Resummation for transverse observables at hadron colliders

Pier Francesco Monni
CERN

Mainly based on
1604.02191 with E. Re and P. Torrielli
and
1705.09127 with W. Bizon, E. Re, L. Rottoli, and P. Torrielli
and ongoing work

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The quest for precision at the LHC

- The need for theory precision is twofold
  - As a precision machine, the LHC is providing us with %-accurate measurements of SM parameters/dynamics (couplings, PDFs, masses, …). A full exploitation of this data requires a deep understanding of the theory
  - Precision can allow for indirect constraints on new physics (NP) through mild distortions in kinematic distributions
    - Sensitivity is often improved by looking at exclusive regions of phase space where underlying QCD activity needs to be minimised (e.g. boosted kinematics, vetoes, …)
    - A careful assessment of the SM background is essential in most cases. This already reaches the few-percent level in some scenarios

Fixed-order vs. All-order

- Fixed-order calculations of radiative corrections are formulated in a well established way (technically very challenging, but well posed problem):
  - compute amplitudes at a given order
  - provide an effective subtraction of IRC divergences
  - compute any IRC-safe observable

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\mathrm{Born}}} \frac{d\sigma}{dv'} dv' \sim \overset{\text{LO}}{1} + \overset{\text{NLO}}{\alpha_s} + \overset{\text{NNLO}}{\alpha_s^2} + \dots$$

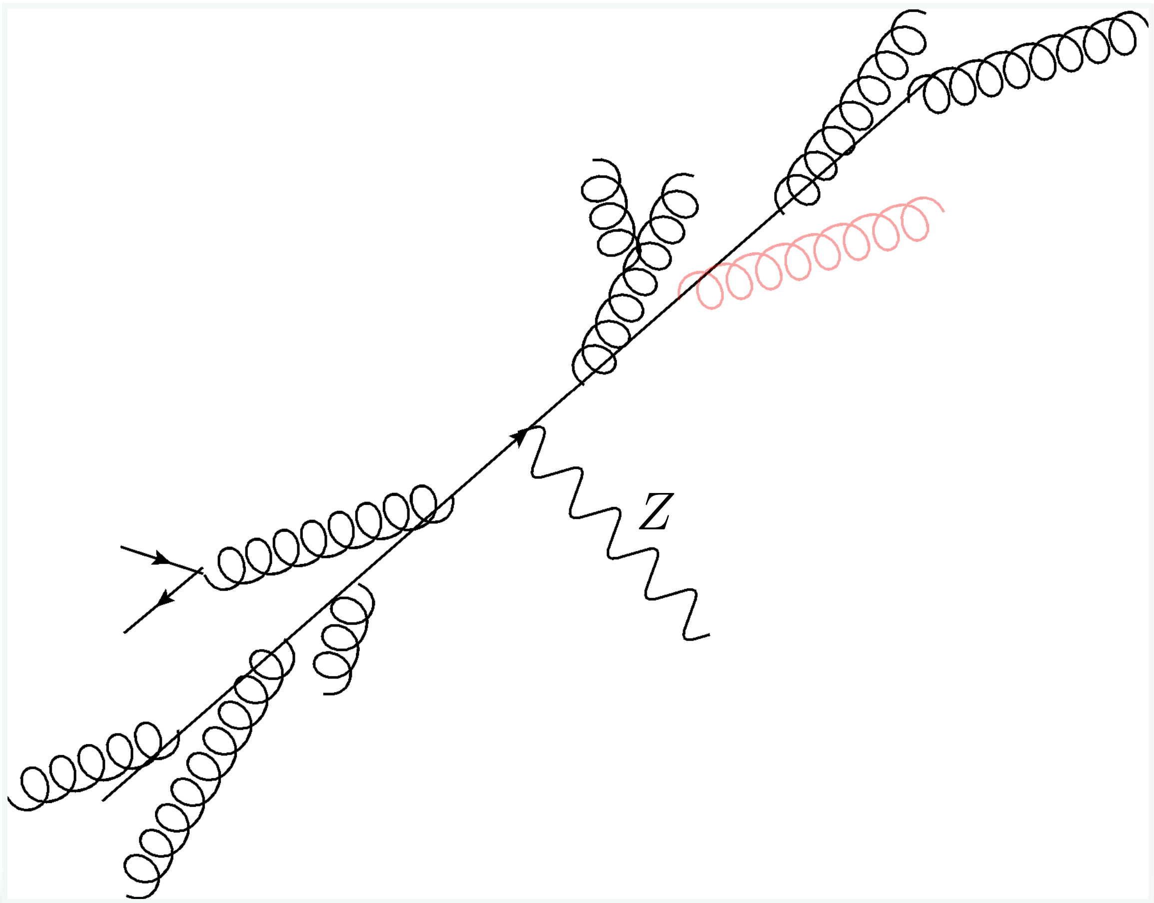
- All-order calculations are still at an earlier stage of evolution
  - Each different observable has its own type of sensitivity to IRC physics, it is hard to formulate a general method that works for all at a generic perturbative order
  - Higher-order resummations are therefore often formulated in an observable-dependent way, for few well-behaved collider observables

$$\Sigma(v) = \int_0^v \frac{1}{\sigma_{\mathrm{Born}}} \frac{d\sigma}{dv'} dv' \underset{v \to 0}{\sim} e^{\overset{\text{LL}}{\alpha_s^n L^{n+1}} + \overset{\text{NLL}}{\alpha_s^n L^n} + \overset{\text{NNLL}}{\alpha_s^n L^{n-1}} + \dots}$$

Path towards resummation: factorisation

- In the logarithmic regime, Born amplitudes receive radiative corrections from virtual diagrams (unconstrained), and soft/collinear real radiation
- The QCD amplitude (almost always) factorises in these kinematic limits
  - This is a necessary condition to formulate an all-order perturbative calculation (otherwise new structure would arise at each new order)

e.g. emission of a soft gluon



$$\mathcal{M}(k_1, k_2, \ldots, k, \ldots, k_n)$$

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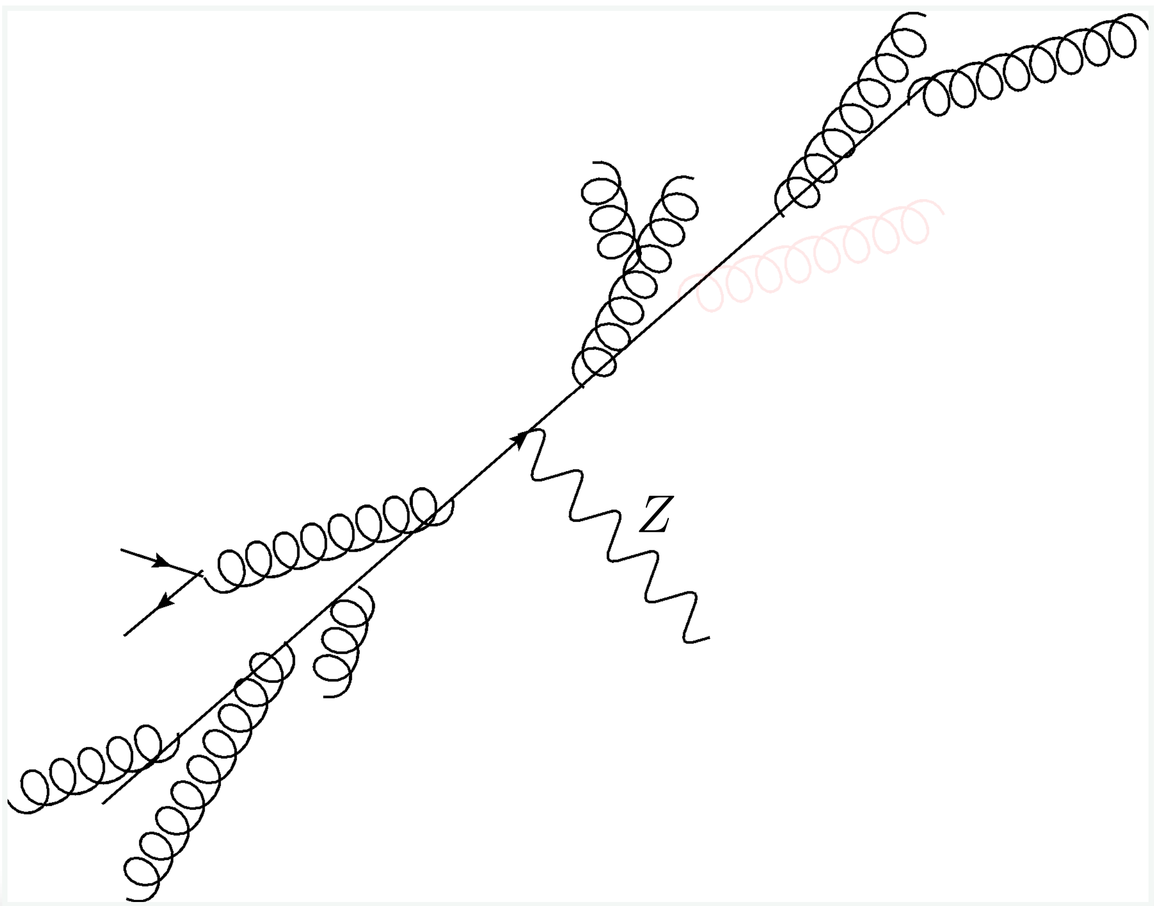


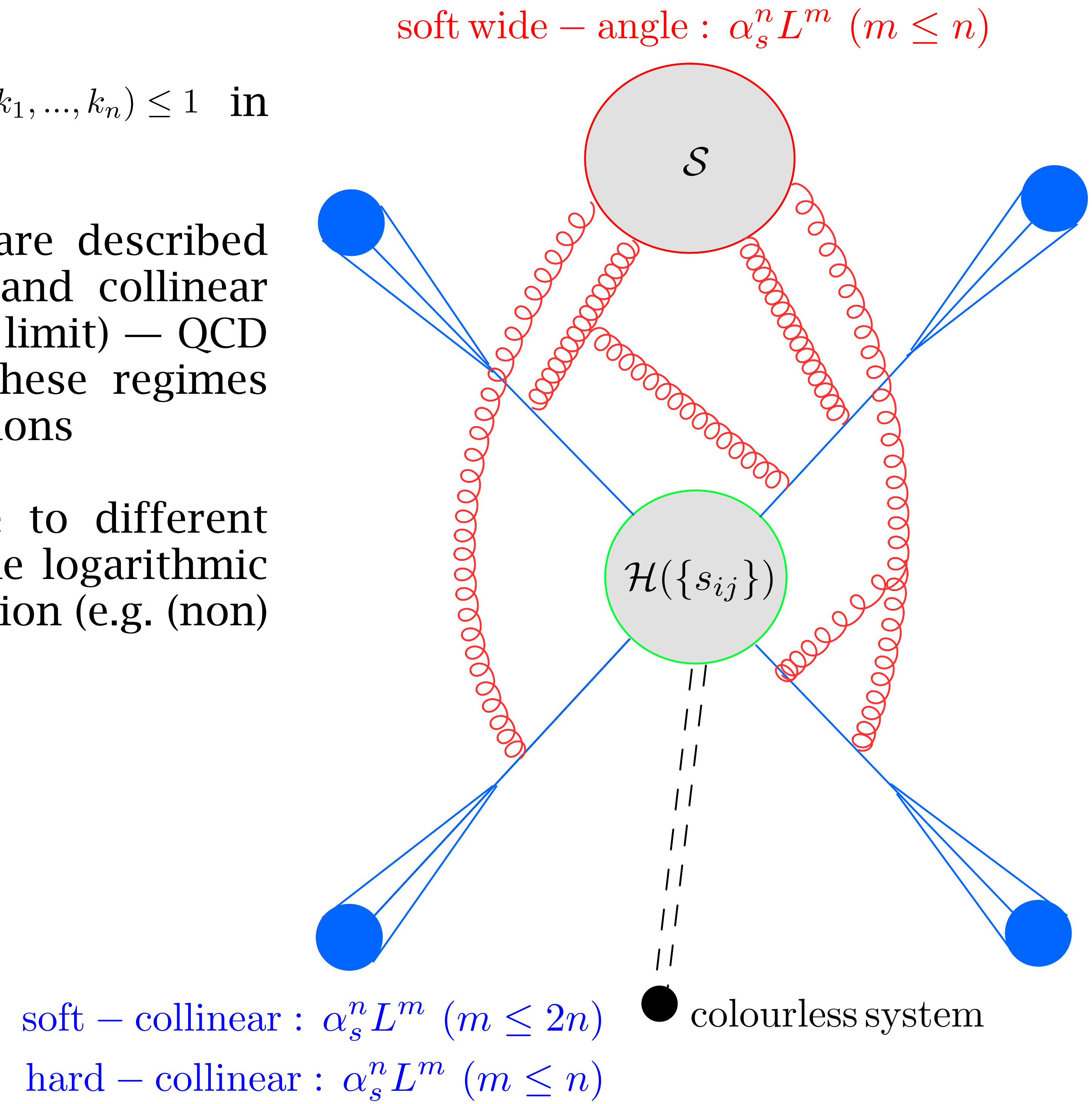
$$\mathcal{M}(k_1, k_2, \ldots, k, \ldots, k_n)$$
$$\simeq \mathcal{M}(k_1, k_2, \ldots\ldots, k_n)\mathcal{M}_{\rm soft}(k)$$

Factorisation of amplitudes in the IRC*

- Consider a IRC observable $V = V(\{\tilde{p}\}, k_1, ..., k_n) \leq 1$ in the Born-like limit $V \to 0$

- In this limit radiative corrections are described exclusively by virtual corrections, and collinear and/or soft real emissions (singular limit) — QCD squared **amplitudes factorise** in these regimes w.r.t. the Born, up to regular corrections

