# $B^0 - \bar{B}^0$ Mixing: Multi-Loop QCD Sum-Rule Analysis

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# Motivation

SM is now complete: Era of precision tests

Flavor physics and B-physics is the El Dorado –
plenty of experimental data: LHCb, Belle II,...

In theory, a stumbling block – QCD, twofold difficulties:

i) perturbative analysis:
SM has quite a few of different scales, $m_t \sim 170\ \mathrm{GeV}$, $m_b \sim 5\ \mathrm{GeV}$, $m_s \sim 0.1\ \mathrm{GeV}$, $\Lambda_{\mathrm{QCD}} \sim 0.5\ \mathrm{GeV}$,
expansions are in $\alpha_s \ln(m_t/m_b)$ and
eventually $\alpha_s \ln(m_t/\mu)$ with $\mu \sim \Lambda_{\mathrm{QCD}}$
ii) Non-perturbative aspects:
Quarks and Gluons vs. Hadrons

The point (i) is technical (rather difficult),
while point (ii) still (un/partly) solved

# Mixing

$B^0 - \bar{B}^0$ is a two-state flavor system with $\Delta B = 2$.

In recent years, progress has been made in both

- Perturbation Theory (problem (i), NNLO, 2-3 loops)
- Hadronic Matrix Element (problem (ii), sophisticated SR, lattice simulations)

This talk is based on

*A.Grozin,R.Klein,ThM,AAP, Phys.Rev. D94, 034024 (2016)*

*A.Grozin,ThM,AAP, Phys.Rev. D96(2017)074032*

*A.Grozin,ThM,AAP, arXiv:1806.00253, PRD, to appear (2018)*

*ThM, B.D. Pecjak, AAP, Eur.Phys.J. C71 (2011) 1607*

both (i) and (ii) are discussed.

# $B^0 - \bar{B}^0$ mixing: phenomenology

$b$ and $\bar{b}$ quarks hadronize into flavor eigenstates $(B^0, \bar{B}^0)$ which then evolve as

$$i\frac{d}{dt}\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix} = H_{eff}\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix}$$

with $H_{eff}$ being a $2 \times 2$ (nondiagonal !) matrix

$$H_{eff} = (M - i\Gamma/2)_{ij}\,, \quad i, j = 1, 2$$

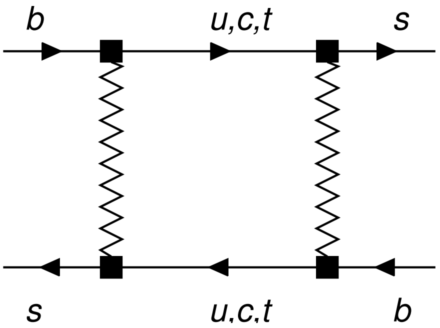
Eigenstates are $(B_L, B_H)$ with “fuzzy” beauty

Observables of $B^0 - \bar{B}^0$ system:

mass difference: $\Delta m = M_{heavy} - M_{light} \approx 2\,|M_{12}|$
decay rates difference:
$\Delta\Gamma = \Gamma_L - \Gamma_H \approx -2\,|\Gamma_{12}|\cos\Phi$, $\Phi = \arg(-M_{12}/\Gamma_{12})$

# $B^0 - \bar{B}^0$: SM (EW/flavor) picture

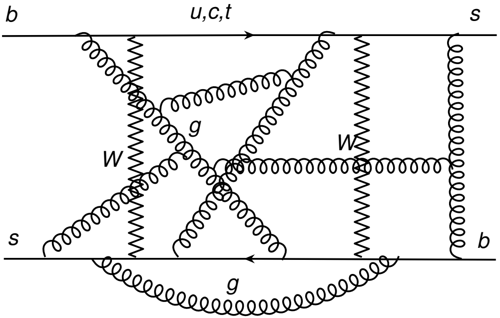
Double W-boson exchange gives $\Delta F = 2$ process




Famous box diagram at EW level

# $B^0 - \bar{B}^0$ mixing: SM (EW+QCD) picture

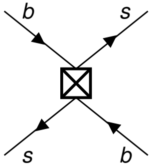
Full SM diagrams with QCD corrections




$m_t = 170$ GeV, $m_W = 90$ GeV,
$m_b = 5$ GeV, $m_s \approx 0$ GeV, $\Lambda_{\text{QCD}} \sim 0.5$ GeV
Expansion parameter enhanced $\sim \alpha_s \ln(m_t/m_b)$

