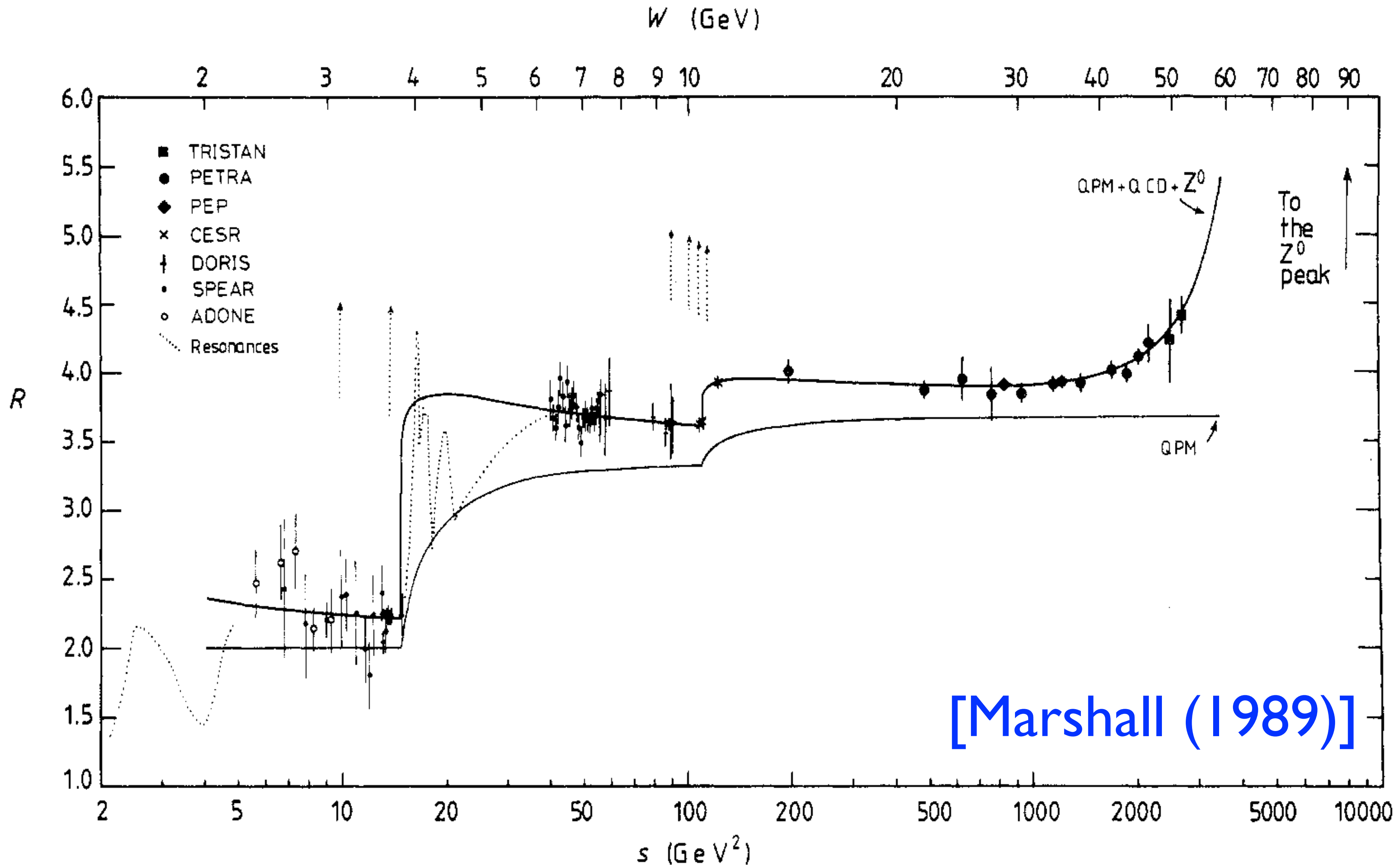


Figure 8. A comparison of R measurements with global fit.

$$f(\beta) = \frac{1}{2}\beta(3 - \beta^2)$$

Threshold velocity term
(quarks are not massless)

- large errors in each experiment
- NLO correction clearly needed
- combining results from 20 to 65 GeV:

$$\alpha_S(35 GeV) = 0.146 \pm 0.030 \longrightarrow \alpha_S(M_Z) = 0.124 \pm 0.021$$

➡ although not very sensitive to QCD effects
(perturbative expansion start at α_S^0)

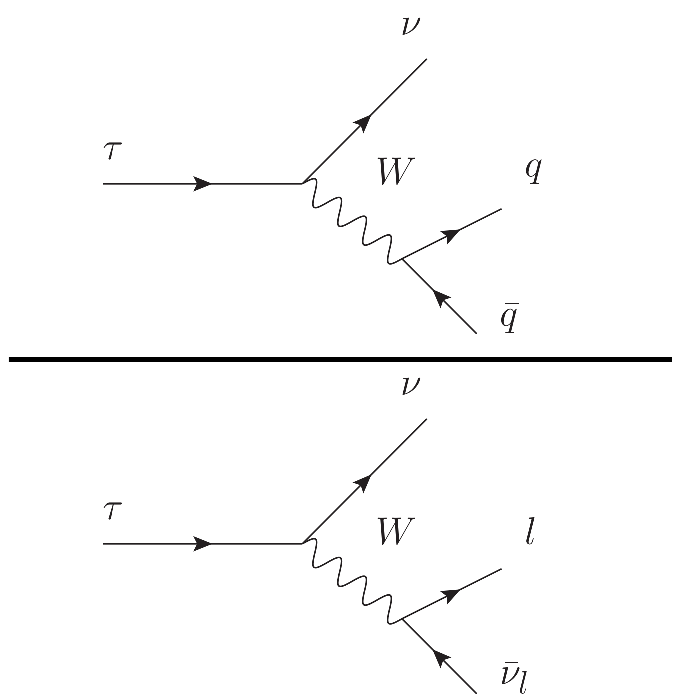
α_S determination:

- measuring the total hadronic decay width at the Z pole at LEP

$$\Gamma(Z^0 \to had) \ \longrightarrow \ \alpha_S(M_Z) = 0.122 \pm 0.009$$

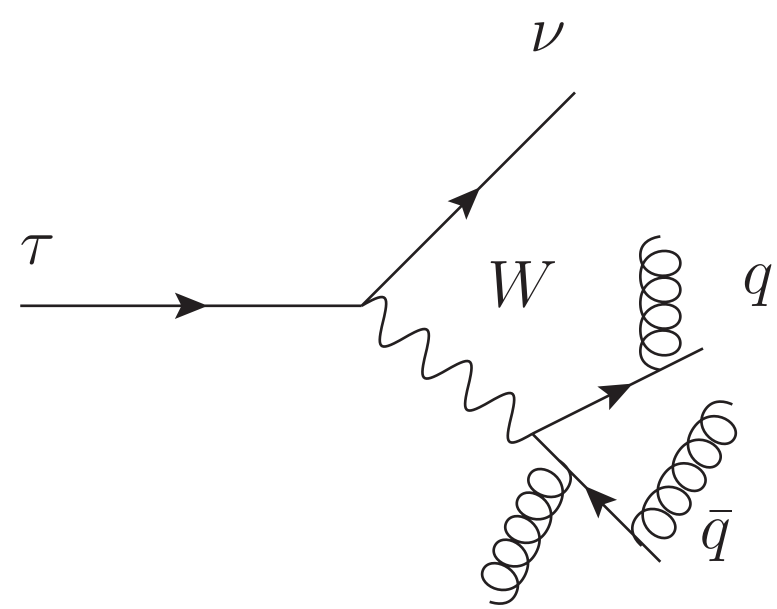
- comparing leptonic and hadronic tau decays one has:

α_S^0)



= 3

α_S)



- here QCD corrections desplaces value from 3

➡ $\alpha_S(M_\tau) = 0.36 \pm 0.05 \ \longrightarrow \ \alpha_S(M_Z) = 0.122 \pm 0.005$

How to compute more features of the final state

- indeed, we have combined diagrams for producing two quarks and two quarks and a gluon in the final state, that have never been observed

- leading final state particles bring colour quantum number that has never been observed neither

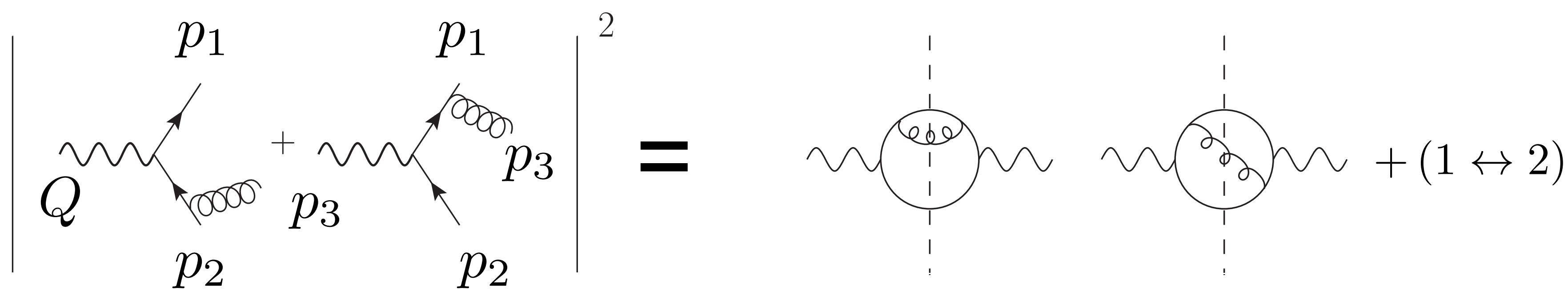
- we have to find the conditions that allows for the description of the final state in term of an evolution from the elementary processes (among quark and gluons) to the hadrons that are observed

- let's go through the details of the radiative corrections for $e^+e^- \rightarrow$ hadrons

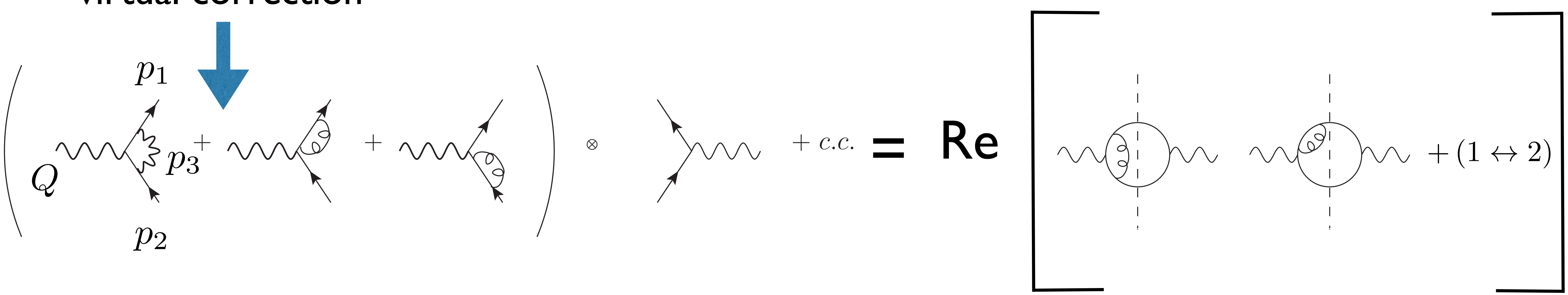
α_S corrections $e^+e^- \to$ hadrons

- first perturbative correction to e^+e^- annihilation into hadrons (NLO QCD) is computed adding real and virtual emission diagrams
- that guarantees that the result is finite
- the technical difficulty is that the contributions leave in different phase spaces