- Different observables are sensitive to different singular modes which determine the logarithmic structure of the perturbative expansion (e.g. (non) global, hard-collinear logarithms, …)

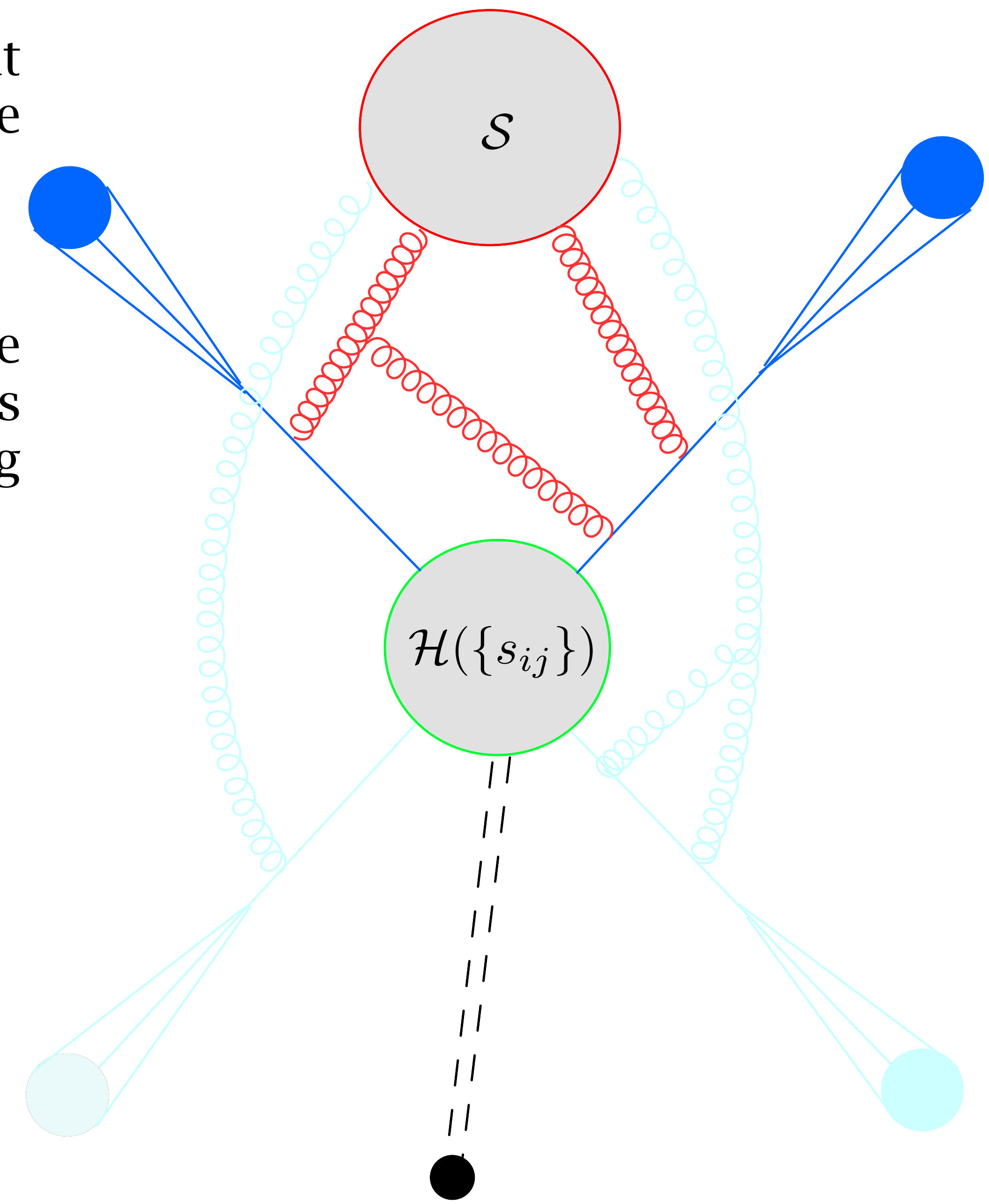


* Modulo Glauber gluons

Two-emitter processes

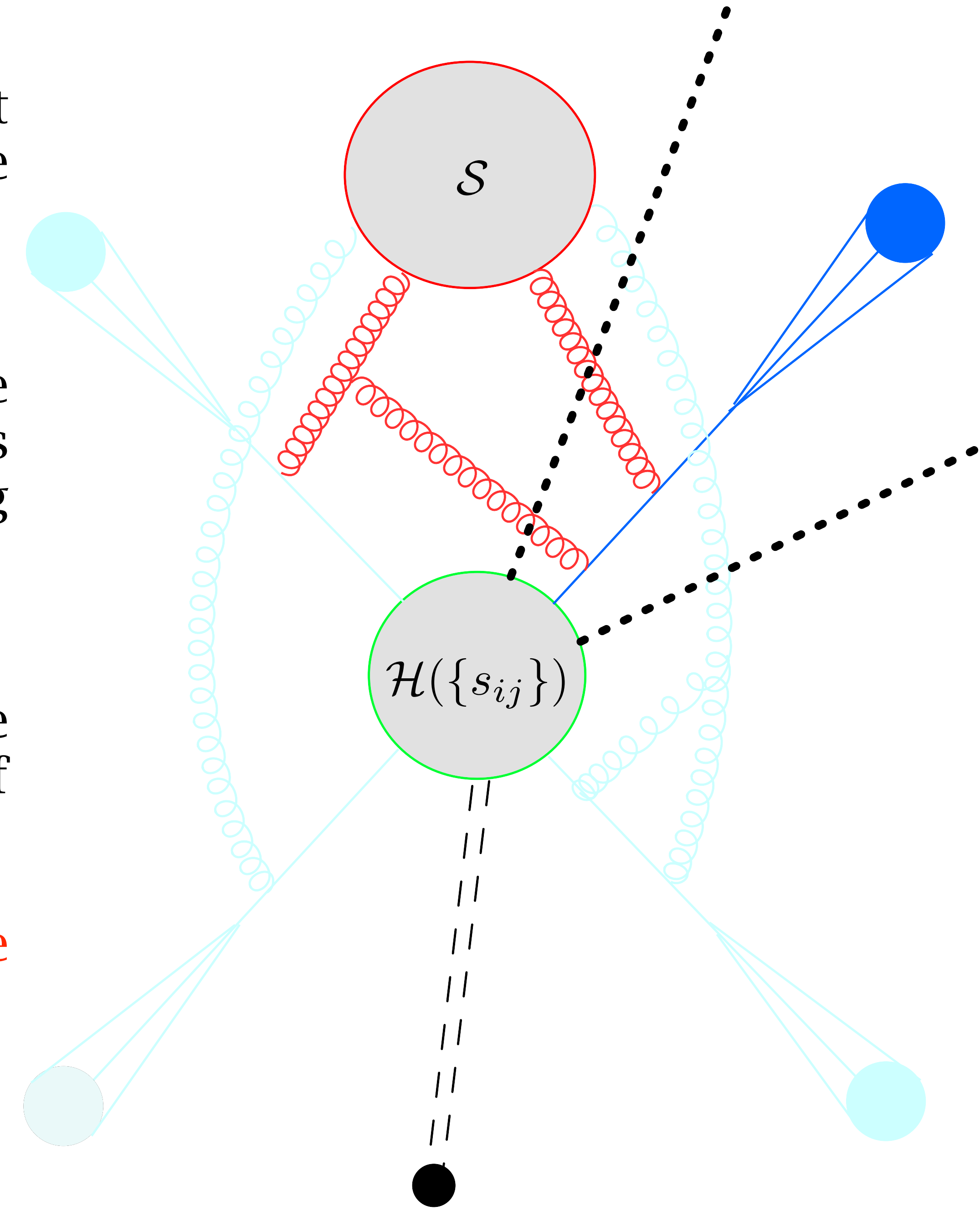
- The strong angular separation between different modes ensures they evolve independently at late times after the collision

- The structure of the coherent soft radiation at large angles (interference between emitters) gets increasingly complex with the number of emitting legs



Non-Global observables

- The strong angular separation between different modes ensures they evolve independently at late times after the collision
- The structure of the coherent soft radiation at large angles (interference between emitters) gets increasingly complex with the number of emitting legs
- For non-global observables one is always sensitive to the full evolution of the soft radiation outside of the resolved phase-space region
  - Both soft and collinear modes are present in the general case
  - Collinear modes can be absent



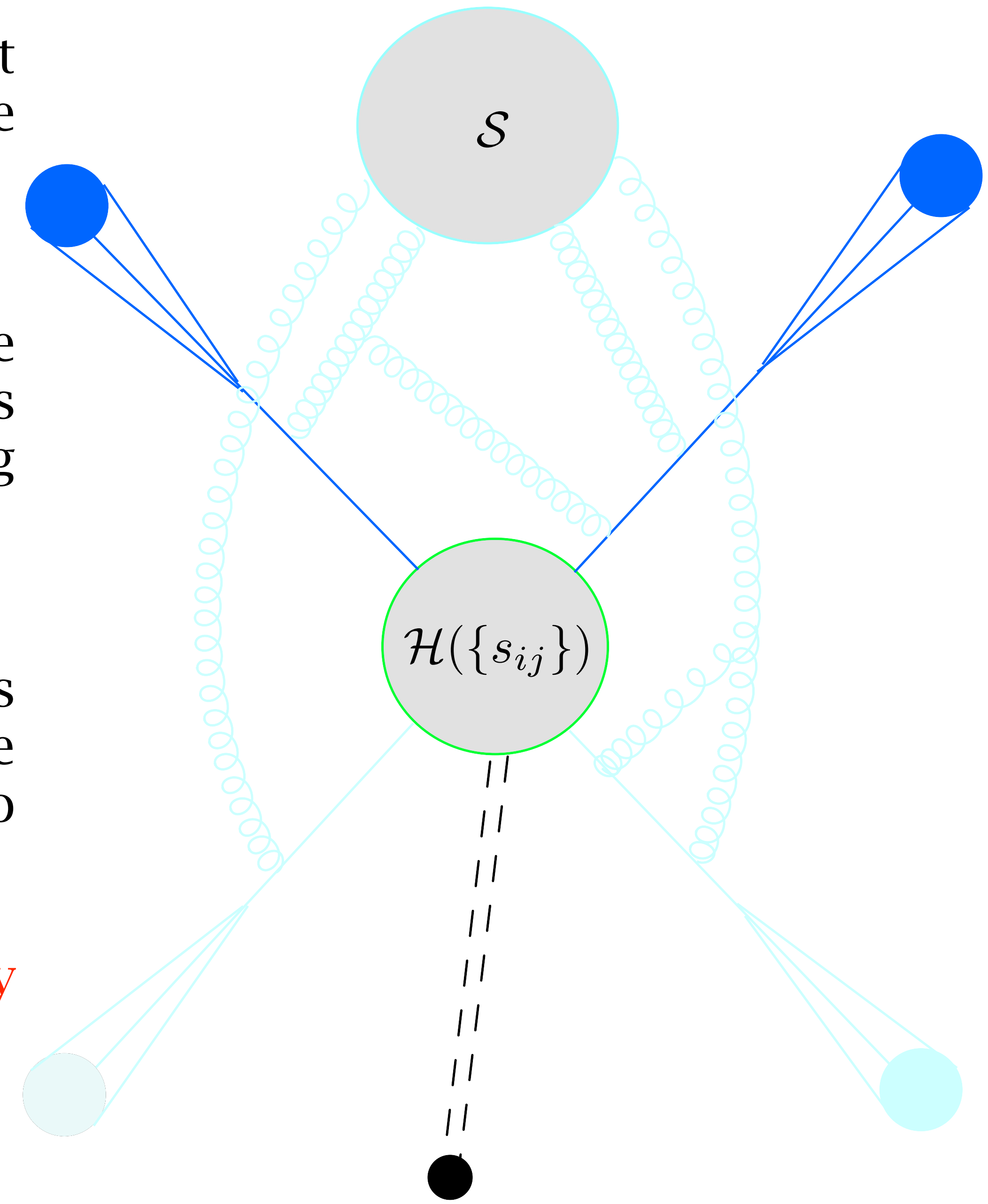
[Dasgupta, Salam ’01; Banfi, Marchesini, Smye ’02]
[Caron-Huot ’15-‘16; Larkoski, Moult, Neill ’15; Becher, Neubert, Rothen, Shao ’15-‘16]

Two-emitter processes

- The strong angular separation between different modes ensures they evolve independently at late times after the collision

- The structure of the coherent soft radiation at large angles (interference between emitters) gets increasingly complex with the number of emitting legs

- For continuously global observables in processes with two emitters, colour coherence forces the effect of soft modes exchanged with large angles to vanish
  - Only collinear (soft/hard) modes effectively remain
  - Soft modes can be absent in specific cases



Factorisation of the observable

- Factorisation of the amplitude is not enough as the all-order radiation is tangled by the observable

$$\Sigma(v) = \int d\Phi_{\rm rad} \sum_{n=0}^{\infty} |\mathcal{M}(k_1,\dots,k_n)|^2 \Theta(v - V(k_1,\dots,k_n))$$

- In order to perform an all-order calculation, one needs to ***break*** the observable too into hard, soft and collinear pieces. This can be done for some observables which treat the radiation rather inclusively
  - Resummation can be performed, e.g., by formulating a soft-collinear EFT of the singular modes (SCET)

[Bauer, Fleming, Pirjol, Stewart '01]

[Beneke, Chapovsky, Diehl, Feldmann '02]

e.g. the Thrust event shape $\tau \equiv 1 - T = 1 - \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$

$$\Theta(Q^2\tau - \bar{k}^2 - k^2 - wQ) = \frac{1}{2\pi i}\int_C \frac{d\nu}{\nu} \mathrm{e}^{\nu\tau Q^2}\mathrm{e}^{-\nu k^2}\mathrm{e}^{-\nu \bar{k}^2}\mathrm{e}^{-\nu w Q}$$

$\bar{n}-$collinear

soft

$n-$collinear

$$\Sigma(\tau) = |\mathcal{H}|^2 \frac{1}{2\pi i}\int_C \frac{d\nu}{\nu} e^{\nu\tau Q^2} S(wQ)\mathcal{J}_n(k^2)\mathcal{J}_{\bar{n}}(\bar{k}^2) + \mathcal{O}(\tau)$$

Eluding observable factorisation

- Factorisation is a powerful tool, but limited to observables that have a simple analytic expression in the relevant limits or do not mix soft and collinear radiation (e.g. jet rates)
- Ultimately, we want to use the modern knowledge of IRC dynamics to make more accurate generators. At present a general framework to assess the accuracy of Parton Showers is missing
  - It is of primary importance to formulate a link between higher-order resummation and PS
- Can we devise a formulation without a factorisation formula ?
  - ***recursive* IRC safety**: simple set of criteria for the observable that allows one to formulate the resummation at NLL for global observables without the need for an explicit factorisation.