# Effective theory approach

Heavy fields $(t, W)$ are integrated out:




Effective (local) Hamiltonian

$$H_{eff} = \frac{G_F^2 M_W^2}{4\pi^2} \left(V_{tb}{}^* V_{td}\right)^2 C(m_t, m_W, m_b, \alpha_s)\{\bar{b}_L \gamma_\sigma s_L \bar{b}_L \gamma^\sigma s_L\}$$

$C(m_t, m_W, m_b, \alpha_s(\mu))$ is computable in QCD perturbation theory, known at NLO (two loop graphs)

Hadronic part is a matrix element of a local operator

$$\langle \bar{B}^0 | \bar{b}_L \gamma_\sigma s_L \bar{b}_L \gamma^\sigma s_L | B^0 \rangle_{(\mu \sim m_b)}$$

# Aspects of Problem (i) and (ii)

(i) Coefficient $C(m_t, m_W, m_b, \alpha_s(\mu))$

- Matching of the SM onto Fermi theory with $\Delta F = 2$
- Computation in dimensional regularization: and Extensions of the operator basis needed.
- Several four quark operators, only one is physical, other have vanishing matrix elements (evanescent operators)

(ii) Matrix Element $\langle \bar{B}^0 | \bar{b}_L \gamma_\sigma s_L \bar{b}_L \gamma^\sigma s_L | B^0 \rangle_{(\mu \sim m_b)}$

- Lattice QCD or QCD Sum Rules
- Full, unquenched Lattice Simulations with dynamic $b$ quarks available
- ... Still matching to the lattice operators at 1 loop only.
- Here: New QCD Sum Rule Analysis

## QCD Sum Rule approach for the Matrix Element

Since $m_b \gg \Lambda$ one can use Heavy Quark Expansions

- Matching of the QCD operator to HQET:

$$\bar{b}_L \gamma_\sigma s_L \bar{b}_L \gamma^\sigma s_L = C_1 \{ \bar{h}^+ \gamma_\sigma s_L \bar{h}^- \gamma^\sigma s_L \} + C_2 \{ \bar{h}^+ s_R \bar{h}^- s_R \}$$

- The operators $h^+, h^-$ are the static fields for quark b and anti-quark b
- Perturbative Result for the matching coefficients:

$$C_1 = 1 - \frac{7}{2}\frac{\alpha_s}{\pi}, \quad C_2 = -\frac{3}{2}\frac{\alpha_s}{\pi}$$

- HQET operators are at the scale of order $\Lambda_{\mathrm{QCD}}$
  non-perturbative methods required:
  Lattice or QCD Sum Rules

# Matrix Element in the Heavy Quark Limit

Sum Rule Set up: Three-point correlator

$$K = \int d^d x_1\, d^d x_2\, e^{ip_1 x_1 - ip_2 x_2} \langle 0 | T \tilde{\jmath}_2(x_2) \tilde{Q}_1(0) \tilde{\jmath}_1(x_1) | 0 \rangle$$

- four-quark operator $\tilde{Q}_1 = \bar{h}_+ \gamma_\beta s_L\, \bar{h}_- \gamma^\beta s_L$
- Interpolating currents $\tilde{\jmath}_1(\mu) = \bar{s}\gamma_5 h_+,\ \tilde{\jmath}_2(\mu) = \bar{s}\gamma_5 h_-$
- Overlap with the static $B$ meson: $\langle 0 | \tilde{\jmath}_1 | \bar{B}(p) \rangle_\mu = F(\mu)$

Dispersion relation for Euclidean times $\tau_{1,2}$ ($\tau = it$)

$$K(\tau_1, \tau_2) = \int_0^\infty d\omega_1\, d\omega_2\, e^{-\omega_1 \tau_1 - \omega_2 \tau_2}\, \rho(\omega_1, \omega_2)$$

determines the spectral density $\rho(\omega_1, \omega_2)$

Hadronic picture: $B$-meson pole plus continuum

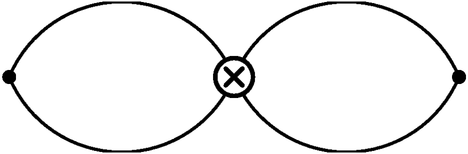
$$\rho_H(\omega_1, \omega_2) = F^2 \langle \bar{B}^0 | \tilde{Q}_1 | B^0 \rangle \delta(\omega_1 - \bar{\Lambda}) \delta(\omega_2 - \bar{\Lambda}) + \rho_{\rm cont}(\omega_1, \omega_2)$$