- real emission



- virtual correction



- we already know that the correction to the total cross section is finite from unitarity

  and from explicit calculation it is: $\left(\sigma_0 \dfrac{\alpha}{\pi}\right)$

Real radiation closely

- energy fractions: $x_i = \dfrac{2p_i \cdot Q}{Q^2} \quad = \quad \dfrac{2E_i}{Q}$ (c.m. frame) $\qquad x_i > 0$

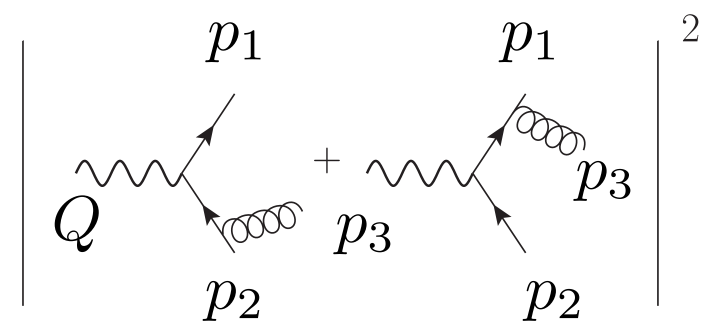
- energy conservation: $x_1 + x_2 + x_3 = \dfrac{2\sum_i p_i \cdot Q}{Q^2} = 2$

- angles: $2\,p_1 \cdot p_3 = (p_1 + p_3)^2 = (Q - p_2)^2 = Q^2 - 2\,p_2 \cdot Q$

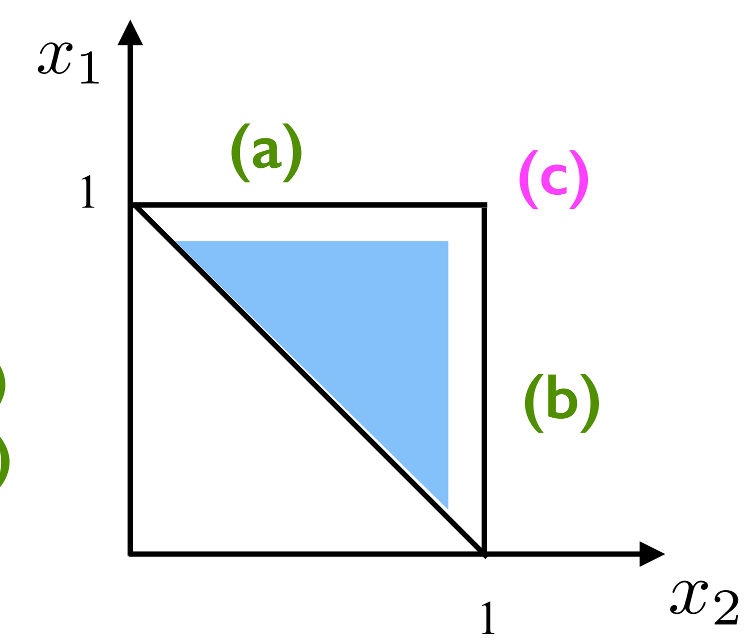
  ➡ $2\,E_1 E_3 (1 - \cos\theta_{13}) = Q^2 (1 - x_2) \qquad x_i < 1$

  ➡ $\theta_{13} \to 0 \quad \Longleftrightarrow \quad x_2 \to 1$

$$\sigma^R = \int_0^1 dx_1 dx_2 dx_3 \delta(2 - x_1 - x_2 - x_3) |M_R(x_1, x_2, x_2)|^2$$



$$|M_R(x_1, x_2, x_2)|^2 = \sigma_0 C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}$$



- singular:

$x_1 \to 1 \quad \Longleftrightarrow \quad \theta_{23} \to 0$ collinear (a)

$x_2 \to 1 \quad \Longleftrightarrow \quad \theta_{13} \to 0$ collinear (b)

$\dfrac{1 - x_1}{1 - x_2} \sim const.$ $\; x_3 \to 0 \quad \Longleftrightarrow \quad E_3 \to 0$ soft (c)

Real radiation closely

- in reality, from the running of the strong coupling and the experimental evidence we already know that there is a physical cut-off for the energy fraction, a change of regime:

$$\varepsilon \sim \frac{M_{had}}{Q} \sim \frac{\Lambda_{QCD}}{Q}$$

$$\infty = \alpha_S(Q) \int_0^1 \frac{dx}{1-x} \quad \Longrightarrow \quad \alpha_S(Q) \int_0^{1-\varepsilon} \frac{dx}{1-x} \sim \alpha_S(Q) \log \frac{1}{\varepsilon} \sim \alpha_S(Q) \frac{1}{\alpha_S(Q)}$$

including more orders…

$$➡ \quad \sigma \sim \sigma_0 \left(1 + \alpha_S(Q) \frac{1}{\alpha_S(Q)} + ...\right) \sim \sigma_0 (1 + 1 + ...)$$

- ultimately the divergences that we see are related to the non perturbative regime of the theory

- non perturbative physics is logarithmically enhanced

Real radiation closely

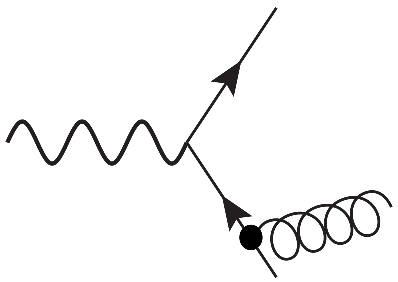
- structure of the divergences:

$$\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} = \frac{1+(1-x_3)^2}{(1-x_1)(1-x_2)} - 2 \qquad \frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left(\frac{1}{1-x_1}+\frac{1}{1-x_2}\right)$$

$$d\sigma_R = \sigma_0 dW_{13} + \sigma_0 dW_{23} + \; finite\, terms$$

- splitting probability:

$$P_{qg}(x_3) = C_F \frac{\alpha_S}{2\pi}\frac{1+(1-x_3)^2}{x_3}$$



$$\sim \sigma_0 dW_{23}$$

$$dW_{23} = \frac{dx_1}{1-x_1} dx_3 P_{qg}(x_3) = \frac{d\cos\theta_{23}}{1-\cos\theta_{23}} dx_3 P_{qg}(x_3) \xrightarrow[E_3\to 0]{\theta_{23}\to 0} \frac{d\theta_{23}^2}{\theta_{23}^2}\frac{dE_3}{E_3}$$

virtual

$$\sigma_R + \sigma_V = finite + \sigma_0 \int_{-1}^{+1} \frac{d\cos\theta_{23}}{1-\cos\theta_{23}} \int_0^1 dx_3 P_{qg}(x_3)\left[\frac{1}{1-\frac{x_3(1-\cos\theta_{23})}{2}} - 1\right] + (1 \leftrightarrow 2)$$

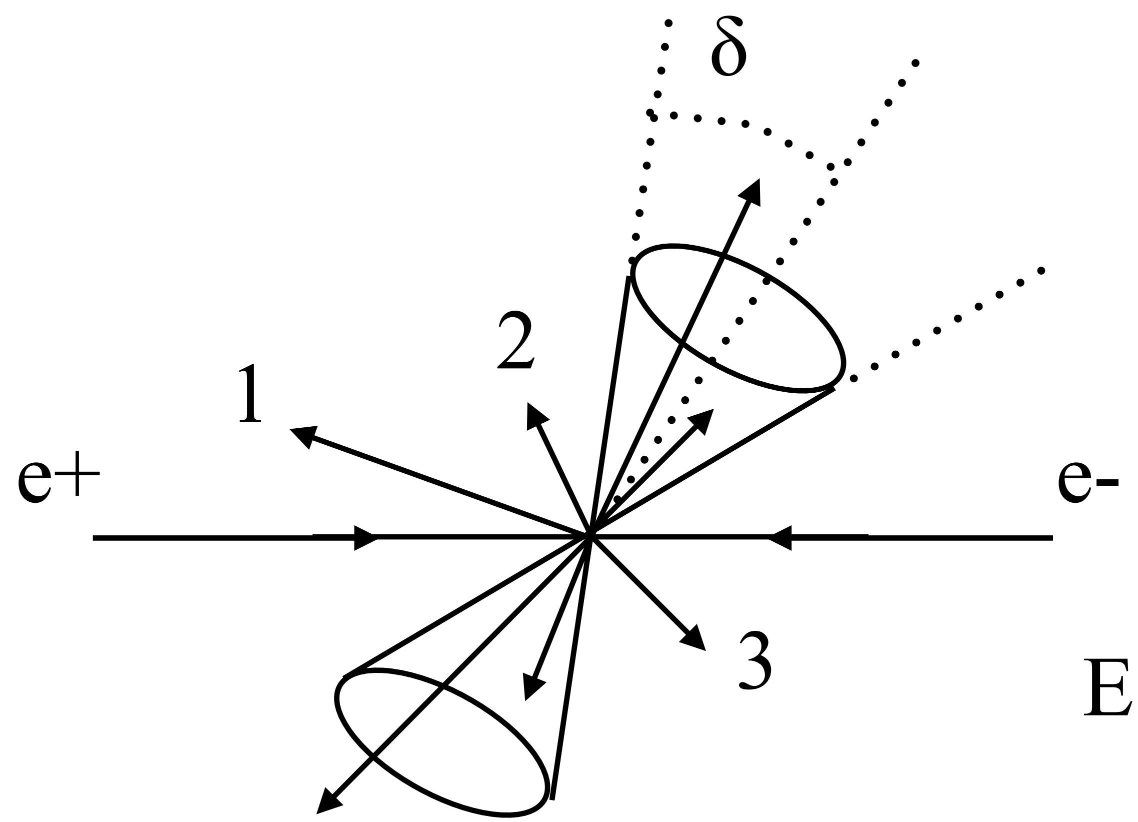
$$\left[\frac{1}{1-\frac{x_3(1-\cos\theta_{23})}{2}} - 1\right] = \frac{\frac{x_3(1-\cos\theta_{23})}{2}}{1-\frac{x_3(1-\cos\theta_{23})}{2}} \xrightarrow[\substack{x_3\to 0\\ \theta_{23}\to 0}]{} x_3\theta_{23}^2$$

The only way to produce a differential prediction is to use variables such that the contribution of the virtual and the divergent part of the real are indistinguishable (KLN theorem in QM)

Sterman-Weinberg jets

- first example of infrared and collinear safe variable for e+e- annihilation

- the cross section for the production of Sterman-Weinberg jets is the cross section for producing events in which all but at most a fraction ε of the total energy (E) is contained in two opposite cones of size δ