    [Banfi, Salam, Zanderighi ’01-’04]
  - Most of modern global observables fall into this category. Exceptions exist: e.g. rIRC unsafe observables (e.g. old JADE and Geneva algorithms), Sudakov-safe observables. No general structure beyond LL for these is known yet
  - The method can be reformulated and systematically extended at higher logarithmic orders

[Banfi, McAslan, PM, Zanderighi ’14-’16]
[PM, Re, Torrielli ’16]
[Bizon, PM, Re, Rottoli, Torrielli ’17]

A case study: transverse observables

- Transverse observables in colour-singlet production offer a clean experimental and theoretical environment for precision physics:

$$\boxed{V(\{\tilde{p}\},k) \equiv V(k) = d_\ell \, g_\ell(\phi) \left(\frac{k_t}{M}\right)^a}$$

  - SM measurements (e.g. W, Z, photon,…): parton distributions, strong coupling, W mass,…
  - BSM measurements/constraints (e.g. Higgs): light/heavy NP, Yukawa couplings,…

- Of this class, the family of inclusive observables probes directly the kinematics of the colour singlet:

$$\boxed{V(\{\tilde{p}\},k_1,\ldots,k_n) = V(\{\tilde{p}\},k_1+\cdots+k_n)}$$

  - sensitive to non-perturbative effects (hadronisation, intrinsic kt) only through transverse recoil
  - very little/no sensitivity to multi-parton interactions
  - measured precisely at experiments

- Experimental uncertainty is already at the % level (or less) in some cases (e.g. Z production). Perturbation theory must be pushed to its limits

e.g. Z/H at small transverse momentum

- Study of small-pt region received a lot of attention in collider literature. Theoretically, it offers a clean environment to test/calibrate exclusive generators against more accurate predictions. Experimentally, shape is sensitive to light-quark Yukawa couplings

- Theoretically interesting observable. Two mechanisms compete in the $p_t \to 0$ limit
  - Sudakov (exponential) suppression when $k_{ti} \sim p_t$
  - Azimuthal cancellations (power suppression, dominant) when $k_{ti} \gg p_t$

- Standard solution obtained in impact-parameter space. Information on the radiation entirely lost

$$\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + ... + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}},$$

$$\frac{d^2\Sigma(p_t)}{d\Phi_B dp_t} = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_S(b_0/b)) H_{\rm CSS}(M) \mathbf{C}^{c_2}_{N_2}(\alpha_S(b_0/b)) \mathbf{f}(b_0/b)$$
$$\times \exp\left\{ -\sum_{\ell=1}^{2} \int_{b_0/b}^{M} \frac{dk_t}{k_t} \mathbf{R}'_{{\rm CSS},\ell}(k_t) \right\}.$$

[Parisi, Petronzio ’79]
[Collins et al. ’85]
[Bozzi et al. ’05]
[Becher et al. ‘10+’12]

- Coefficient functions and anomalous dimensions known up to N^{3}LL, except for four-loop cusp

[Catani, Grazzini ’11][Catani et al. ‘12][Gehrmann, Luebbert, Yang ‘14][Davies, Stirling ‘84]
[De Florian, Grazzini ’01][Becher, Neubert ‘10][Li, Zhu ’16][Vladimirov ’16]

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  - Su
  - A

- Stand ... ely lost

Is it possible to obtain a more exclusive solution in momentum space ?

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Direct space: virtual corrections

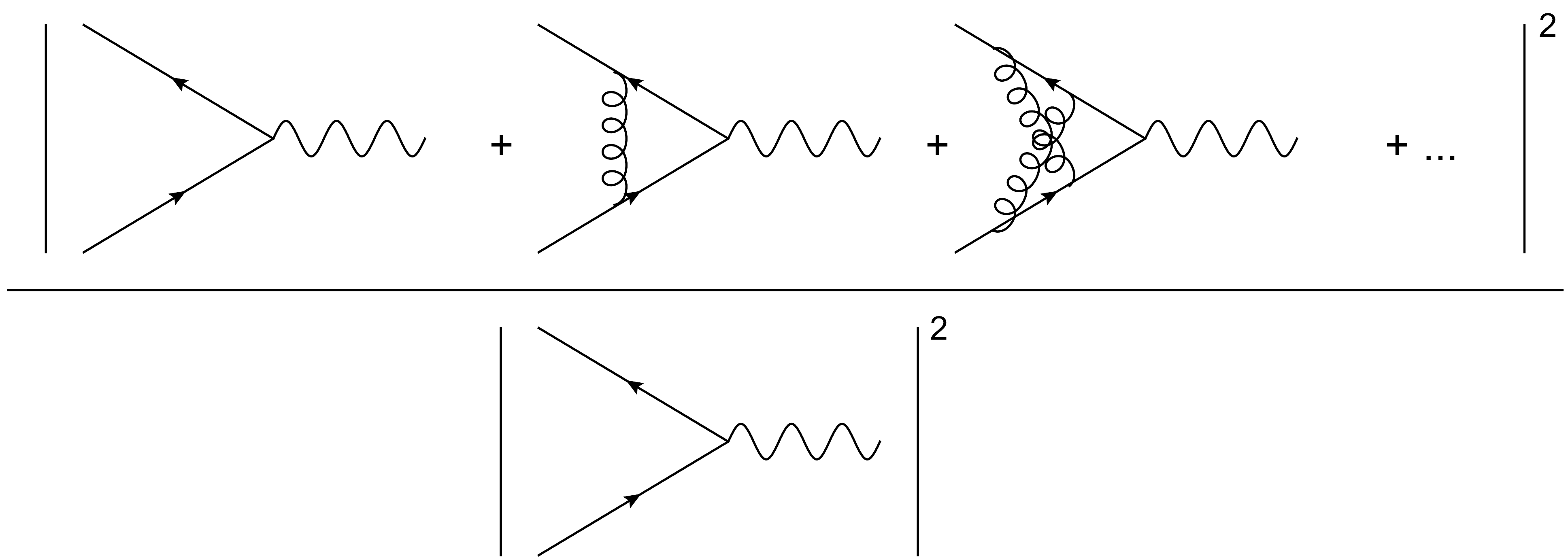
- Write all-order cross section as ($V(\{\tilde{p}\}, k_1, \ldots, k_n) = |\vec{k}_{t1} + \cdots + \vec{k}_{tn}|$)

$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 \, \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

All-order form factor

e.g. [Dixon, Magnea, Sterman ’08]

$\mathcal{V}(\Phi_B) =$



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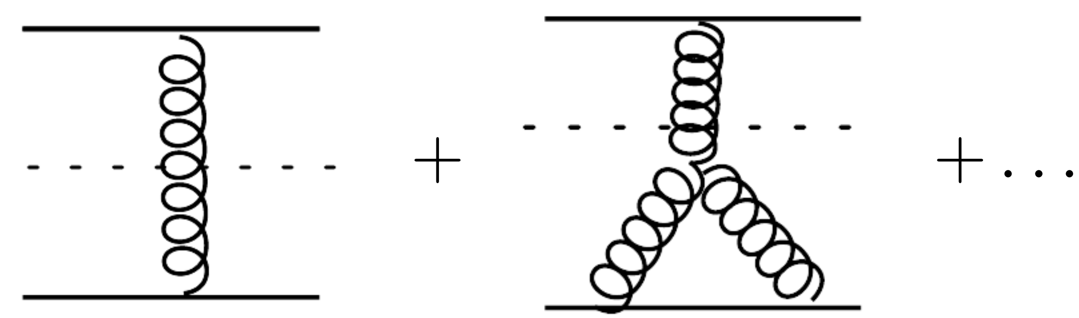
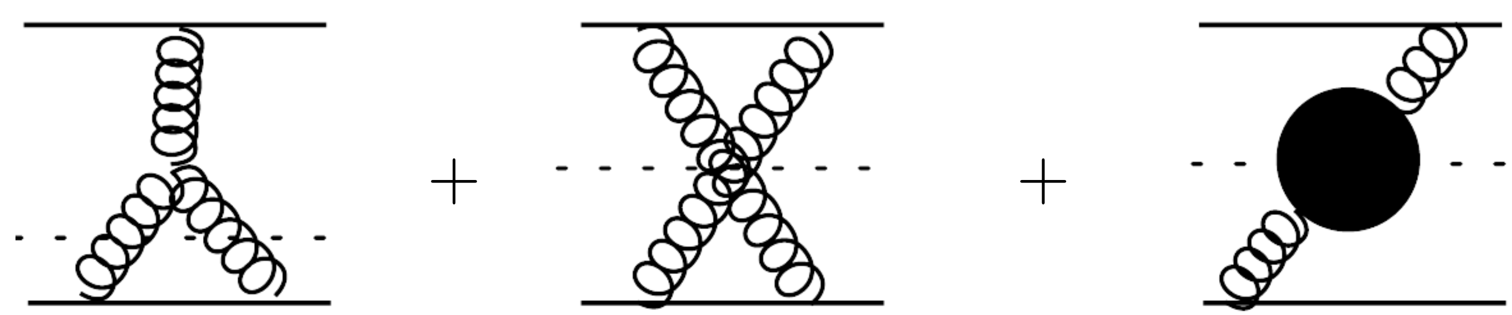
- Logarithmic counting: we need a logarithmic hierarchy in the squared amplitudes (resummation means iteration of lower-order amplitudes)

$$|\tilde{M}(k_a)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} = |M(k_a)|^2 \longrightarrow$$



$$|\tilde{M}(k_a, k_b)|^2 = \frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} - \frac{1}{2!}|M(k_a)|^2 |M(k_b)|^2 \longrightarrow$$



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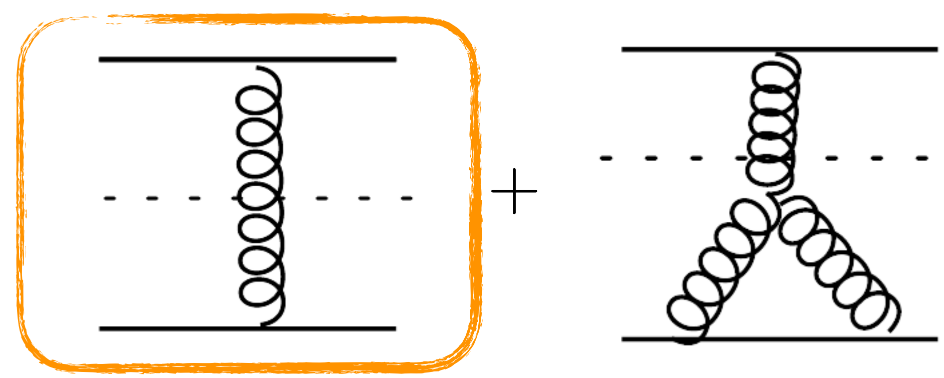

$+ \ldots$

$\alpha_s L^2$

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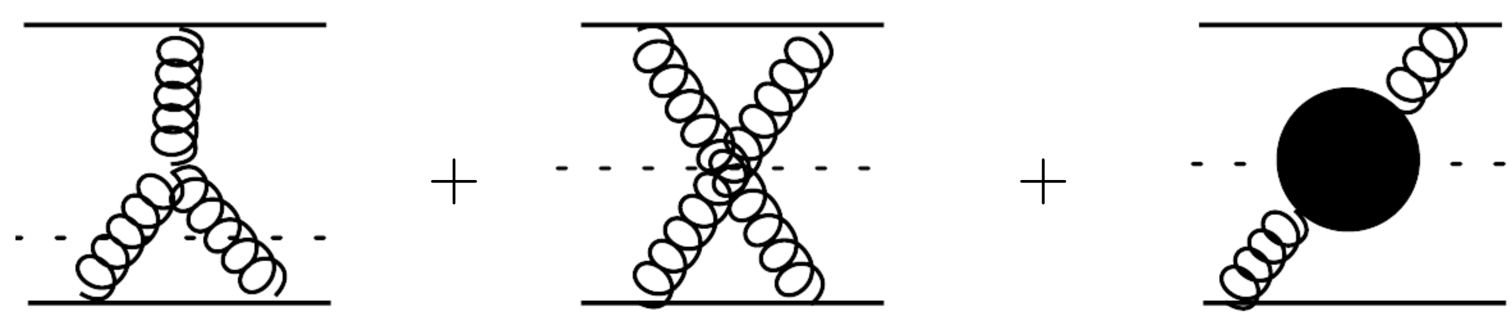