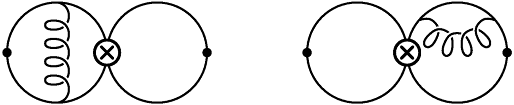
- Lattice computes $K(\tau_1, \tau_2)$ and fits B-contribution
- Sum-Rule method uses Operator Product Expnsion (OPE) to explicitly computes $K(\omega_1, \omega_2)$ and analytically continues it to find

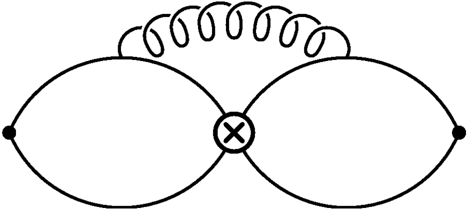
$$F^2 \langle B | \tilde{Q}_1 | B \rangle = \int d\omega_1 d\omega_2 \, \rho^{\rm OPE}(\omega_1, \omega_2).$$

$\boxed{\text{Why are Sum Rules still competitive quantitatively?}}$

# OPE diagrams

LO:  .

NLO fact:  .

NLO nonfact: 

A. Grozin, R. N. Lee: JHEP 02 (2009) 047 [arXiv:0812.4522]

# Structure of OPE diagrams

OPE diagrams fall into two categories

$$K(\omega_1,\omega_2) = K_{\text{fac}}(\omega_1,\omega_2) + \Delta K(\omega_1,\omega_2)$$

The factorized part has an explicit form

$$K_{fac}(\omega_1,\omega_2) = \left(1+\frac{1}{N_c}\right) \times \Pi(\omega_1)\Pi(\omega_2)$$

with $\Pi(\omega_i)$ - a 2-point correlator

$$p^\alpha \Pi(\omega) = i \int dx e^{ipx} \langle T \tilde{j}(x) \bar{h} \gamma^\alpha (1-\gamma_5) d(0) \rangle$$

SR for the factorized piece $K_{\text{fac}}(\omega_1,\omega_2)$ yields $B = 1$.

Sum Rules compute the deviation from $B = 1$
(at a precision level of 20%)

# Results 1: Analytical Expressions

We have computed three loop diagrams for three point correlator (NLO result)

$$\rho(\omega_1,\omega_2) = \left(1+\frac{1}{N_c}\right)\rho(\omega_1)\rho(\omega_2) + \Delta\rho(\omega_1,\omega_2)$$

$$= \left(1+\frac{1}{N_c}\right)\rho(\omega_1)\rho(\omega_2)\left(1-\frac{\alpha_s}{4\pi}\frac{N_c-1}{2N_c}\left(\frac{4}{3}\pi^2-5\right)\right)$$

*A.Grozin,R.Klein,ThM,AAP, Phys.Rev. D94, 034024 (2016)*

- The result is rather simple and $\omega$ independent (only for the *LL* operator)
- Numerically $\left(1-2.72\frac{\alpha_s}{4\pi}\right) = 1-0.68\frac{\alpha_s(\mu\sim 1\ \mathrm{GeV})}{\pi}$

# Results 2: Numerical Values

The various NLO contributions:

- Perturbative contribution (3-loop)
  $\Delta B_{PT} = -0.10 \pm 0.02 \pm 0.03$

  *A.Grozin,R.Klein,ThM,AAP, Phys.Rev. D94, 034024 (2016)*
- Quark condensate contribution (2-loop)
  $\Delta B_{q} = -0.002 \pm 0.001$

  *A.Grozin,R.Klein,ThM,AAP, Phys.Rev. D94, 034024 (2016)*
- Other condensates (tree-level+2-loop gluon cond)
  $\Delta B_{nonPT} = -0.006 \pm 0.005$

  *ThM, B.D. Pecjak, AAP, Eur.Phys.J. C71 (2011) 1607*

$\boxed{\text{Total } \Delta B = -0.11 \pm 0.04 \pm 0.03}$

# Comparison to Lattice Calculations

Bag parameter from the sum rule: $B = 1 - (0.11 \pm 0.04)$

For comparison:
Use the RG invariant definition $\hat{B} = ZB$ with

$$Z = \alpha_s(m_b)^{-\frac{\gamma_0}{2\beta_0}} \left(1 + \frac{\alpha_s(m_b)}{4\pi} \left(\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2}\right)\right)$$