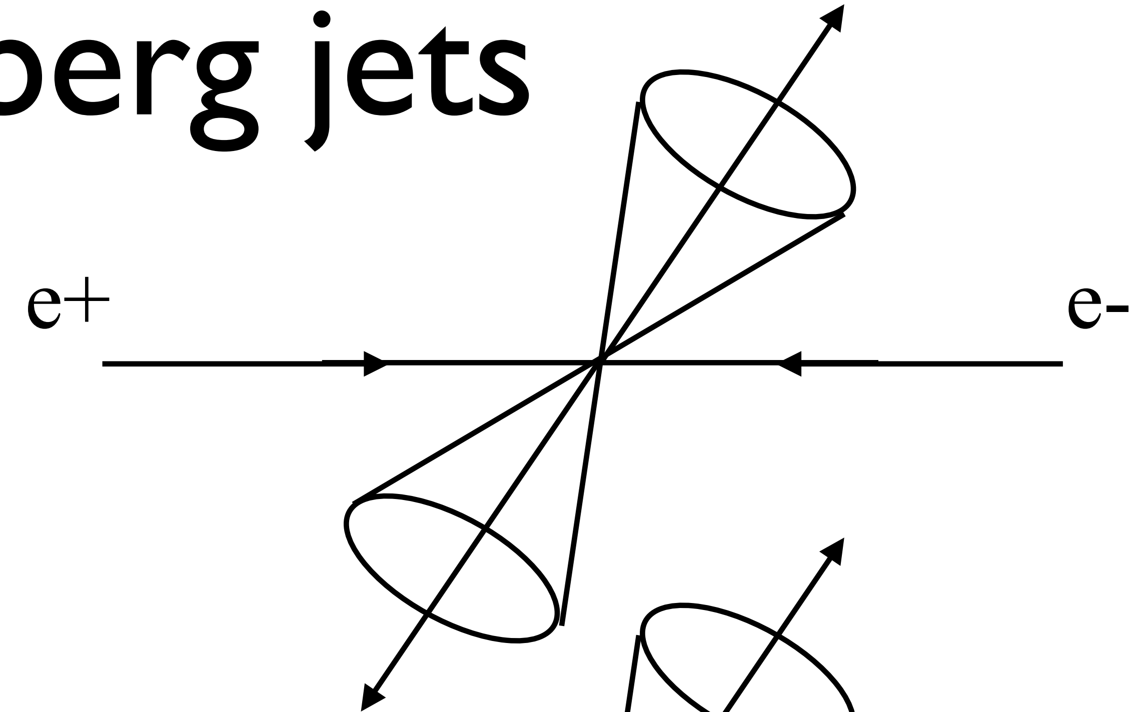


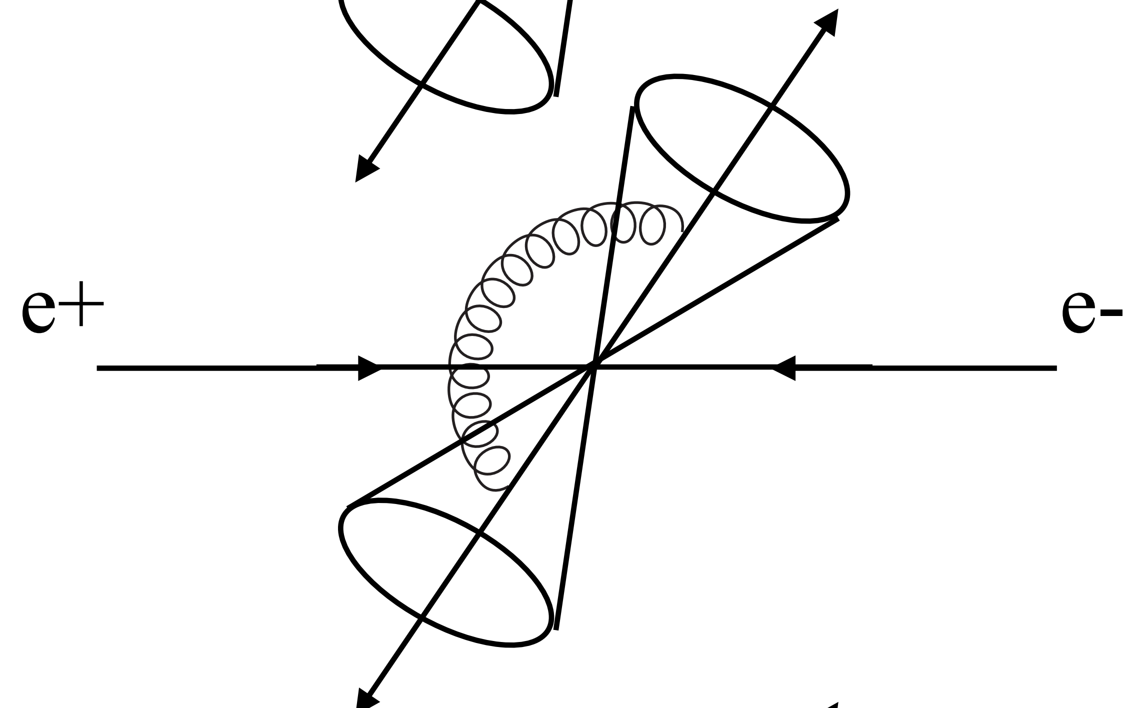
$$E_1 + E_2 + E_3 < \varepsilon E$$

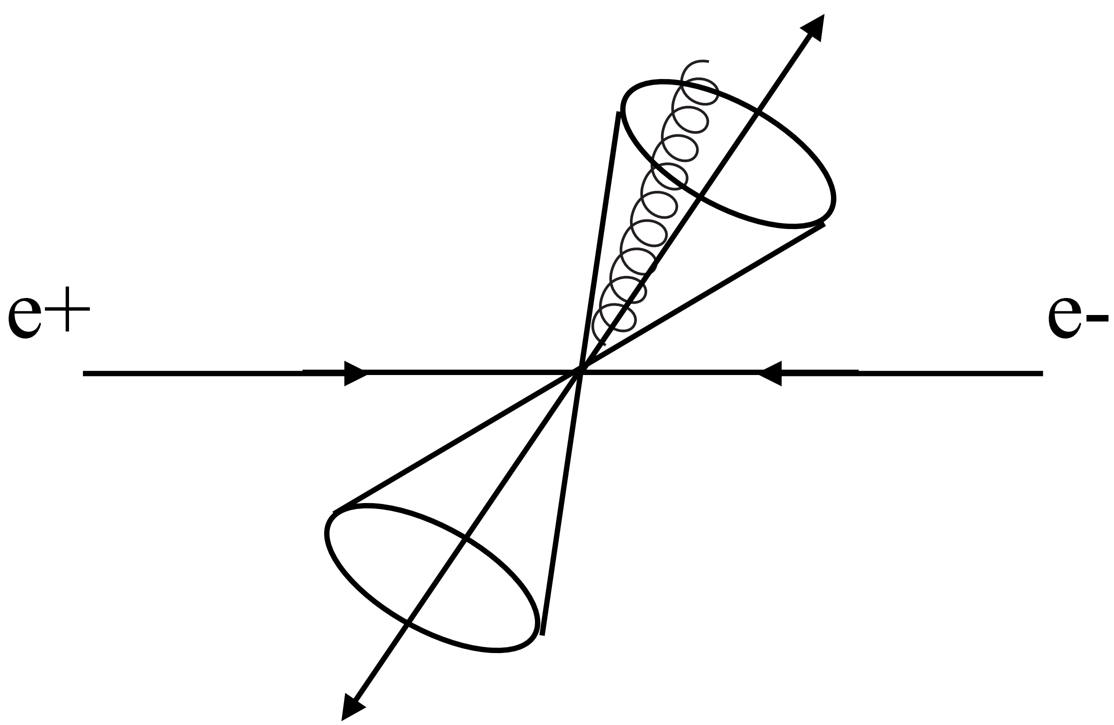
- an event will contribute to the Sherman-Weinberg cross section if we can find a cone that satisfies the above conditions

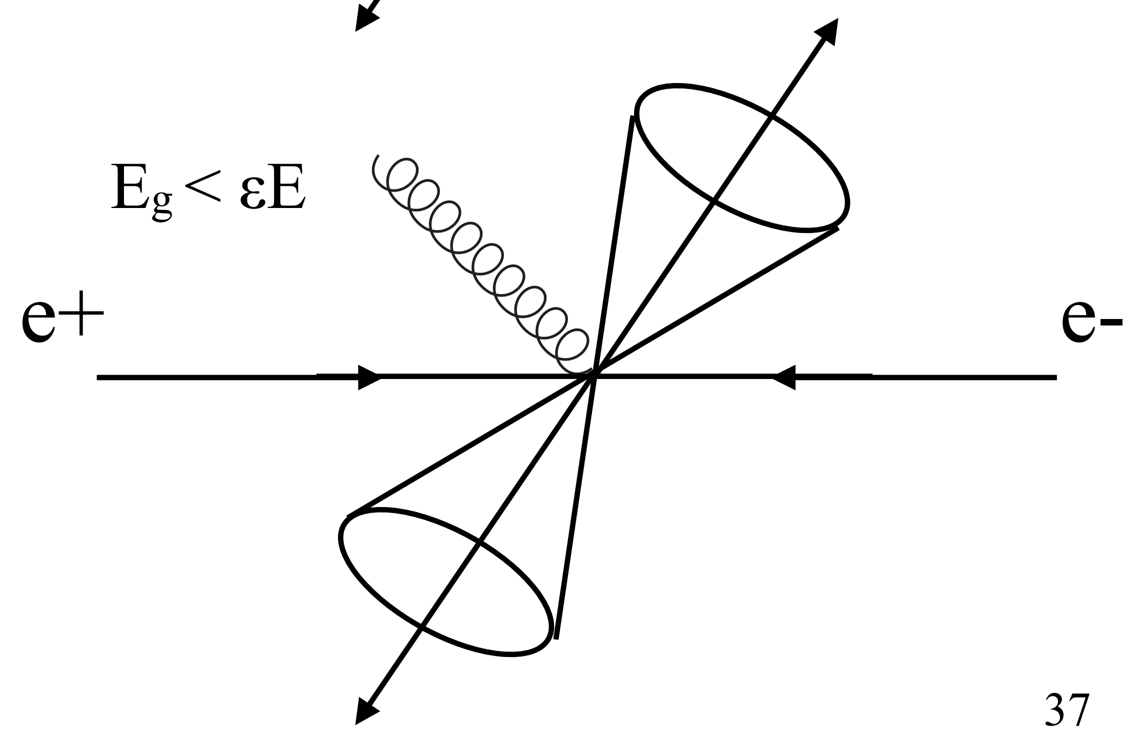
Sterman-Weinberg jets

- let’s see what enters in this jet definition:
  - ➡ all the Born cross section contributes, being a two particle final state
  - ➡ that’s true for the virtual as well, of course
  - ➡ as for the real radiation we include all the collinear emission and also all the soft one!