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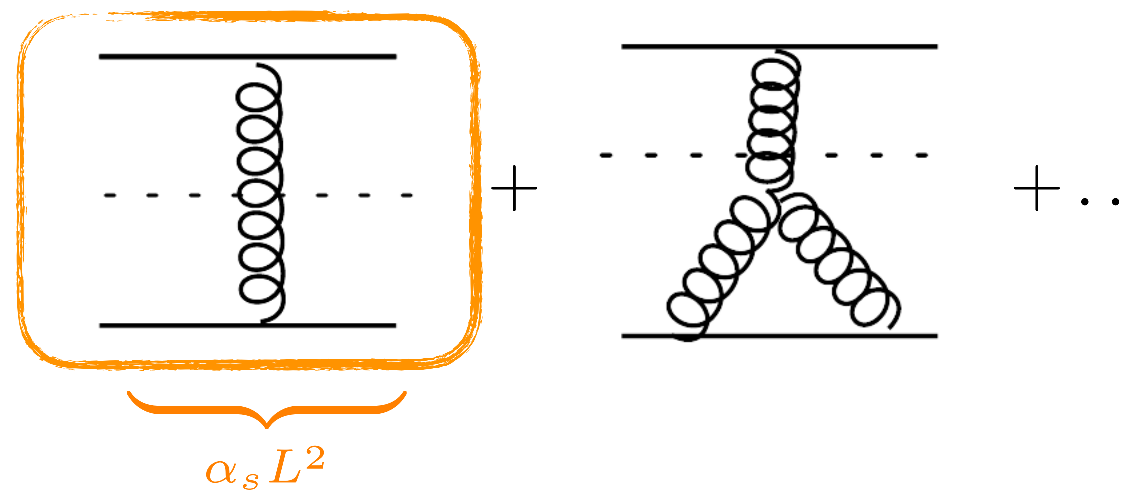

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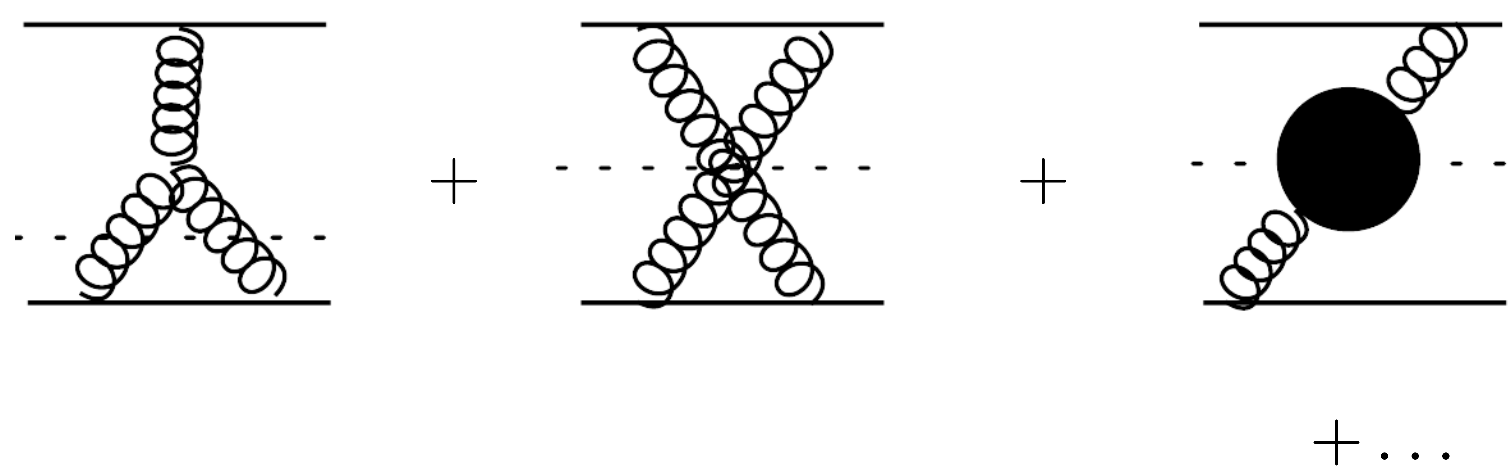

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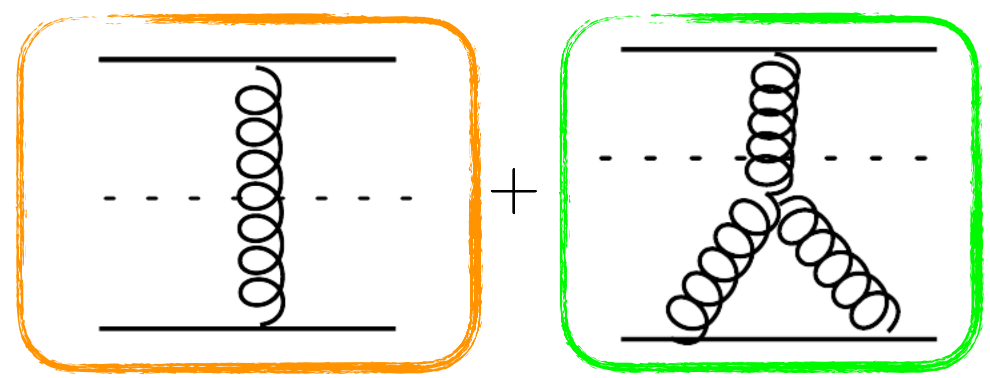

$+ \dots$

$\alpha_s L^2$

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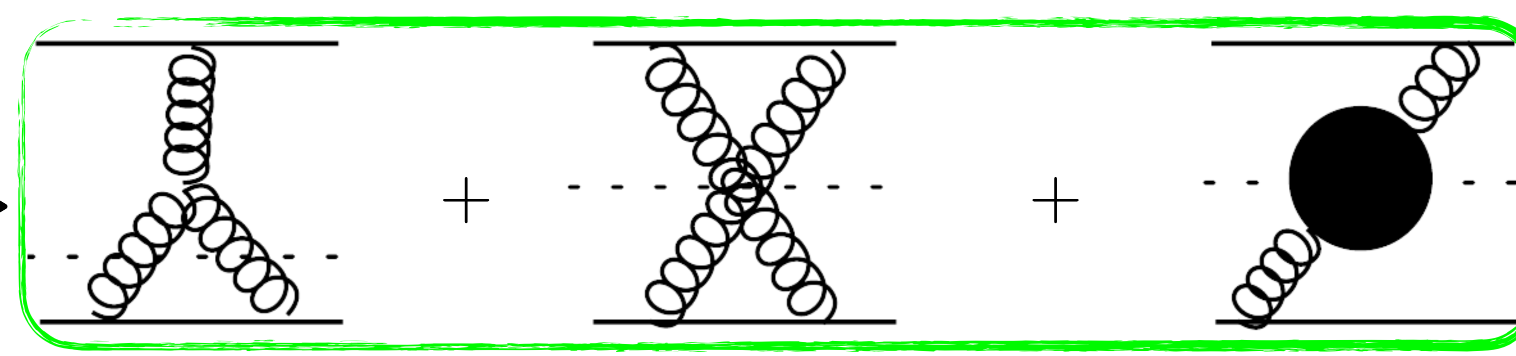

$\alpha_s^2 L^3 + \alpha_s^2 L^2$

$+ \dots$

this LL is absorbed in the resummation of $|M(k)|^2$

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- Recast all-order squared ME for *n* real emissions as iteration of *correlated blocks*
- Scaling of the observable in the presence of radiation *must* preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \left\{ \left(\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right) + \right.$$

$$\left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 + \right.$$

$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots \right]$$

$$\left. + \left[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b, k_c) \right|^2 + \ldots \right] + \ldots \right\},$$

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$$\left[\sum_{a>b} \frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 + \right.$$
$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots \right]$$
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$$\Sigma(v) = \int d\Phi_B \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] \underbrace{|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2}_{\text{Real emissions}} \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_n)\right)$$

- Recast all-order squared ME for *n* real emissions as iteration of *correlated blocks*
- Scaling of the observable in the presence of radiation *must* preserve the above hierarchy

e.g. soft radiation (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 = |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \Bigg\{ \underbrace{\left(\frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2\right)}_{\text{LL}} +$$

$$\Bigg[\sum_{a>b} \underbrace{\frac{1}{(n-2)!} \left(\prod_{\substack{i=1 \\ i \neq a,b}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b)\right|^2}_{\text{NLL}} +$$

$$\sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b)\right|^2 \left|\tilde{M}(k_c, k_d)\right|^2 + \ldots}_{\text{NLL}} \Bigg]$$

$$+ \Bigg[\sum_{a>b>c} \frac{1}{(n-3)!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a, k_b, k_c)\right|^2 + \ldots \Bigg] + \ldots \Bigg\},$$

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$$\left. \sum_{a>b} \sum_{\substack{c>d \\ c,d \neq a,b}} \underbrace{\frac{1}{(n-4)!2!} \left(\prod_{\substack{i=1 \\ i \neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left| \tilde{M}(k_a, k_b) \right|^2 \left| \tilde{M}(k_c, k_d) \right|^2 + \ldots}_{\text{NLL}} \right]$$

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In addition to this counting, requiring that the observable is recursively IRC safe allows one to construct a (simpler) all-order subtraction scheme

All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2$ and $\mathcal{V}(\Phi_B)$:
  - introduce a phase-space resolution scale (slicing parameter) $Q_0 = \epsilon k_{t1}$
  - real correlated blocks with total transverse momentum $k_{ti} < \epsilon k_{t1}$(unresolved) do not modify the observable, and can be ***ignored*** in the measurement function
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$$\sum_{n=0}^{\infty}|M(\tilde{p}_1,\tilde{p}_2,k_1,\ldots,k_n)|^2\longrightarrow|M_B(\tilde{p}_1,\tilde{p}_2)|^2$$
$$\times\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\prod_{i=1}^{n}\left(|M(k_i)|^2+\int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)\right.\right.$$
$$\left.\left.+\int[dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\ldots\right)\right\}$$

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$$\prod_{i=1}^{n}\int[dk_i]\mathcal{V}(\Phi_B)\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\prod_{i=1}^{n}\Bigg(|M(k_i)|^2+\int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)\right.$$
$$\left.+\int[dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\ldots\Bigg)\Theta(\epsilon k_{t1}-k_{ti})\right\}$$

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$$\propto\int\frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(\epsilon k_{t1})}R'(k_{t1})$$

All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}[dk_i]|M(\tilde{p}_1,\tilde{p}_2,k_1,\dots,k_n)|^2$ and $\mathcal{V}(\Phi_B)$:
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$$R(\epsilon k_{t1})\equiv\sum_{\ell=1}^{2}\int_{\epsilon k_{t1}}^{M}\frac{dk_t}{k_t}R'_\ell(k_t)=\sum_{\ell=1}^{2}\int_{\epsilon k_{t1}}^{M}\frac{dk_t}{k_t}\left(A_\ell(\alpha_s(k_t))\ln\frac{M^2}{k_t^2}+B_\ell(\alpha_s(k_t))\right)$$

Anomalous dimensions start differing from b-space ones at N³LL

$$\prod_{i=1}^{n}\int[dk_i]\mathcal{V}(\Phi_B)\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\prod_{i=1}^{n}\left(|M(k_i)|^2+\int[dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)\right.\right.$$
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  - compute *unresolved* reals and *virtuals* analytically in D dimensions (*much* easier than full observable)

$$\boxed{\begin{aligned}\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = &\left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M\frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi}\\ &\times e^{-\mathbf{R}(\epsilon k_{t1})}\exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t))+\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}\end{aligned}}$$

DGLAP anomalous dims

RGE evolution of coeff. functions

Sudakov radiator: integral of single inclusive block.

All-order subtraction of IRC singularities

- Subtraction of the IRC poles between $\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}[dk_i]|M(\tilde{p}_1,\tilde{p}_2,k_1,\ldots,k_n)|^2$ and $\mathcal{V}(\Phi_B)$:
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$$\begin{aligned}
\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = {} & \boxed{\left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0))H(\mu_R)\mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right]\int_0^M\frac{dk_{t1}}{k_{t1}}\int_0^{2\pi}\frac{d\phi_1}{2\pi} \\
& \times e^{-\mathbf{R}(\epsilon k_{t1})}\exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0}\frac{dk_t}{k_t}\mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}} \\
& \boxed{\sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right) \\
& \times\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=2}^{n+1}\int_{\epsilon}^{1}\frac{d\zeta_i}{\zeta_i}\int_0^{2\pi}\frac{d\phi_i}{2\pi}\sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi}\mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right) \\
& \times\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right),}
\end{aligned}$$