$Z = 1.51$ at $\alpha_s(m_b) = 0.2$. Thus: $\boxed{\hat{B}_{SR} = 1.34 \pm 0.06}$

Latest lattice result (A.Bazavov et al. (2016)) $\boxed{\hat{B}_{latt} = 1.38(12)(6)}$

Other lattice results

(S.Aoki et al., Review, 2016) $\hat{B}_{latt} = 1.26(9)$

(2009 (P.Lepage), 2015 (Y.Aoki)) $\hat{B}_{latt} = 1.30(6)$

# NNLO Matching to HQET

Matching to leading order in $1/m$ (including evenescent operators)

$$c\bar{b}_L\gamma_\sigma s_L\,\bar{b}_L\gamma^\sigma s_L =$$
$$C_1\{\bar{h}^+\gamma_\sigma s_L\bar{h}^-\gamma^\sigma s_L\} + C_2\{\bar{h}^+ s_R\bar{h}^- s_R\} + C_E O_E$$

NLO coefficients $C_1 = \left(1 - \frac{7}{2}\frac{\alpha_s}{\pi}\right)\,, \quad C_2 = -\frac{3}{2}\frac{\alpha_s}{\pi}$

Chose a convenient operator basis: ($\gamma^n$: antisymmetrized product of $n$ $\gamma$'s)

$$O_n = \bar{h}^+\gamma_\perp^n q\;\bar{h}^-\gamma_\perp^n q \quad \text{with} \quad \gamma_\perp = \gamma - v\not{v}$$
$$O_l = O_1 - O_0 \qquad O_p = \frac{3}{4}O_0 + \frac{1}{4}O_1$$

NNLO coefficients

$$\begin{aligned} C_l(m_b) &= 1 - 12a_s - 175.6a_s^2, \\ C_p(m_b) &= -8a_s - 311.2a_s^2, \qquad\qquad a_s = \frac{\alpha_s}{4\pi} \end{aligned}$$

*A.Grozin,ThM,AAP, arXiv:1806.00253, PRD, to appear (2018)*

Convergence seems to be marginal but

- This is an artefact of the $\overline{\mathrm{MS}}$ scheme
- Relation between physical quantities seems to converge better

$$\Delta m = const(1 - 6.4a_s - (4.9 + x_l^{(2)})a_s^2))f_B^2$$

- $x_l^{(2)}$: NNLO contribution to SR ($\sim$ 1, not 100)

# Discussion

To take home

- The accuracy of the QCD Sum Rule for the ($\hat{B}$ parameter) is better than 10%
  $\hat{B}_{SR} = 1.34 \pm 0.06, \qquad \hat{B}_{latt} = 1.26(9), 1.38(12)(6)$
- New feature in phenomenology: NLO for the perturbative coefficients is not sufficient for a precision better than 10%.

$C^{(1)}_{\mathrm{QCD}\to\mathrm{HQET}} = 1 - \frac{7}{2}\frac{\alpha_s}{\pi} \approx 1 - 0.19 + ? \to 1 - 0.19 - 0.04$
$C^{(2)}_{\mathrm{QCD}\to\mathrm{HQET}} = \alpha_s(1 - 10\frac{\alpha_s}{\pi}) \approx \alpha_s(1 - 0.16)$

# Conclusion

Precision flavor physics is promising for BSM searches!
Theory requires two parts:

- Perturbation Theory, at NNLO at least
- Calculation of Matrix Elements, tools are available (lattice QCD and [to some extend] QCD Sum Rules)
- Matching of lattice to continuum needs to be improved
- $\overline{\mathrm{MS}}$ as a renormalization scheme? Bad convergence? Relations between physical quantities seem to be ok.

We have realised this program for $B\overline{B}$ Mixing at NNLO
The results are competitive with current lattice results!

# Outlook

- SR technology works and is numerically competitive
- One can compute also other operators (not only LL):
  - for width differences

    *M. Kirk, A. Lenz, T. Rauh, JHEP 1712 (2017) 068*
  - for new physics

    *L. Di Luzio, M. Kirk, A. Lenz, Phys.Rev. D97 (2018) no.9, 095035*
  - It is useful as an independent check/confirmation of lattice results
- Include the strange mass for a calculation of $B_s\overline{B}_s$ Mixing *see talk by T. Rauh at this conference*