Sterman-Weinberg jets

- from the computational point of view the only case that has not to be included is just a part of the real contribution where the radiated particle is outside the cone and has a fractional energy greater then ε

- using the expression:

$$\sigma_R + \sigma_V = finite + \int_{-1}^{+1} \frac{d\cos\theta_{23}}{1-\cos\theta_{23}} \int_0^1 dx_3 P_{qg}(x_3) \left[\frac{1}{1 - \frac{x_3(1-\cos\theta_{23})}{2}} - 1 \right] + (1 \leftrightarrow 2)$$

  with a bit of algebra one finds that the accepted cross section is well approximated by:

$$\sigma = \sigma_0 \left(1 - \frac{4C_F\alpha_S}{2\pi} \log\varepsilon \log\delta^2 \right) + \ subleading\ terms$$

- for reasonable values of ε and δ almost all the cross section is made of Sterman-Weinberg jets, we are assuming that qualitatively quarks and gluons describe the average behaviour of the hadrons we measure

- at high energy the coupling decrease, ε and δ can be chosen smaller so jets become thinner

- the 2 jet angular distribution will be the one of σ_0, that is the typical $1+\cos^2(\theta)$

Subtraction methods

$$\sigma_{NLO} = \int_m d\sigma_V + \int_{m+1} d\sigma_R$$

- separately divergent on different phase spaces

- thanks to the factorisation of the amplitudes in the IR and C limits is it possible to define auxiliary cross sections that has the same divergences of the real contribution, but are integrable over the degrees of freedom of the radiated parton

$$\sigma_{NLO} = \int_m \left[d\sigma_V + \int_1 d\sigma_A \right] + \int_{m+1} [d\sigma_R - d\sigma_A]$$

[Ellis, Ross, Terrano 1980]

- the explicit poles of the virtual amplitude are explicitly cancelled in the first integral, while the divergences of the real matrix element in the second integral are cancelled locally

- the systematic of this construction can be hard (there are several versions), but has to be done only once for every NLO computation (Catani-Seymour, Frixione-Kunszt-Signer)

- several extension to compute NNLO corrections developed (NNLO revolution)

InfraRed and Collinear safety

- let's refine the statement:

  ➡ in every computation that is not just the total cross section, the differential cross section is convoluted with a phase space function (F) that defines the physical quantity we want to compute (including experimental cuts)

$$\sigma_B = \int_m d\sigma_B \qquad d\sigma_B = d\phi^{(m)} |M_m^{(tree)}|^2 F^{(m)}(p_i)$$

  ➡ in both real and virtual, divergences come from infrared and collinear emissions

$$\sigma_{NLO} = \int_m \left[d\sigma_V + \int_1 d\sigma_A \right] F^m(p_i) + \int_{m+1} \left[d\sigma_R \, F^{m+1}(p_i) + d\sigma_A \, F^m(p_i) \right]$$

- IR/C cancellation will take place if and only if :

$$F^{m+1}(..., p_i, ..., p_j, ...) \simeq F^m(..., p_i + p_j, ...)$$

soft limit $\quad p_i \to 0 \qquad \longrightarrow$ IR safety

collinear limit $\quad p_i // p_j \qquad \longrightarrow$ collinear safety

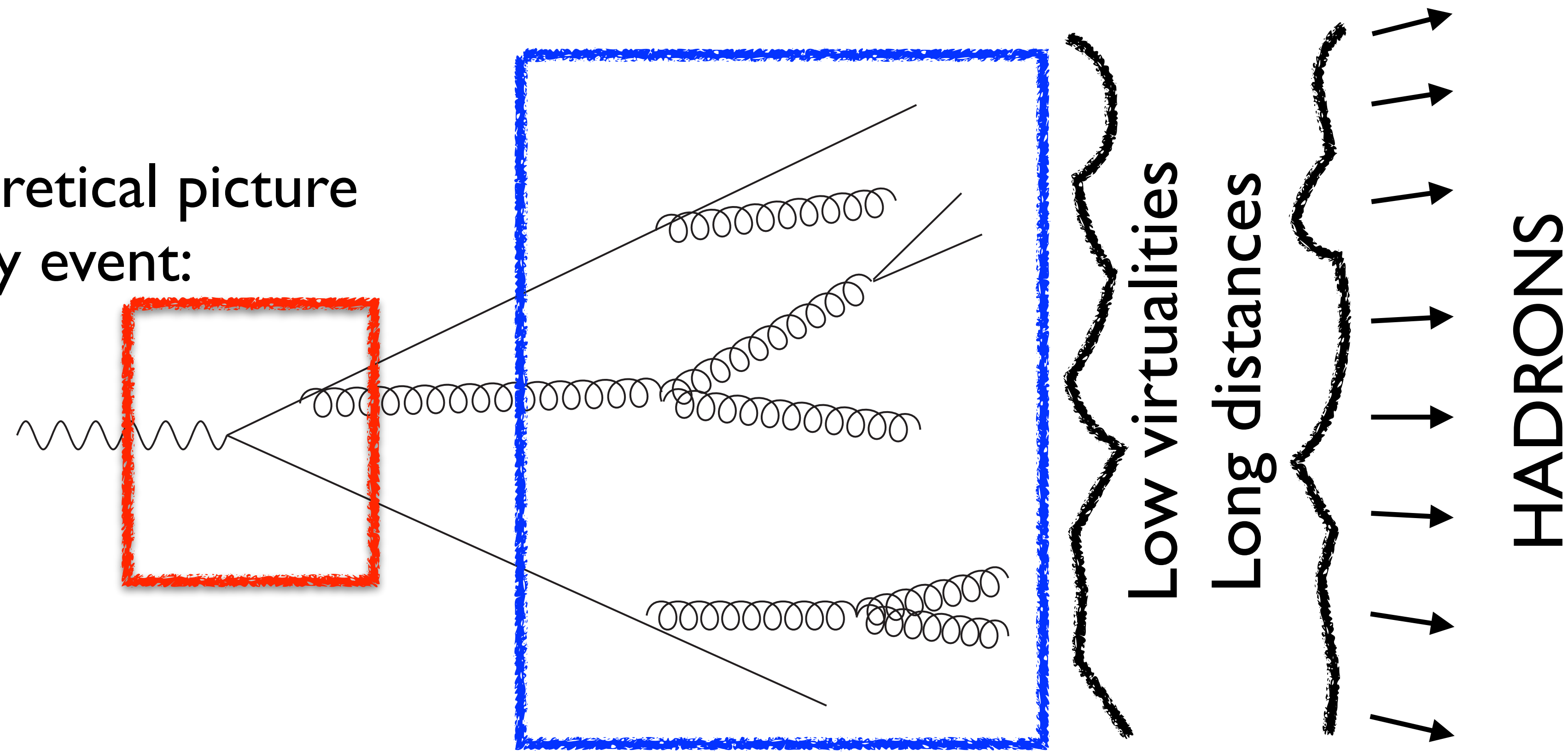
[Sterman, Weinberg 1977]

- ▸ the measured/computed quantity cannot resolve long-distance phenomena
- ▸ its value should remain the same by adding a (many) soft particle or replacing a particle by two (many) collinear particles
- ▸ having cancelled the logarithmic enhancements from real and virtual, non perturbative physics is power suppress!

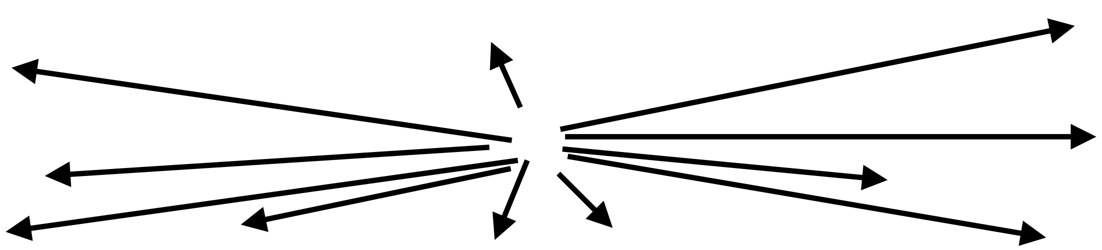
$e^+e^- \to$ hadrons at LEP

- underlining theoretical picture of an high energy event:

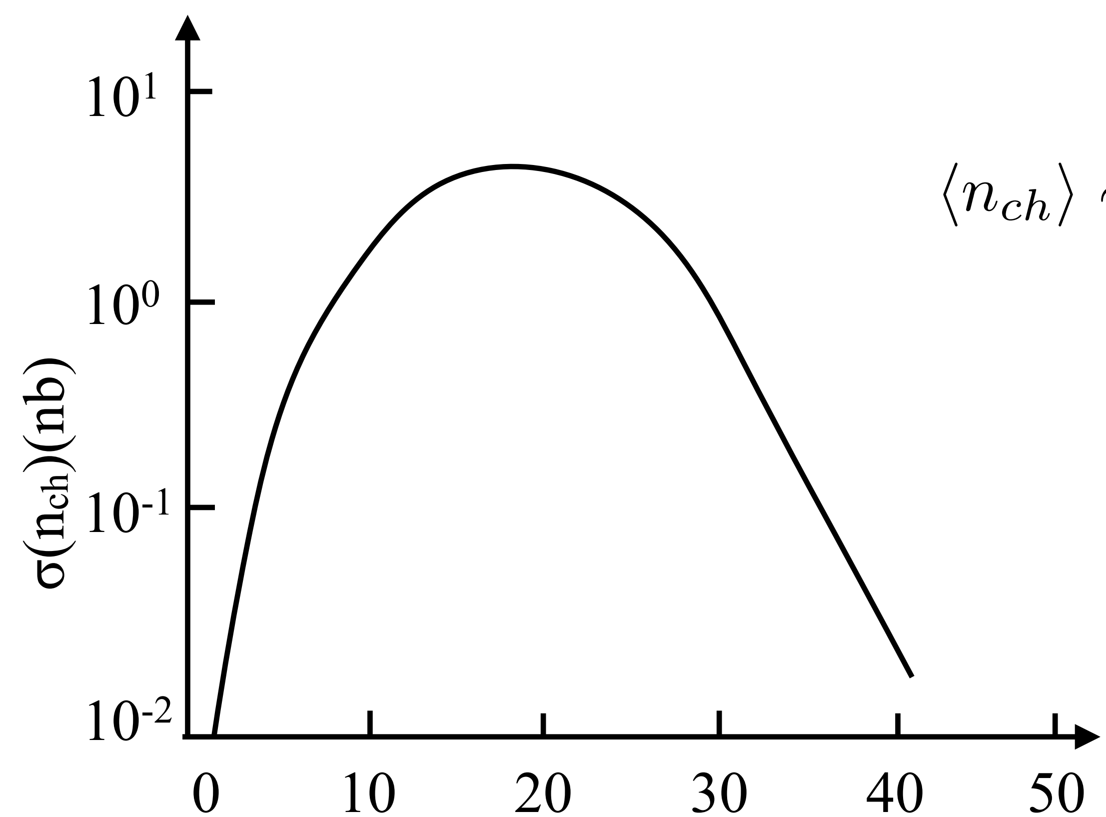
▸ perturbative
▸ approx. pert.
▸ model



- typical event observed



- large multiplicity with large fluctuations



- average mass of the fattest jet $\approx 20\text{GeV}$

➡ QCD predicts:

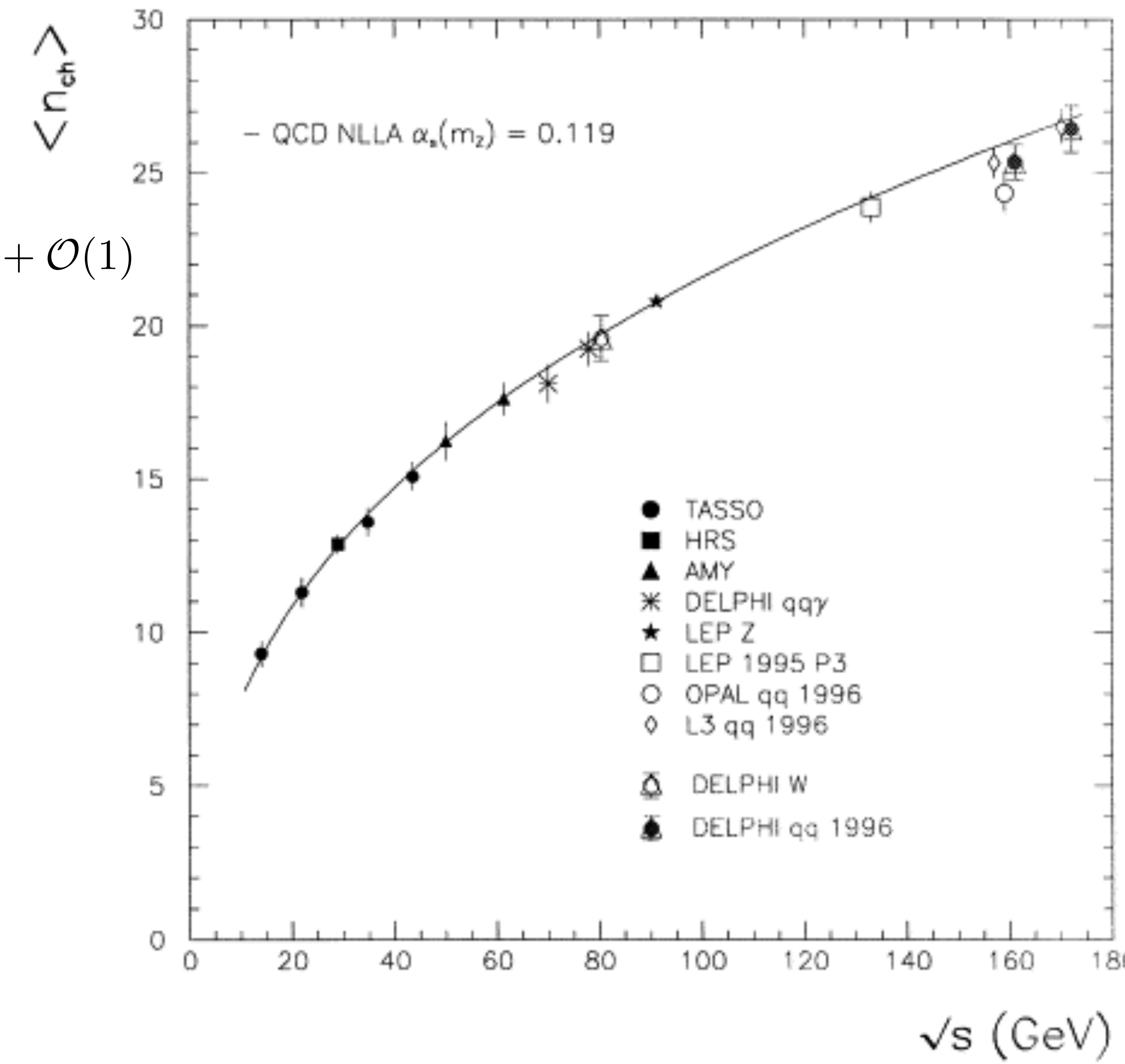
$$\langle \frac{M_j^2}{s} \rangle = \frac{\alpha}{\pi} 1.05 + \mathcal{O}(\alpha^2)$$

$e^+e^- \to$ hadrons at LEP

- hadron multiplicity can be computed assuming that hadron formation happens at low scale of the order of 1GeV and assuming that perturbation theory is valid up to slightly above that scale

- the scaling with the total energy of the average number of parsons at the end of the parton shower cascade can be predicted and is in good agreement with the average number of hadrons

$$\langle n \rangle = \frac{\sqrt{96\pi}}{\beta\sqrt{\alpha_S(Q^2)}} + \left(\frac{1}{4} + 10\frac{n_f}{27\beta}\right)\log \alpha_S(Q^2) + \mathcal{O}(1)$$



Thrust

- event shape distribution
- for each event search for: $$T = Max_n \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$$
- let's add an extra particle to the final state:

➡ in the soft limit for this extra particle it drops out from both the numerator and the denominator

$$T_{m+1} \to T_m \qquad \text{IR safe}$$

➡ in the limit in which the extra particle becomes collinear to another one:

$$p_i = zp, \qquad p_j = (1-z)p, \qquad p_i + p_j = p$$

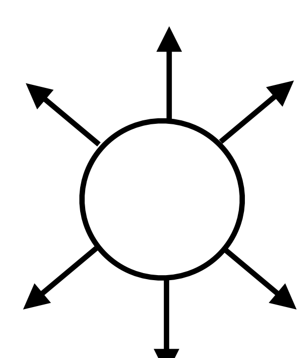
$$|p_i \cdot n| + |p_j \cdot n| = [z + (1-z)]|p \cdot n| = |p \cdot n|$$

$$T_{m+1} \to T_m \qquad \text{Coll. safe}$$

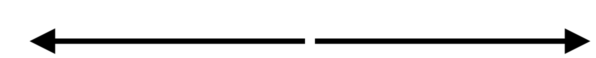
➡ key ingredient is the linearity in particle momenta (many other variables have been defined)

- allowed kinematical range

T=1/2
isotropic event



T=1



2-jet event

Thrust (perturbative uncertainty)

- in perturbation theory the Thrust distribution in e+e- annihilation has the form

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \delta(1-T) + \frac{\alpha_S(\mu)}{2\pi}A(T) + \left(\frac{\alpha_S(\mu)}{2\pi}\right)^2 \left[A(T)2\pi b_0 \log\frac{\mu^2}{Q^2} + B(T)\right] + \mathcal{O}(\alpha_S^3)$$

- radiative corrections are in general quite large

$$\langle 1-T\rangle = \frac{1.05}{\pi}\alpha_S(Q)(1+3\alpha_S(Q))$$

- a common method to estimate the theoretical error due to the truncation is to vary the scale (scale uncertainty)

$$\frac{Q}{2} < \mu < 2Q$$

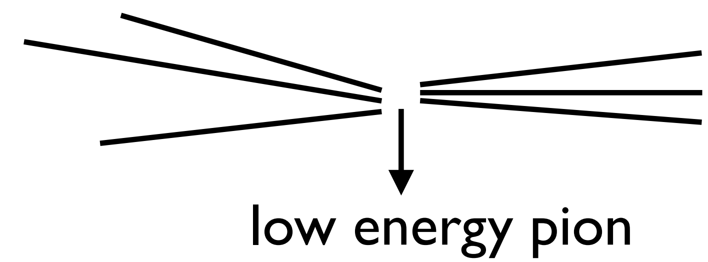
- Higher order terms have to compensate for the variation because physical quantities are scale independent. So higher order terms are at least as large as this variation

Thrust (hadronization effects)

- hadronization is a soft phenomenon that gives power corrections Λ/Q

- for the Trust distribution the emission of a pion with few hundred MeV transverse momentum involving the strong coupling at such small scales has probability 1

➡ effect on Thrust measurement at LEP



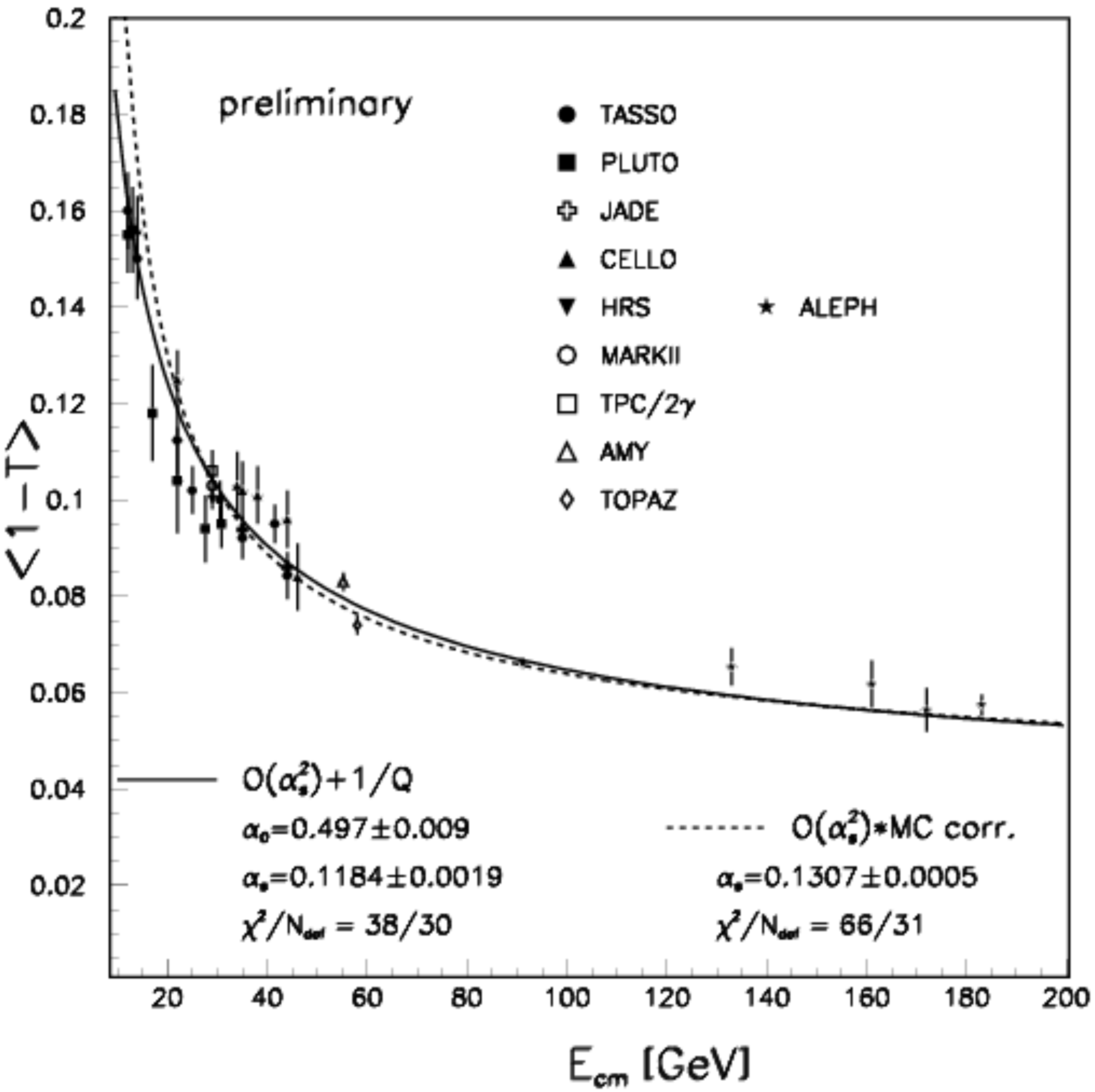
$$\delta T = \frac{\sqrt{m_\pi^2 + p_T^2}}{Q} \simeq \frac{.5}{100}$$