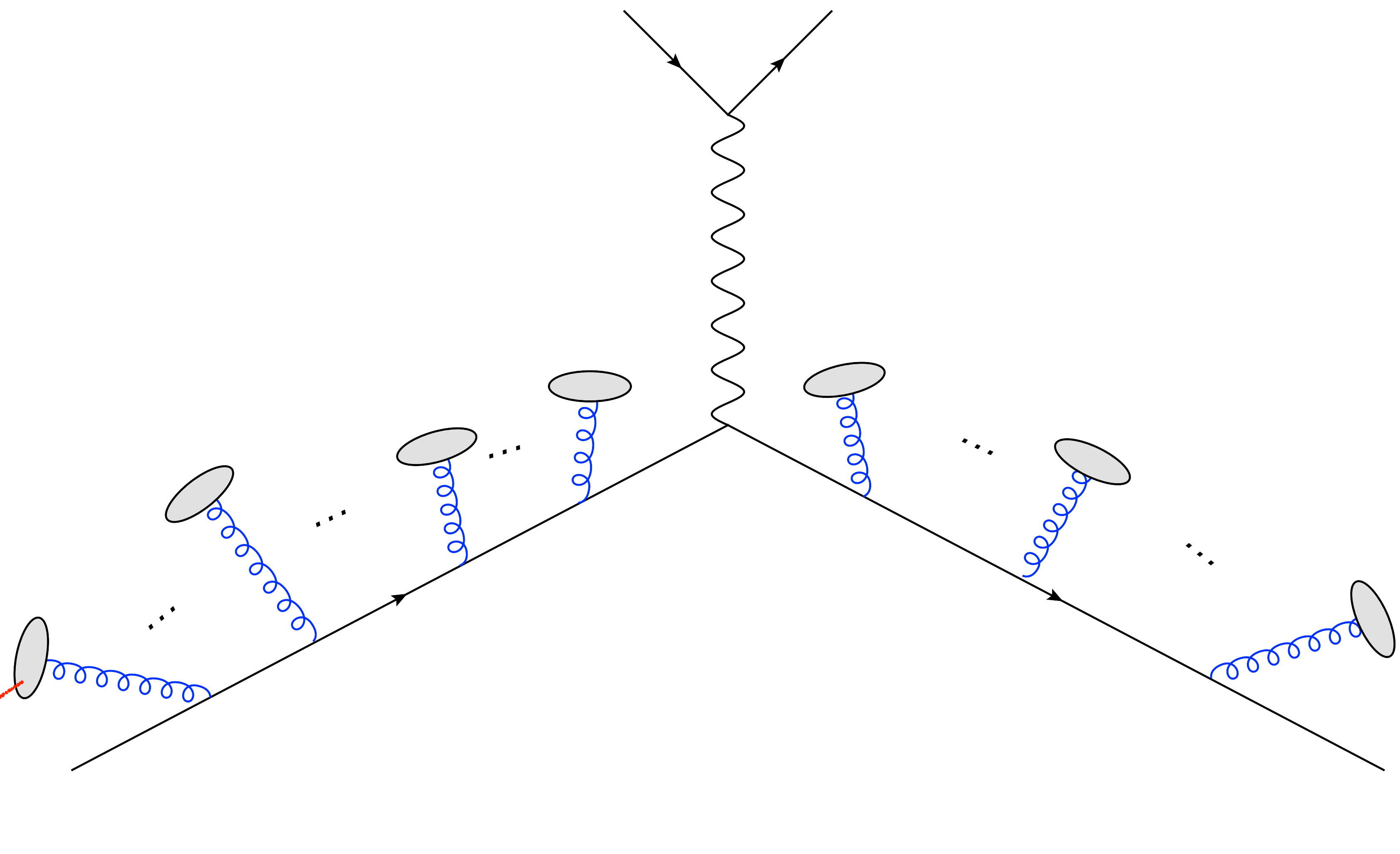
DGLAP anomalous dims

RGE evolution of coeff. functions

- compute *resolved* (reals only) in 4 dim. with $\epsilon \to 0$ (MC events !)

Monte Carlo formulation

- This is, essentially, a *non-fully-exclusive generator* with higher logarithmic accuracy



$$|M(k_i)|^2 + \int [dk_a][dk_b]|\tilde{M}(k_a,k_b)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}-\vec{k}_{ti})\delta(Y_{ab}-Y_i)$$
$$+\int [dk_a][dk_b][dk_c]|\tilde{M}(k_a,k_b,k_c)|^2\delta^{(2)}(\vec{k}_{ta}+\vec{k}_{tb}+\vec{k}_{tc}-\vec{k}_{ti})\delta(Y_{abc}-Y_i)+\dots$$

Monte Carlo formulation

- One great simplification: choice of the resolution variable such that correlated blocks entering at N^{k}LL in the unresolved radiation only contribute at N^{k+1}LL in the resolved case

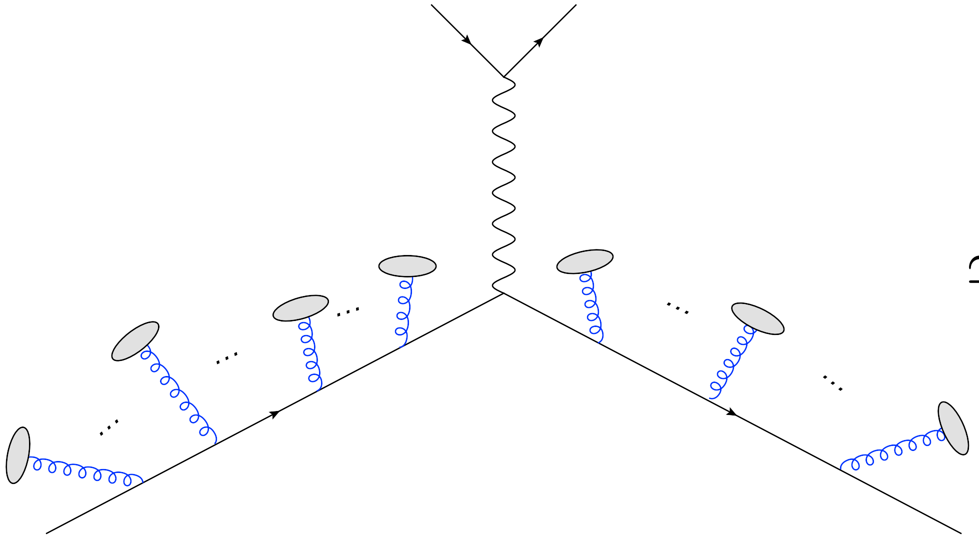
- i.e. we can expand out the cutoff dependence and retain in the Radiator only the terms necessary to cancel the singularities in the resolved radiation

$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln \frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2 \frac{1}{\epsilon} + \ldots$$

$$R'(k_{ti}) = R'(k_{t1}) + R''(k_{t1}) \ln \frac{k_{t1}}{k_{ti}} + \ldots$$

Expansion is safe since in the resolved radiation $k_{t1}/k_{ti} \sim 1$

e.g. at NLL



$$\simeq \int \frac{dk_{t1}}{k_{t1}} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}(k_{t1}) \right) \int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}))$$

$$\int d\mathcal{Z}[\{R'(k_{t1}), k_i\}] = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1})$$

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$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln\frac{1}{\epsilon} + \frac{1}{2} R''(k_{t1}) \ln^2\frac{1}{\epsilon} + \dots$$

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- Corrections beyond NLL are obtained as follows
  - Add subleading effects in the Sudakov radiator and constants
  - Correct *a fixed number* of the NLL resolved emissions:
    - only one at NNLL
    - two at $\mathrm{N^3LL}$
    - …

e.g. at NNLL see:
[Banfi, PM, Salam, Zanderighi ’12]
[Banfi, McAslan, PM, Zanderighi ’14-’16]

Small transverse momentum limit

- CSS result recovered by simply transforming observable into b-space

- Clear physical picture of the dynamics of azimuthal cancellations at small transverse momentum

e.g. NLL with $\mathcal{L}(\mathrm{k}_{t1}) = 1$ for simplicity

$$\frac{d^2\Sigma(v)}{d^2\vec{p}_t d\Phi_B} = \sigma^{(0)}(\Phi_B) \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} R'(k_{t1}) \int d\mathcal{Z}[\{R', k_i\}] \delta^{(2)} \left(\vec{p}_t - \left(\vec{k}_{t1} + \cdots + \vec{k}_{t(n+1)}\right)\right)$$

- Transition from exponential to a power-like suppression at small transverse momentum

$$\frac{d^2\Sigma(v)}{dp_t d\Phi_B} \simeq 4\sigma^{(0)}(\Phi_B)\, p_t \int_{\Lambda_{\mathrm{QCD}}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2\sigma^{(0)}(\Phi_B) p_t \left(\frac{\Lambda_{\mathrm{QCD}}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}}$$

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as $\mathrm{p_t} \to 0$ Sudakov is "frozen" at $\mathrm{k_{t1}} \gg \mathrm{p_t}$
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Random azimuthal orientation of momenta
leads to scaling $\propto 1/\mathrm{k}_{t1}^2$

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Matching to Fixed Order

- Implementation in a MC code (RadISH) up to N3LL
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:

[Anastasiou et al. '15-'16]

[Boughezal et al. '15]
[Caola et al. '15]
[Chen et al. '16]

$$\sigma^{\mathrm{N^3LO}}_{pp\to H} - \Sigma^{\mathrm{NNLO}}_{1-\mathrm{jet}}(p_t^H)$$

  - Additive vs. multiplicative schemes

OLD CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = (\Sigma_{\mathrm{RES}}(p_t))^{Z}\,\frac{\Sigma_{\mathrm{FO}}(p_t)}{(\Sigma_{\mathrm{EXP}}(p_t))^{Z}}$$

$$Z = \left(1-\left(\frac{p_t}{Q_{\mathrm{match}}}\right)\right)^h \Theta(Q_{\mathrm{match}}-p_t)$$

R – SCHEME :

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$$\Sigma_{\mathrm{MAT}}(p_t) = \Sigma_{\mathrm{RES}}(p_t) + \Sigma_{\mathrm{FO}}(p_t) - \Sigma_{\mathrm{EXP}}(p_t)$$

NEW CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = \frac{\Sigma_{\mathrm{RES}}(p_t)}{\mathcal{L}(\mu_F)} \left[\mathcal{L}(\mu_F) \frac{\Sigma_{\mathrm{FO}}(p_t)}{\Sigma_{\mathrm{EXP}}(p_t)}\right]_{\mathrm{EXPANDED}}$$

Matching to Fixed Order

- Implementation in a MC code (RadISH) up to N3LL
  - fully differential in Born kinematics
  - matching to fixed order cumulative distribution, e.g. Higgs:

[Anastasiou et al. '15-'16]

[Boughezal et al. '15]
[Caola et al. '15]
[Chen et al. '16]

$$\sigma^{\mathrm{N^3LO}}_{pp\to H} - \Sigma^{\mathrm{NNLO}}_{1-\mathrm{jet}}(p_t^H)$$

  - Additive vs. multiplicative schemes

OLD CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = (\Sigma_{\mathrm{RES}}(p_t))^{Z}\,\frac{\Sigma_{\mathrm{FO}}(p_t)}{(\Sigma_{\mathrm{EXP}}(p_t))^{Z}}$$

$$Z = \left(1 - \left(\frac{p_t}{Q_{\mathrm{match}}}\right)\right)^h \Theta(Q_{\mathrm{match}} - p_t)$$

R − SCHEME :

$$\Sigma_{\mathrm{MAT}}(p_t) = \Sigma_{\mathrm{RES}}(p_t) + \Sigma_{\mathrm{FO}}(p_t) - \Sigma_{\mathrm{EXP}}(p_t)$$

NEW CHOICE :

$$\Sigma_{\mathrm{MAT}}(p_t) = \frac{\Sigma_{\mathrm{RES}}(p_t)}{\mathcal{L}(\mu_F)}\left[\mathcal{L}(\mu_F)\frac{\Sigma_{\mathrm{FO}}(p_t)}{\Sigma_{\mathrm{EXP}}(p_t)}\right]_{\mathrm{EXPANDED}}$$

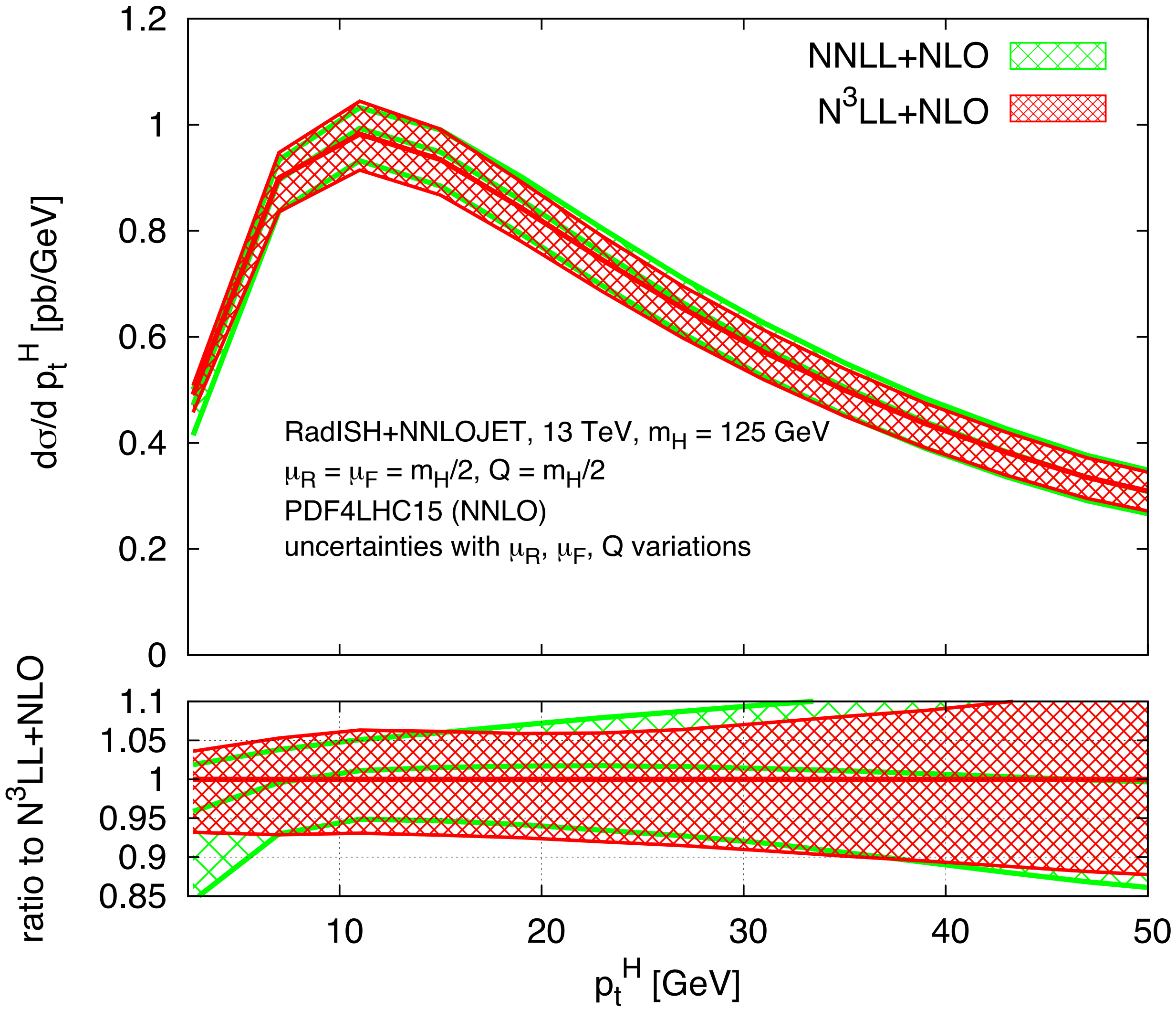
Higher-order (in a log sense) constants from FO in the multiplicative scheme. Only one parameter, i.e. how fast to switch off the logarithms