$$\langle 1 - T \rangle \sim \frac{\alpha}{\pi} \sim .06$$

$$\frac{\delta T}{\langle 1 - T \rangle} \sim 8\%$$

- this effects is parametrised fitting many data with hadronization models

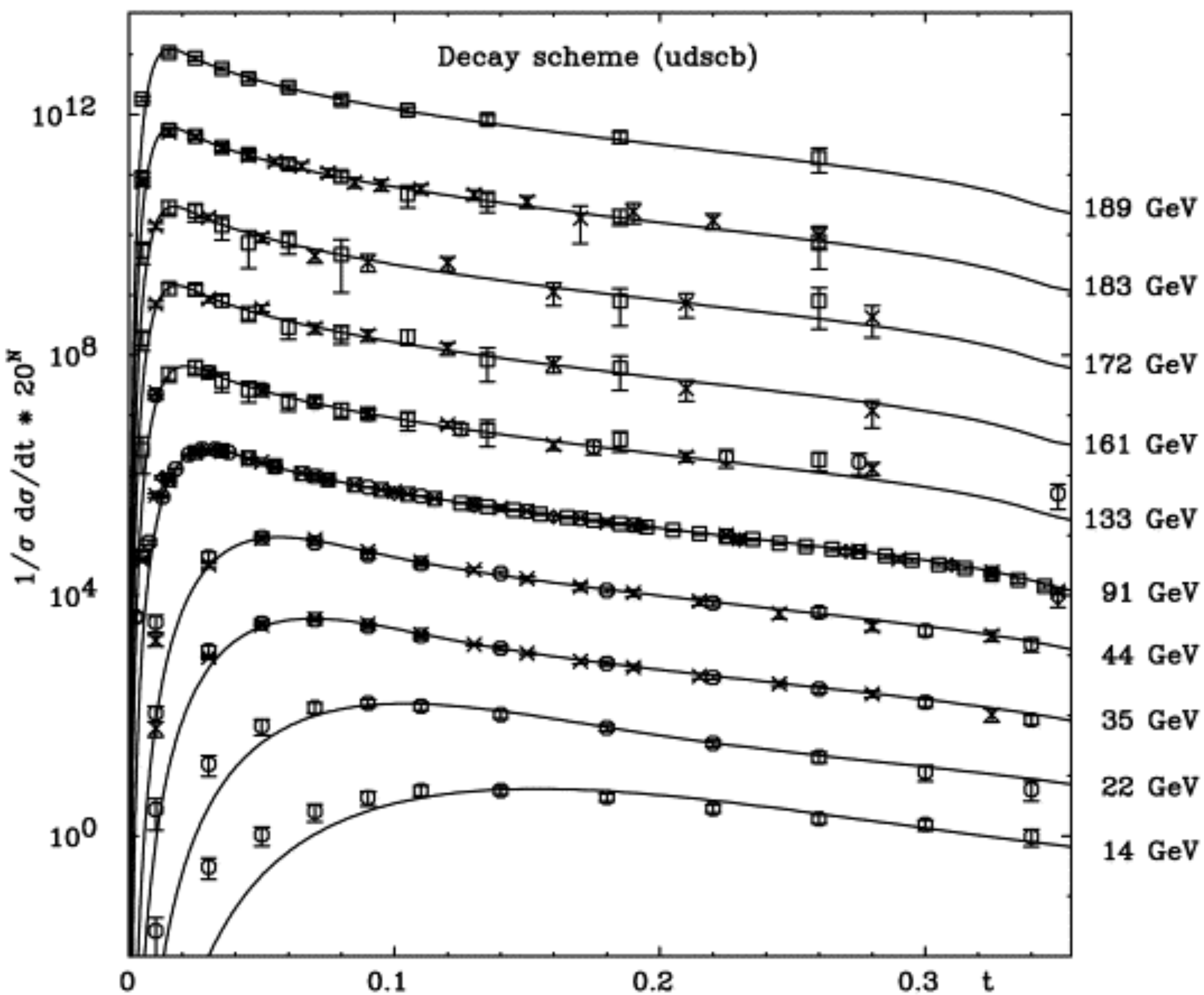
Thrust (data/theory comparison)



- Theory expectation:

$$E_{CM} \to \infty \qquad \Longrightarrow \qquad \langle T \rangle \to 1 \quad (\langle 1-T \rangle \to 0)$$

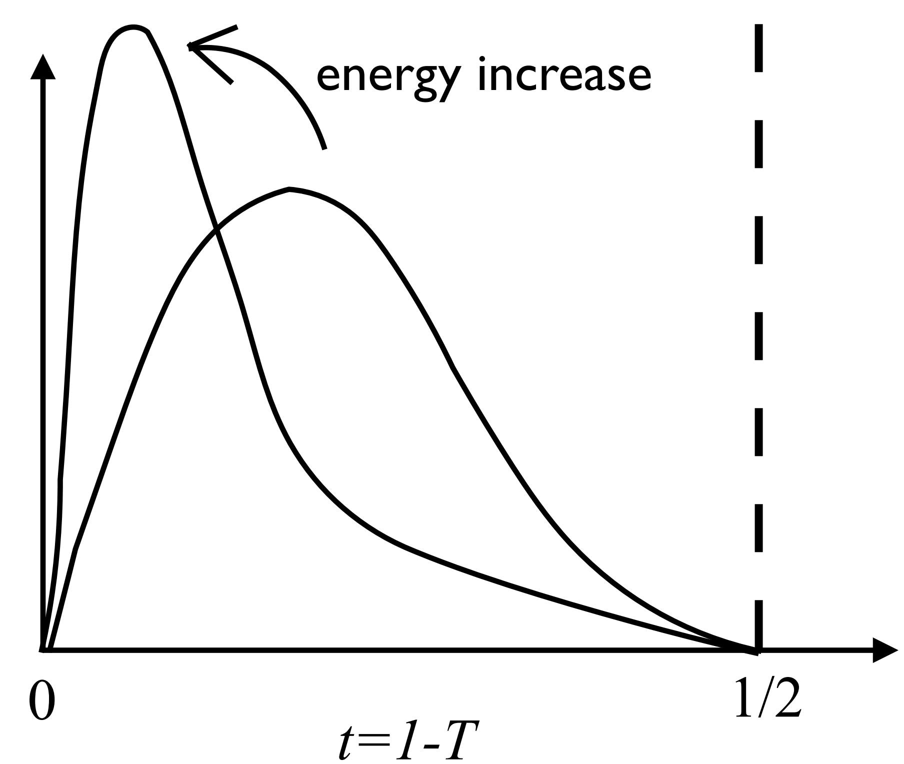
Thrust (data/theory comparison)



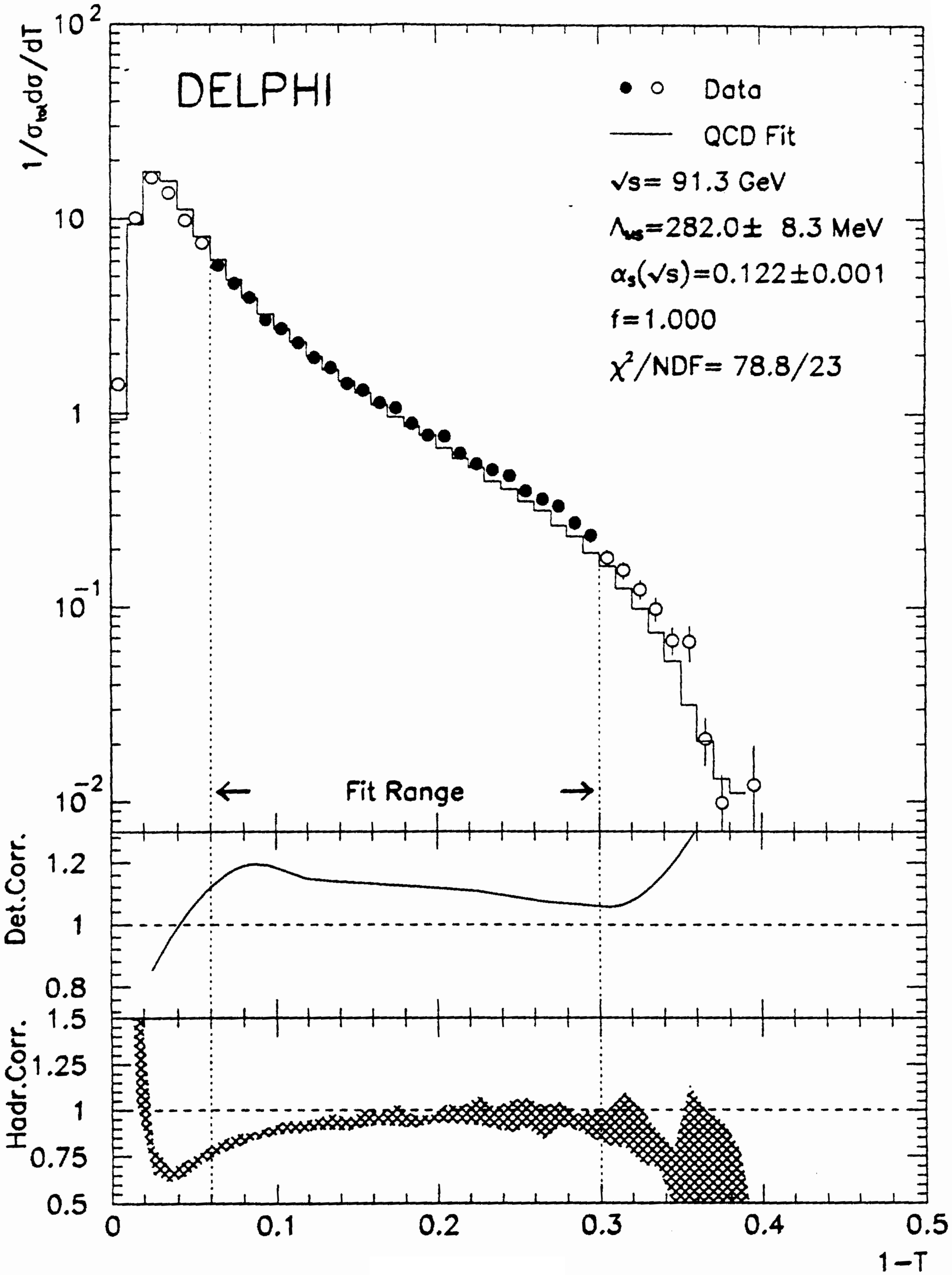
- expectation for distribution of $t = 1 - T$

$$\frac{1}{\sigma}\frac{d\sigma}{dT} \underset{E_{CM}\to\infty}{\sim} \delta(1-T)$$

$$\frac{1}{\sigma}\frac{d\sigma}{dt} \sim \delta(t)$$



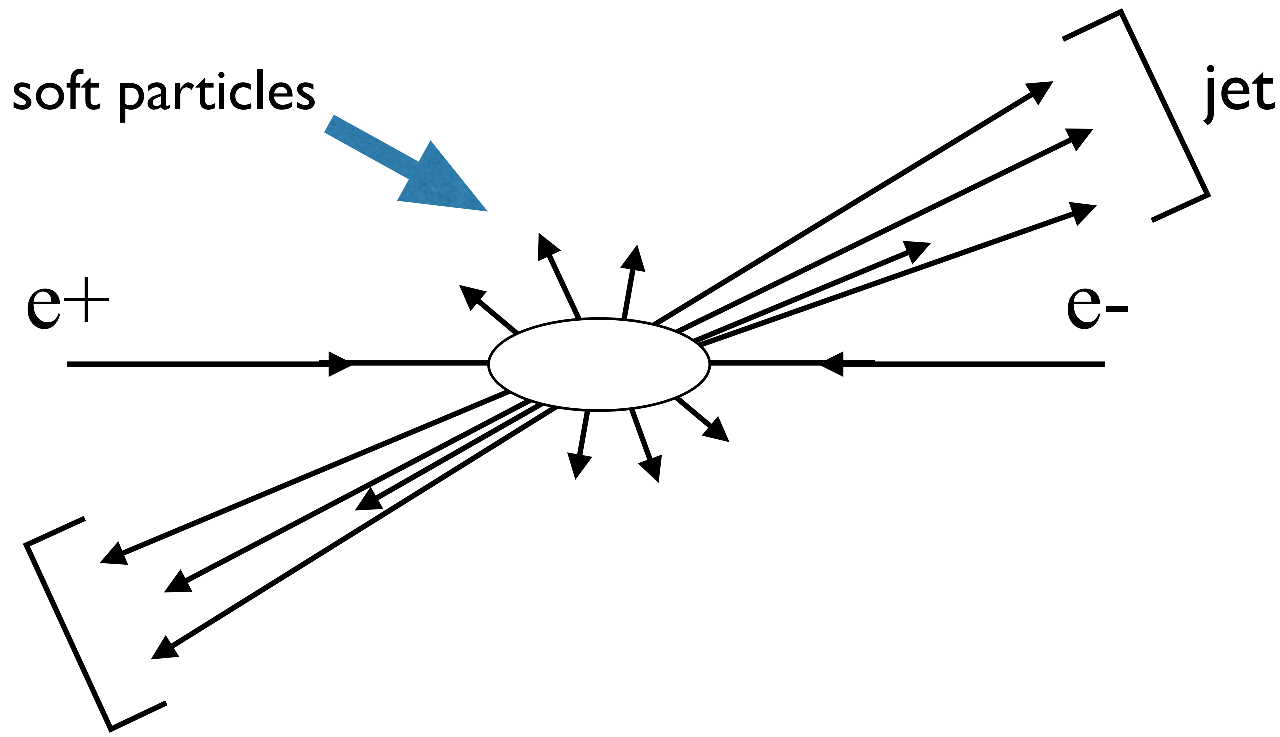
Thrust (strong coupling determination)