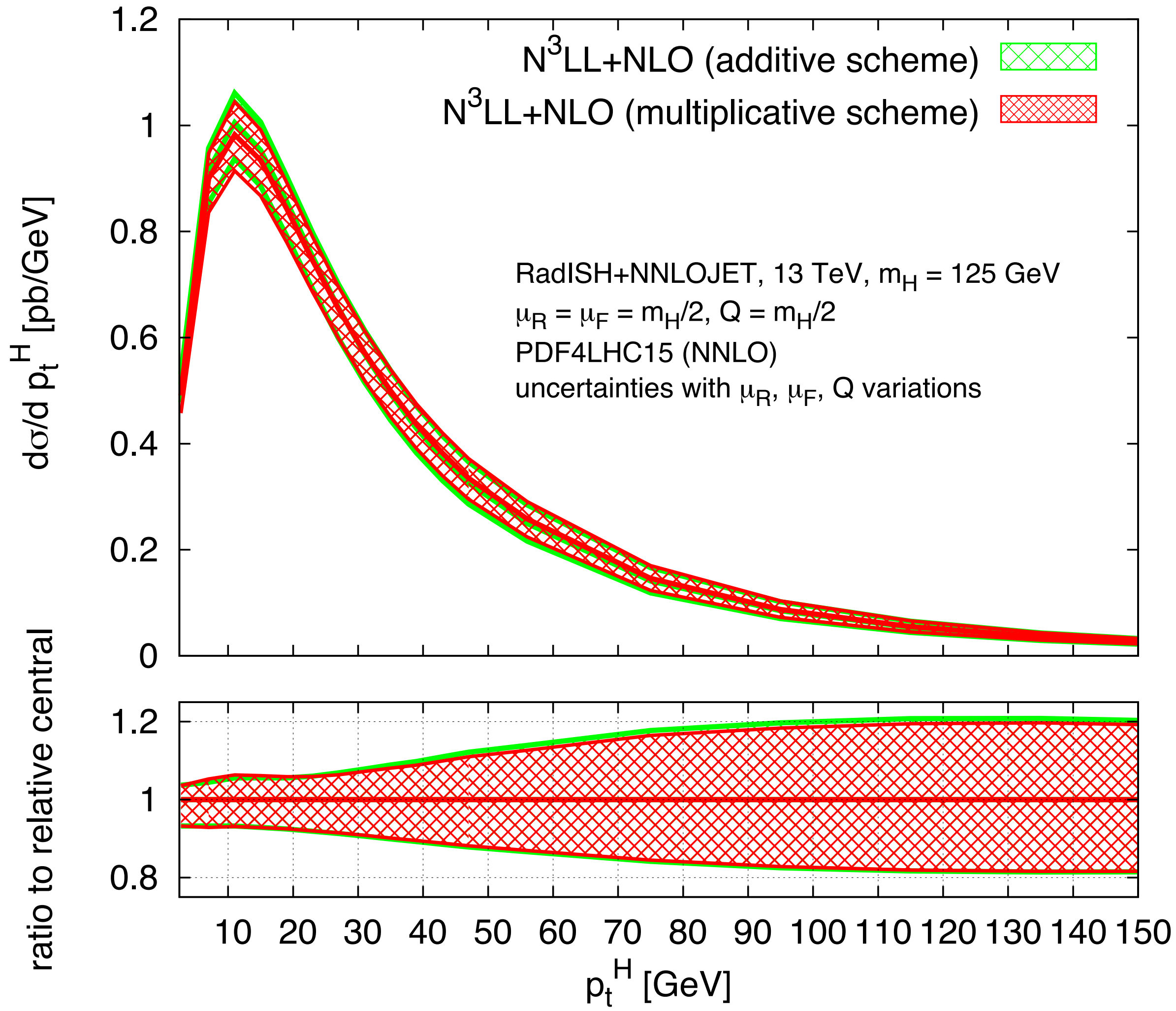
An example: Higgs pT spectrum

- Implementation in a MC code (RadISH) up to N³LL
  - fully differential in Born kinematics

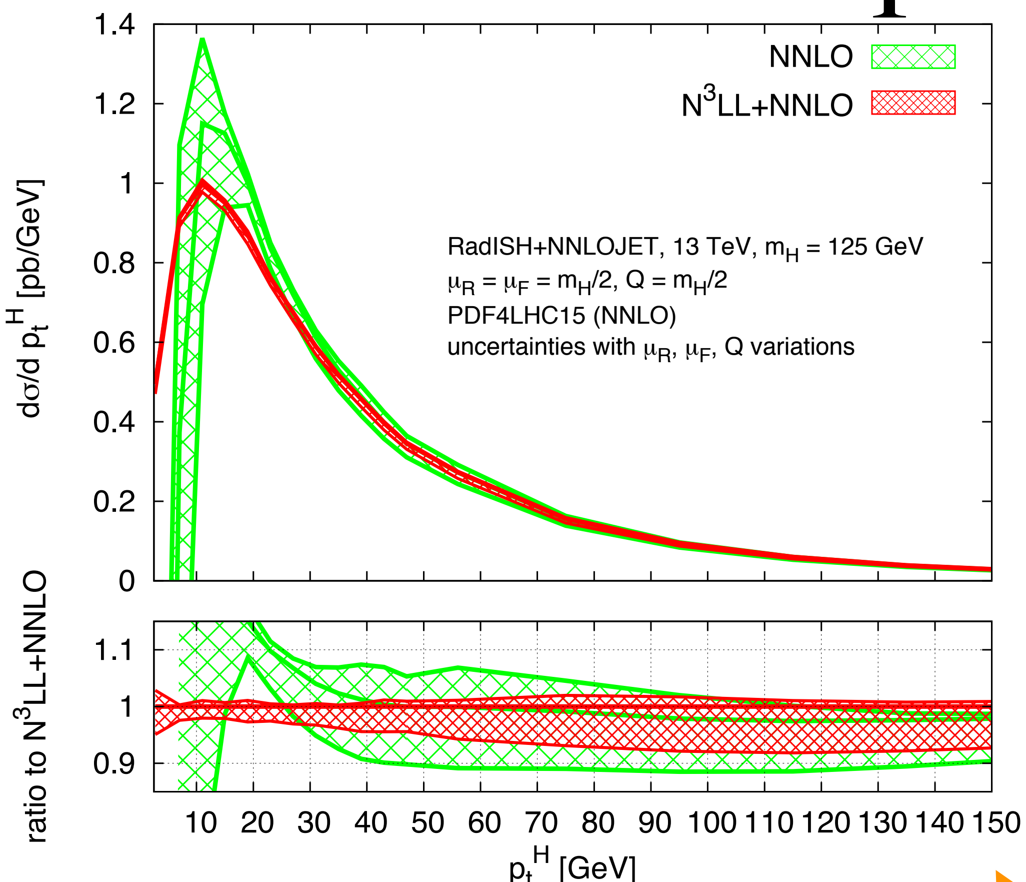
➡N³LL corrections moderate, reduction of uncertainty at small pt

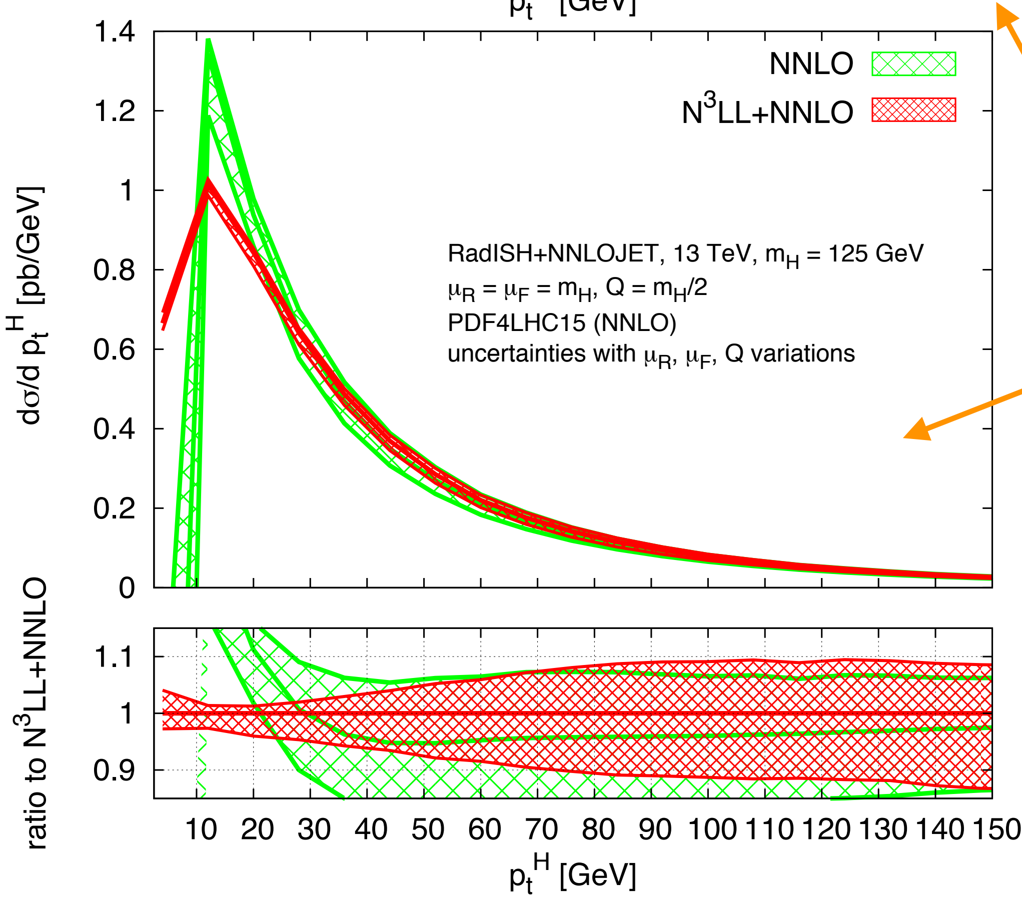
➡Good agreement between different matching schemes, choose multiplicative solution at higher order

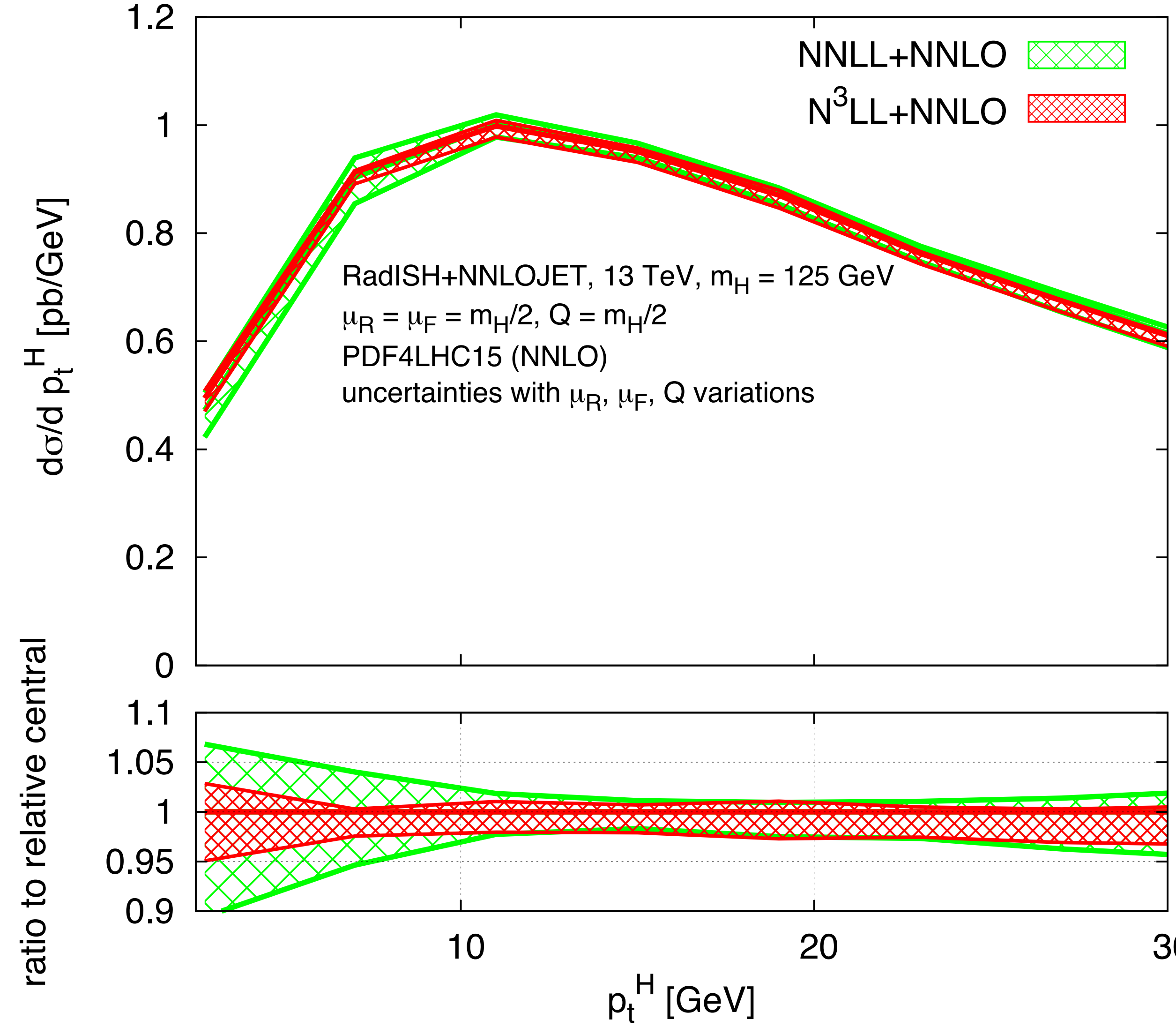




An example: Higgs pT spectrum



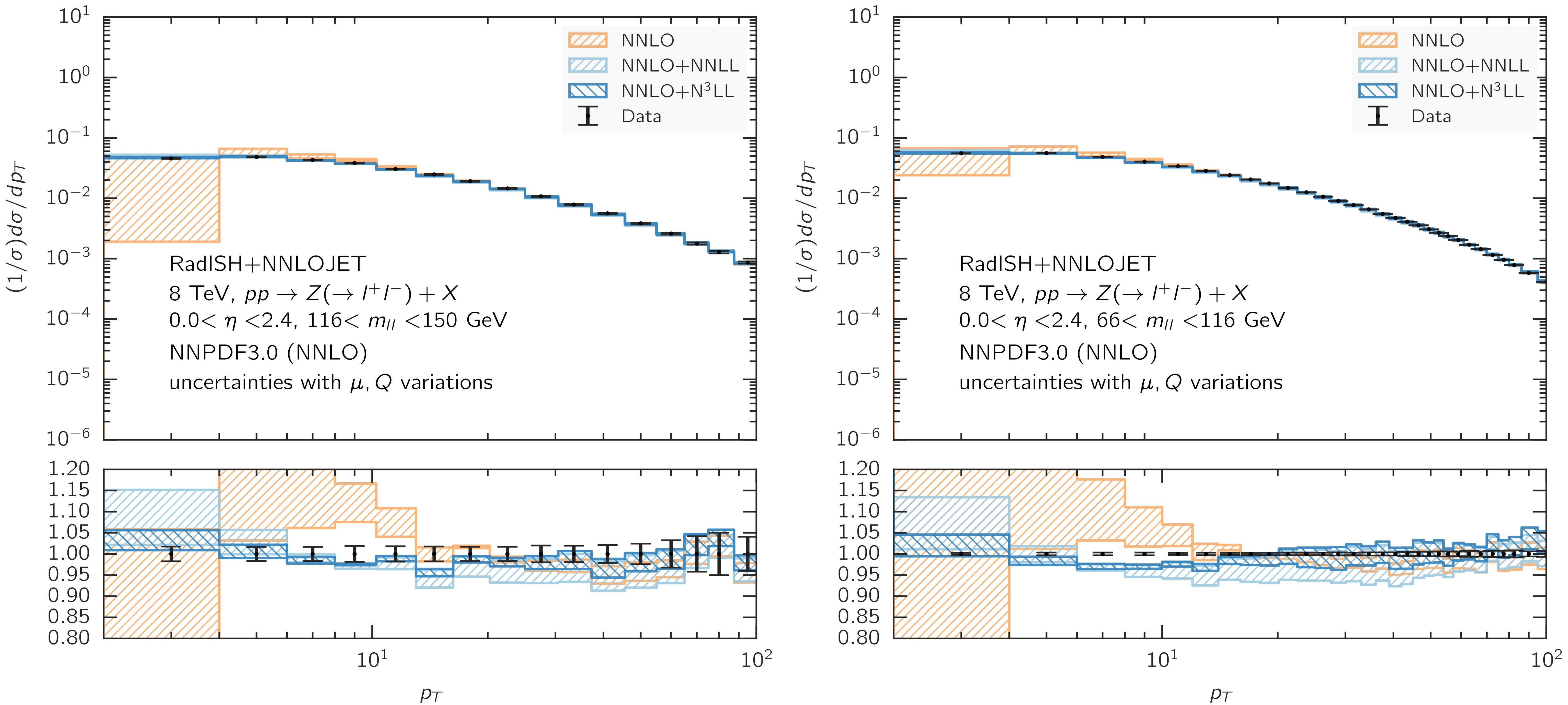




➡ Important cancellations at mH/2 (!), uncertainties likely underestimated at this scale (long known problem)

➡ N^{3}LL corrections amount to a few-% at small pt, reduction of band below 10 GeV consistently with NLO matching

An example: DY distributions (pT)



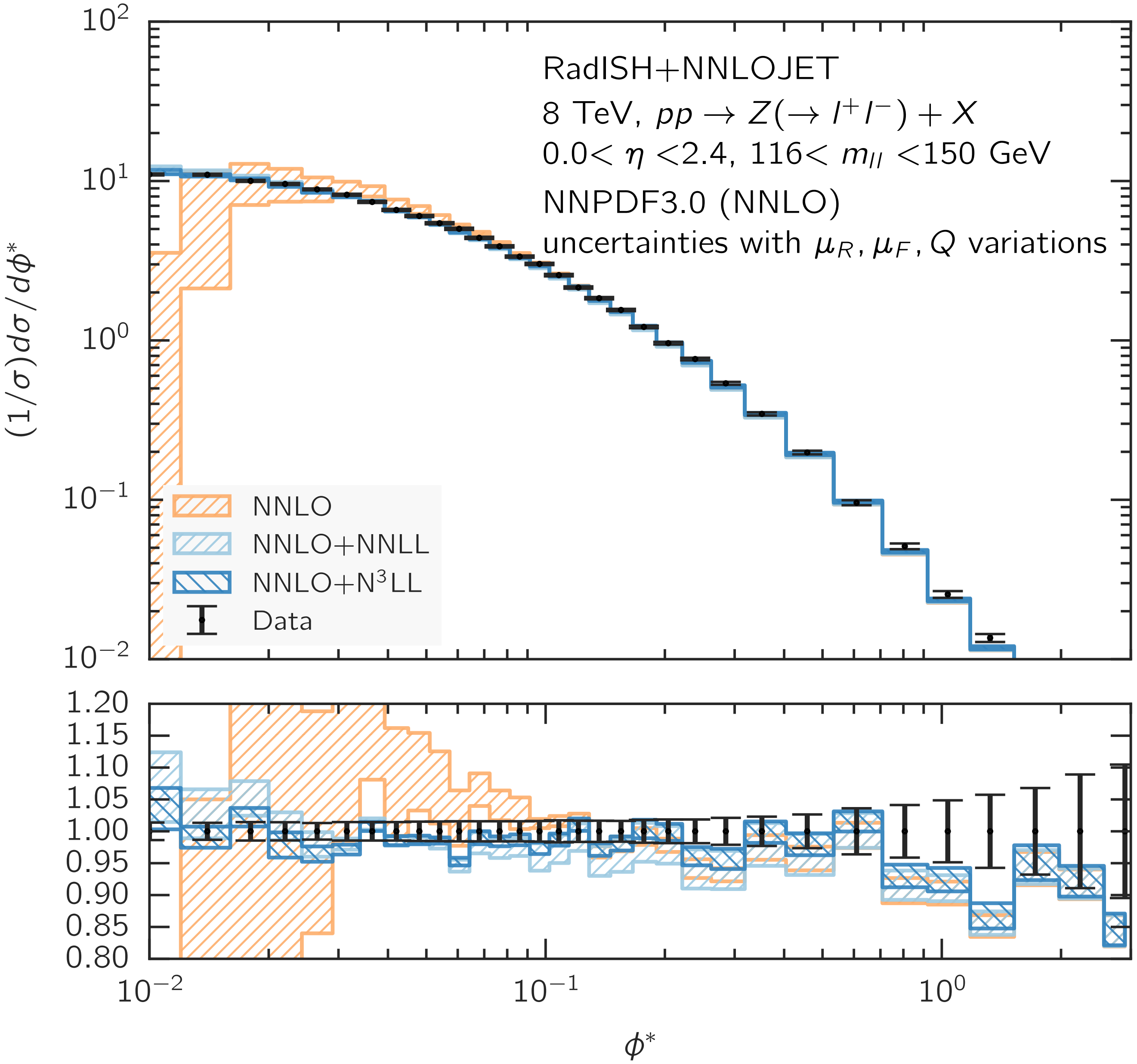
➡Matching to differential NNLO from NNLOJET, assume N³LO correction to total XS is zero (i.e. no as³ constant term included)

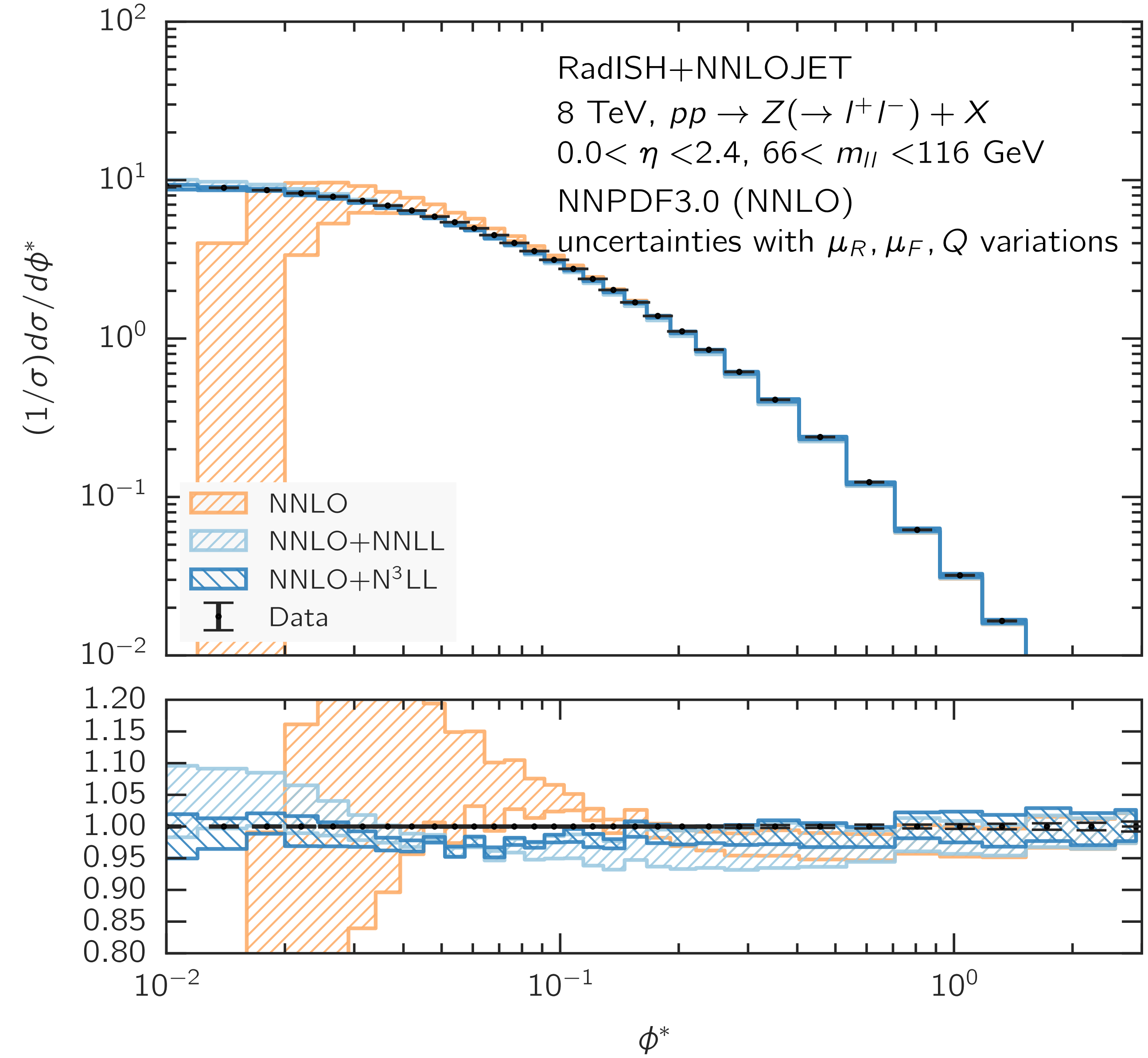
[Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, A. Huss, T.A. Morgan '16]

➡(sub-)percent precision in data, theory can reach ~3-5% accuracy… Other effects important (QED, PDFs, quark masses, hadronisation)

➡Relevant for W-mass studies

An example: DY distributions (phi*)