- Fit range is the results of a balance among:
  - ➡ quality of data
  - ➡ size of perturbative corrections
  - ➡ size of estimated hadronization corrections

Jet cross section

- qualitative definition: collimated spray of high energy hadrons

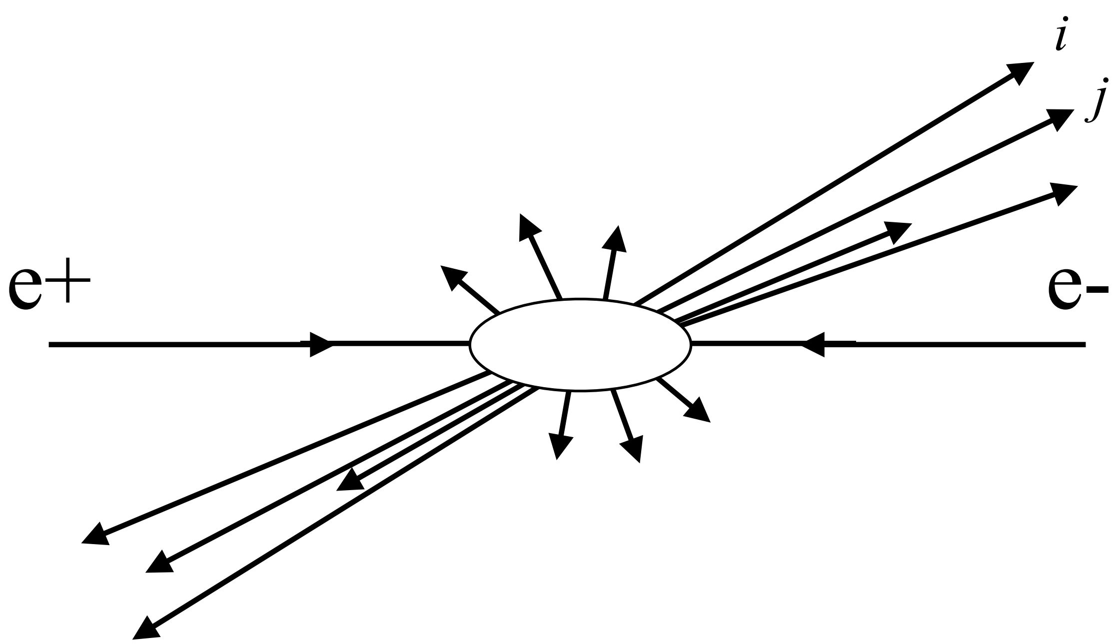


- quantitative studies need of precise definition, in particular low energy particles have to be assigned to jets to have IR safety

  ➡ jet algorithm must fulfil requirements:

    - IR and C safety
    - simple to implement in both experimental analyses and theory predictions
    - small hadronization (non-perturbative) corrections

Jet cross section



1. define a <u>distance</u> between particles

$$y_{ij} = \frac{d_{ij}}{Q^2}$$ ⬅ dimensionfull resolution variable

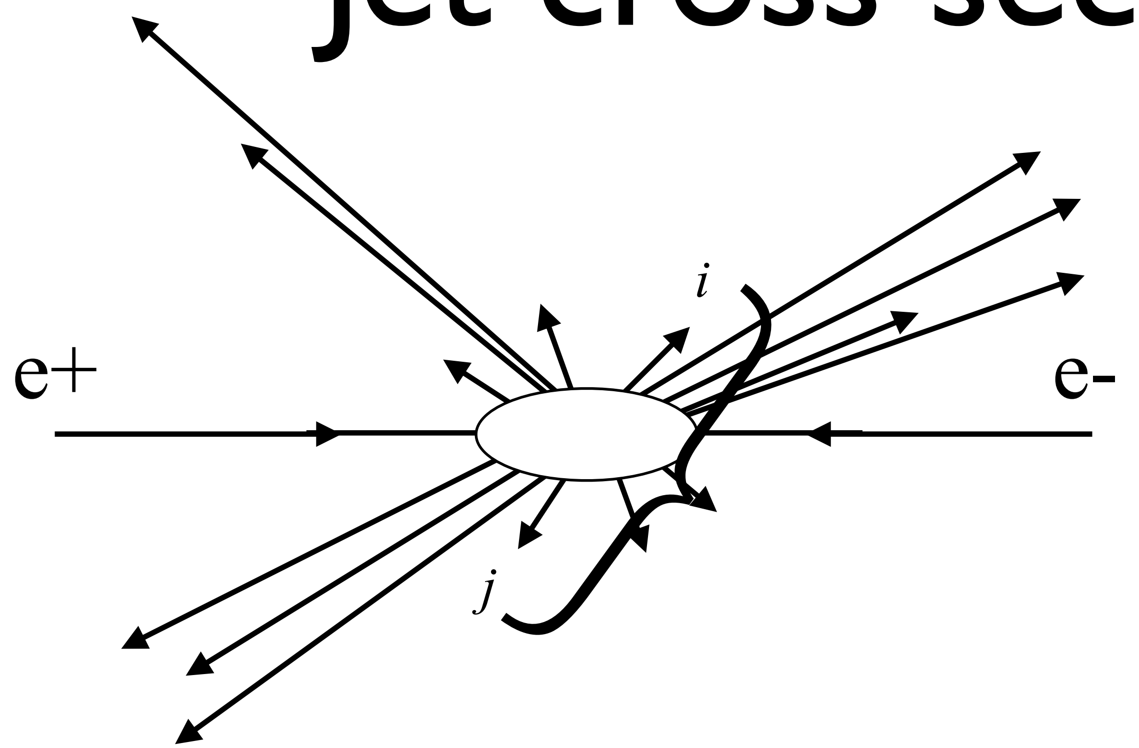
2. <u>merge</u> particles (protojets) with minimum distance until a fixed resolution is reached

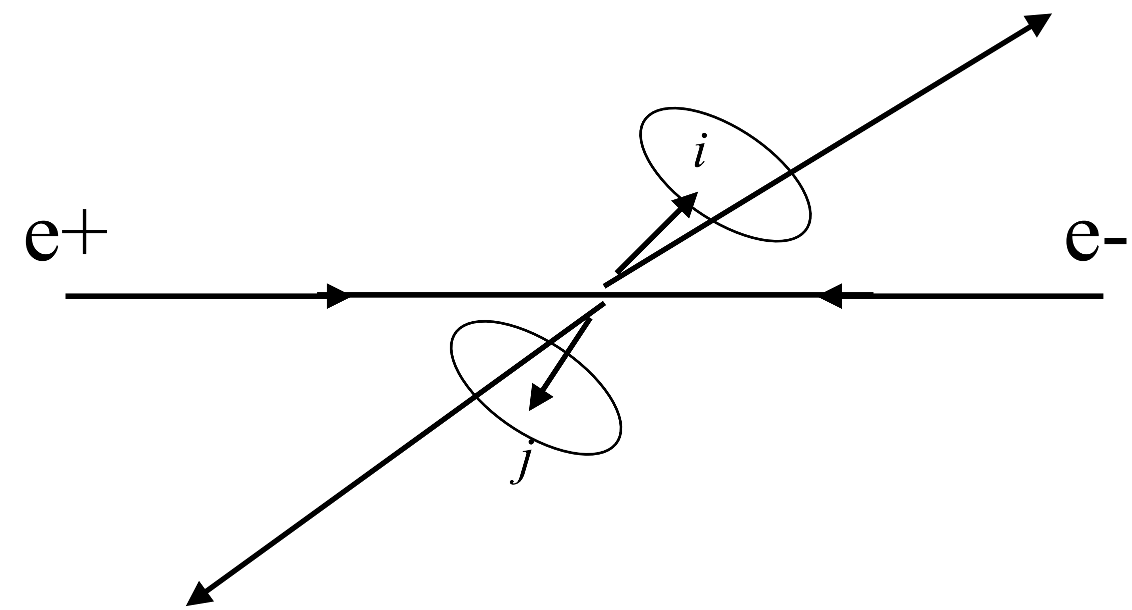
➡ $y_{ij} < y_{cut} \quad \forall i, j$

- Iterative clustering procedure with unambiguous assignment of particles to jets
- The number of jets depends on the resolution y_{cut}
- As for IR/Coll safety it is enough that:

$$\Rightarrow \begin{cases} d_{ij} \underset{E_i \to 0}{\longrightarrow} 0 & \text{IR safe} \\ d_{ij} \underset{\theta_{ij} \to 0}{\longrightarrow} 0 & \text{Coll. safe} \end{cases}$$

Jet cross section (distance)





- JADE algorithm (invariant mass)

$$d_{ij} = 2E_iE_j(1 - \cos\theta_{ij})$$

- soft particles are strongly correlated
- at large angle they are merged in the same jet

- Kt algorithm

$$d_{ij} = 2min(E_i^2, E_j^2)(1 - \cos\theta_{ij}) \quad \overset{\theta_{ij} \to 0}{\sim} \quad k_{\perp ij}$$

- diagonal wrt particle energy
- soft particles are merged with hard particle closest in angle (soft fragments are like to be merged with their parent) avoiding unnatural assignment with creation of soft and wide angle jets

Jet cross section (merge protojets)