➡Similar conclusions for angular distributions

Generalisation to other observables

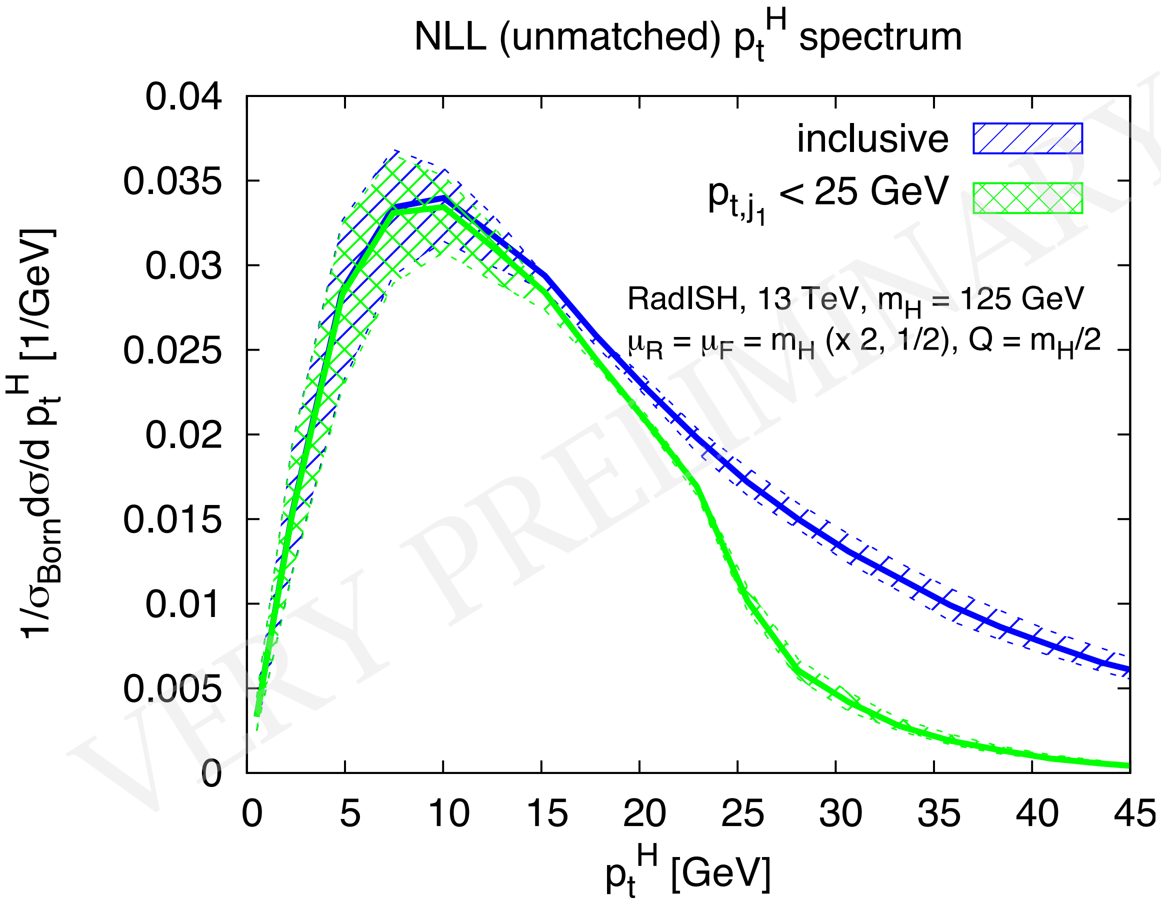
- Extension to non-inclusive observables:
  - Although the resummed formula obtained here is valid for inclusive observables, the Sudakov radiator is universal for all observables which feature the same scaling for a single, soft-collinear emission, i.e. the same LL structure

$$V_{\mathrm{sc}}(\{\tilde{p}\}, k) = \left(\frac{k_t}{M}\right)^a$$

  - The exclusive treatment of resolved correlated blocks (n>1) is simplified by noticing that only a finite number of them must be included in the resolved radiation beyond NLL

- This leads to a general algorithm for all rIRC observables

- Multi-differential cross sections:
  - Not being fully inclusive in the radiation allows one to have more exclusive cuts. The logarithmic accuracy can be easily spoiled (a lot of care is required!)
  - This makes it possible to access exclusive cross sections with higher logarithmic order (?)



Conclusions

- Higher-order resummation can be formulated directly in momentum space without the need for a factorisation for the considered observable
- Currently, two-scale problems in two-emitter processes are solved for all rIRC safe cases
  - Systematic extension to any logarithmic order
  - Efficient implementation in a computer code: automation possible
  - Analytic resummation formulated in a language closer to parton showers
- Current and future directions
  - Joint resummations
  - NNLL sof+/collinear radiator for all rIRC safe observables
  - Study of multi-leg problems beyond NLL
  - Formulation in SCET

Thank you for listening

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$\boxed{\begin{aligned}
\mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) = & \sum_{c,c'} \frac{d|M_B|^2_{cc'}}{d\Phi_B} \sum_{i,j} \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{x_2}^{1} \frac{dz_2}{z_2} f_i\left(k_{t1}, \frac{x_1}{z_1}\right) f_j\left(k_{t1}, \frac{x_2}{z_2}\right) \\
& \left\{ \delta_{ci}\delta_{c'j}\delta(1-z_1)\delta(1-z_2) \left(1 + \frac{\alpha_s(\mu_R)}{2\pi} H^{(1)}(\mu_R) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(2)}(\mu_R)\right) \right. \\
& + \frac{\alpha_s(\mu_R)}{2\pi} \frac{1}{1-2\alpha_s(\mu_R)\beta_0 L} \left(1 - \alpha_s(\mu_R)\frac{\beta_1}{\beta_0} \frac{\ln(1-2\alpha_s(\mu_R)\beta_0 L)}{1-2\alpha_s(\mu_R)\beta_0 L}\right) \\
& \times \left(C_{ci}^{(1)}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \\
& + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1-2\alpha_s(\mu_R)\beta_0 L)^2} \left(\left(C_{ci}^{(2)}(z_1) - 2\pi\beta_0 C_{ci}^{(1)}(z_1) \ln\frac{M^2}{\mu_R^2} \right) \delta(1-z_2)\delta_{c'j} \right. \\
& \left. + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\} \right) + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} \frac{1}{(1-2\alpha_s(\mu_R)\beta_0 L)^2} \left(C_{ci}^{(1)}(z_1) C_{c'j}^{(1)}(z_2) + G_{ci}^{(1)}(z_1) G_{c'j}^{(1)}(z_2) \right) \\
& \left. + \frac{\alpha_s^2(\mu_R)}{(2\pi)^2} H^{(1)}(\mu_R) \frac{1}{1-2\alpha_s(\mu_R)\beta_0 L} \left(C_{ci}^{(1)}(z_1)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\} \right) \right\}
\end{aligned}}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities

Numerical implementation: RadISH

- Since the transverse momenta of the *resolved* reals are of the same order, we can expand the whole integrand about $k_{ti} \sim k_{t1}$ up to the desired logarithmic accuracy
- This expansion allows us to compute higher-order corrections to the NLL *resolved* reals by simply including one correction at a time

e.g. expansion up to NLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$k_{ti}/k_{t1} = \zeta_i = \mathcal{O}(1)$$

$$\boxed{\int d\mathcal{Z}[\{R', k_i\}] G(\{\tilde{p}\}, \{k_i\}) = \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\epsilon}^{1} \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) G(\{\tilde{p}\}, k_1, \ldots, k_{n+1})}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions d*Z* is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

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e.g. expansion up to NNLL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) \right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right)$$

$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \right)$$
$$\times \left(R''(k_{t1}) \ln\frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln\frac{1}{\zeta_s} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\}$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities

- The ensemble of NLL real emissions dZ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Numerical implementation: RadISH

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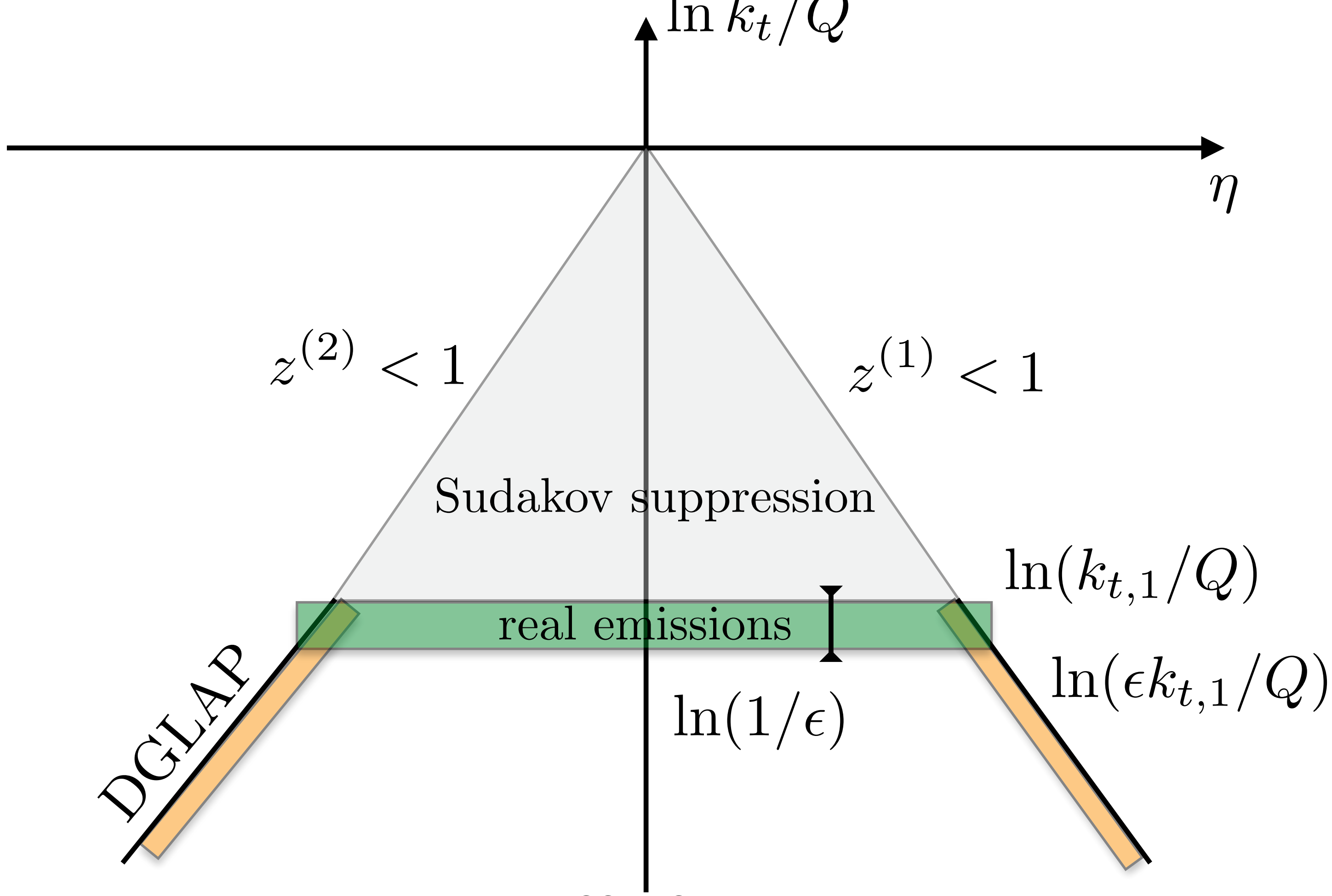
$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) \right)$$
$$\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s} \right) - R'(k_{t1}) \left(\partial_L \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Bigg\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Bigg\}$$

$$+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1})$$
$$\times \Bigg\{ \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1}) \right)^2 \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} - \partial_L \mathcal{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left(\ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}} \right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\}$$
$$\times \Bigg\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1})\right) -$$
$$\Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Bigg\} + \mathcal{O}\left(\alpha_s^n \ln^{2n-6} \frac{1}{v} \right)$$

- Coefficient functions and hard-virtual corrections absorbed into effective parton luminosities
- The ensemble of NLL real emissions dZ is generated as a parton shower. Fast numerical evaluation with Monte-Carlo methods.

Treatment of initial state radiation

- At NLL *resolved* real radiation is soft and collinear, therefore there's no overlapping with the DGLAP evolution (PDFs can be evaluated at kt1)
- Beyond NLL a *resolved* real hard-collinear radiation is allowed; need to perform of DGLAP evolution exclusively for a fixed number of collinear emissions



- e.g. at NNLL expand around the IR cutoff of the last resolved emission

$$q(x, \epsilon k_{t,1}) = q(x, k_{t,1}) - \frac{\alpha_s(k_{t,1})}{\pi} P(z) \otimes q(x, k_{t,1}) \ln \frac{1}{\epsilon} + \mathcal{O}(\mathrm{N^3LL})$$

cutoff dependence cancels against the real counterpart

Equivalence to CSS formula

- Hard-collinear emissions off initial-state legs require some care in the treatment of kinematics. Final result reads

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0)$$

$$\begin{aligned}
\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = & \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \\
& \times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\} \\
& \sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right) \\
& \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi} \mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right) \\
& \times \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right),
\end{aligned}$$

- Formulation equivalent to b-space result, up to a scheme change. Using the delta representation for the distribution one finds

$$\boxed{\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + \ldots + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}}}$$

$$\frac{d\Sigma(v)}{dp_t d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \frac{d\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v)}{dp_t} \mathbf{f}_{N_2}(\mu_0) =$$

$$\boxed{(1 - J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12} \frac{\partial^3}{\partial \ln(Mb/b_0)^3} \Theta(k_t - \frac{b_0}{b}) + \ldots}$$

$$\begin{aligned}
= & \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(b_0/b)) H(M) \mathbf{C}^{c_2}_{N_2}(\alpha_s(b_0/b)) \mathbf{f}(b_0/b) \\
& \times \exp\left\{-\sum_{\ell=1}^{2} \int_0^M \frac{dk_t}{k_t} \mathbf{R}'_\ell(k_t)\,(1 - J_0(bk_t))\right\}.
\end{aligned}$$