- Recombination scheme

  - E-scheme

    ➡ $\widetilde{p}_{ij} = p_i + p_j$

  - E_0-scheme

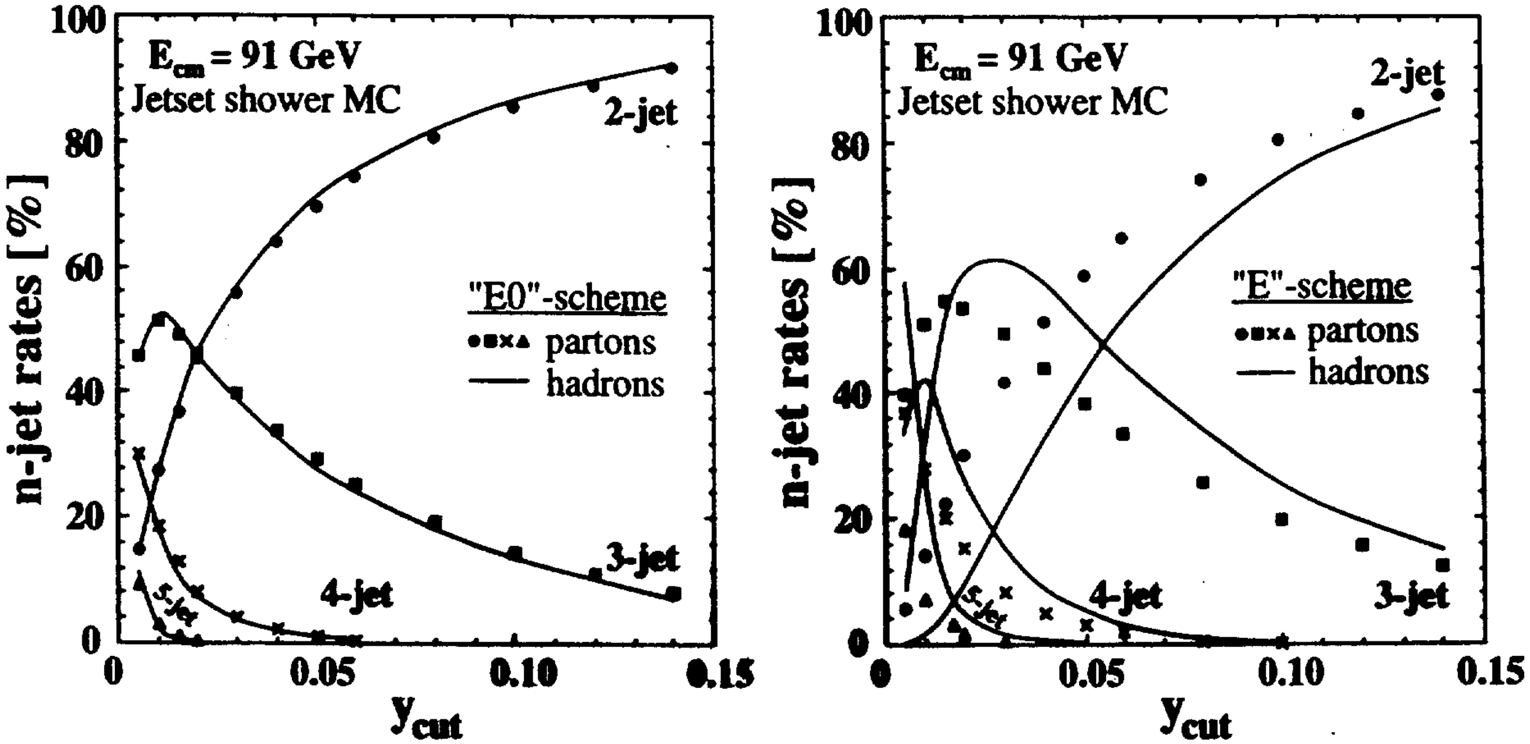
    ➡ $\widetilde{p}_{ij} = \left(E_i + E_j , (E_i + E_j) \dfrac{\vec{p}_i + \vec{p}_j}{|\vec{p}_i + \vec{p}_j|} \right)$

- Momentum of final jets

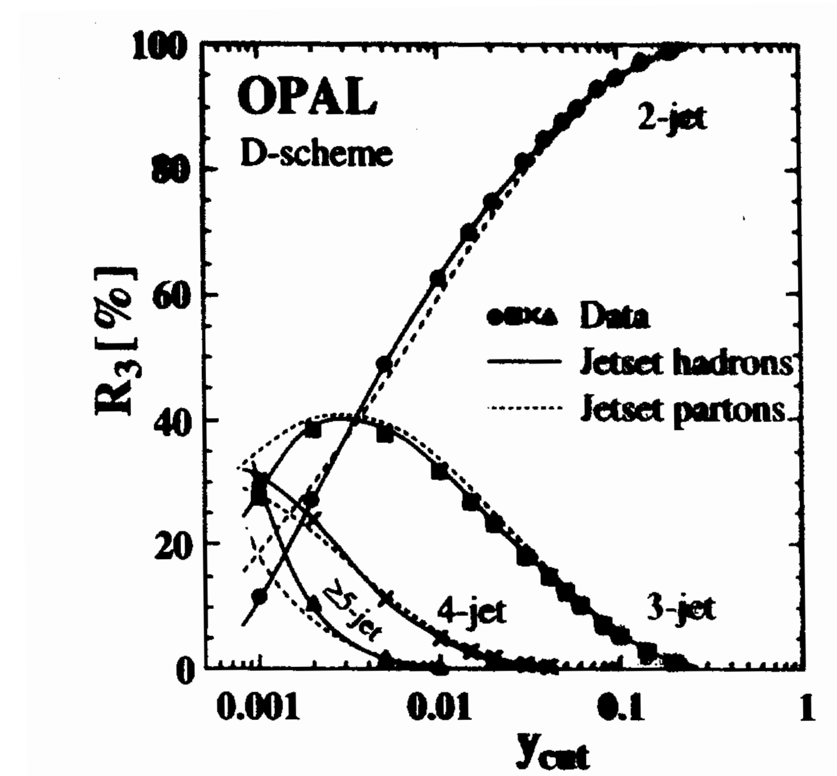
  ➡ $p^{\mu}_{jet} = \sum_i p^{\mu}_i$

- Several others choices for both recombination schemes and final jet momentum in the literature

Jet cross section at LEP



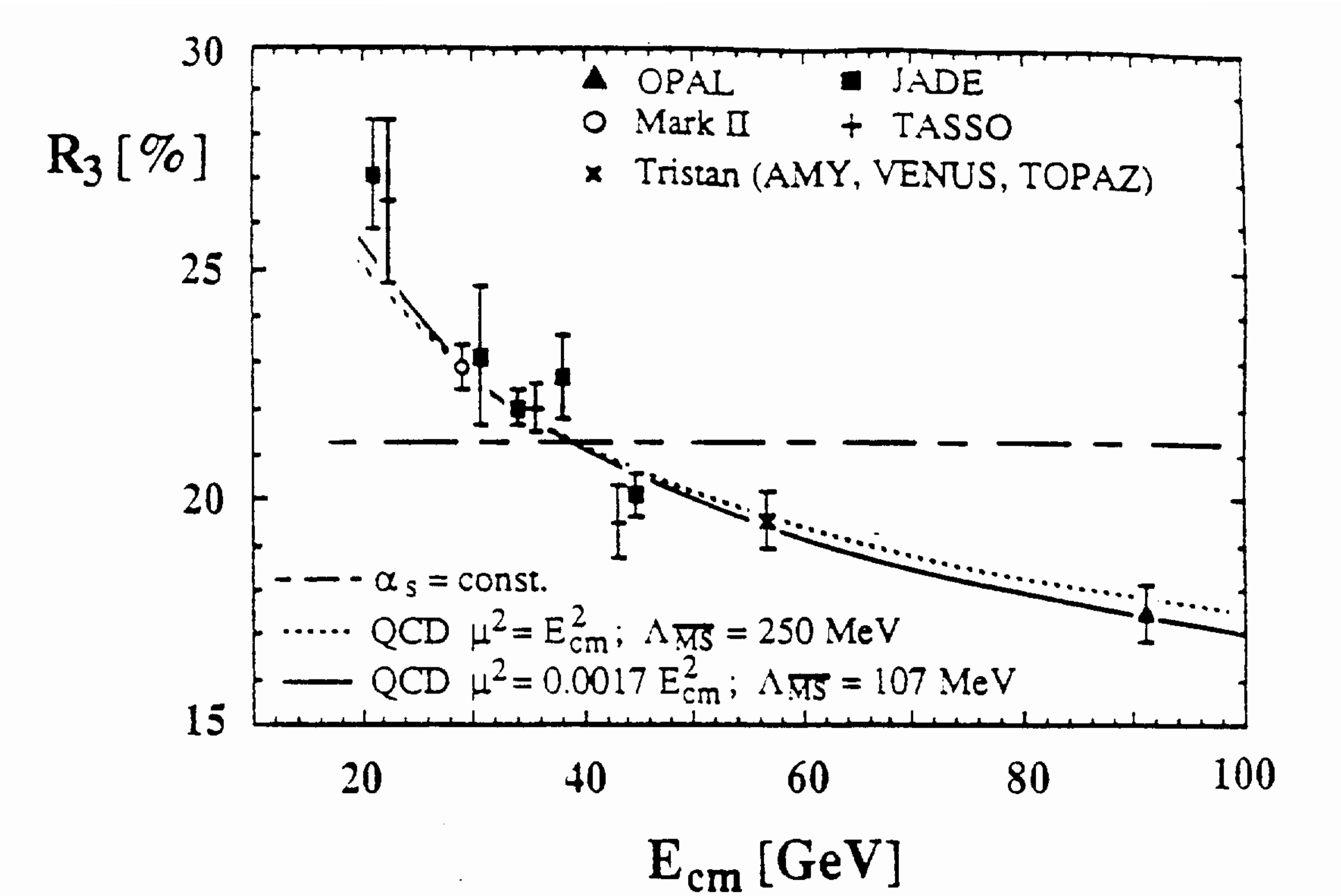
JADE

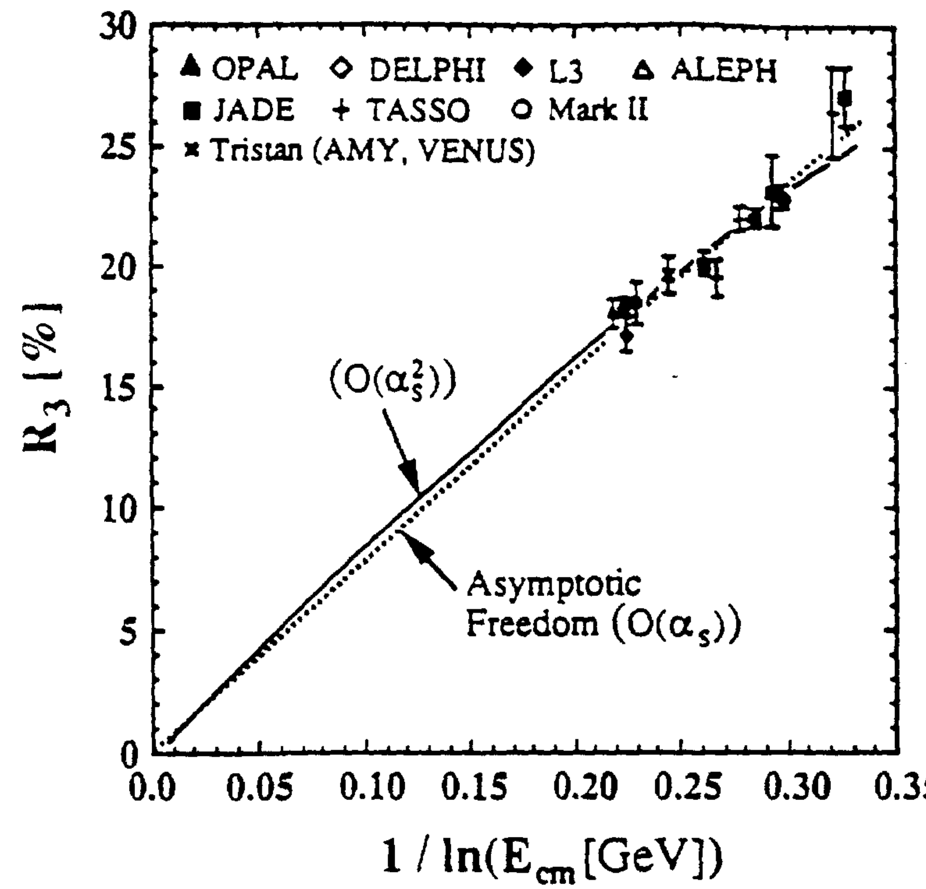


Kt

- multijet rates increase by increasing jet resolution ($y_{cut} \to 0$)
- jet rates depend on the algorithm
- parton vs hadron provides an estimation of non perturbative corrections that are smaller for Kt
- Kt theoretically favoured also because logarithmic enhanced corrections for small y_{cut} can be resumed to all orders in the strong coupling

Jet cross section at LEP





- effect of running coupling on the 3 jet cross section

- experimental way to probe asymptotic freedom

Running of α_S