$|V_{us}|$ from taus

Taku Izubuchi

based on


“Novel $|V_{us}|$ Determination Using Inclusive Strange $\tau$ Decay and Lattice HVPs”

arXiv:1803.07228 [hep-lat]

( RBC and UKQCD Collaborations )
• Tau inclusive decay, Finite Energy Sum Rule
• New method: Combining FESR + Lattice HVP
• Results & Updates
• Conclusion
Tau decay

- $\tau \rightarrow \nu + \text{had}$ through V-A vertex. $V_{ij}$ : CKM, EW correction $S_{EW}$ [Marciano, Sirlin]

$$R_{ij,V/A} = \frac{\Gamma(\tau^- \rightarrow \text{hadrons}_{ij} \nu_{\tau})}{\Gamma(\tau^- \rightarrow e^-\bar{\nu}_e\nu_{\tau})}$$

$$= \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_{\tau}^2} \int_0^{m_{\tau}^2} ds \left(1 - \frac{s}{m_{\tau}^2}\right)^2 \left[(1 + 2\frac{s}{m_{\tau}^2}) \rho_{ij,V/A}^{(0+1)}(s) + \rho_{ij,V/A}^{(0)}(s)\right]$$

- The Spin=0 and 1, vacuum polarization, Vector(V) or Axial (A) current-current 2pt

$$\Pi_{ij,V/A}^{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T J_{ij,V/A}^{\mu}(x) J_{ij,V/A}^{\dagger\nu}(0) | 0 \rangle$$

$$= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{ij,V/A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{ij,V/A}^{(0)}(q^2)$$
Finite Energy Sum Rule (FESR)

[Shifman, Vainstein, Zakharov 1979]

- FESR = Optical theorem (Unitarity) + Dispersion relation (Analyticity)

- Optical theorem relate $S=-1$ spectral function $\rho_{V/A,ij}^{0/1}(s)$ and HVP $\Pi_{V/A,ij}^{0/1}(s)$ for given quantum number: flavor (us or ud), spin (0 or 1), parity (V or A)

$$\frac{1}{\pi} \text{Im}\Pi(s) = \rho(s)$$

- Do finite radius contour integral for arbitrary regular weight function $w(s)$

$$\int_{s_{th}}^{s_0} ds \rho(s)w(s) = + \frac{i}{2\pi} \oint_{|s|=s_0} ds \Pi(s)w(s)$$

- Real axis integral is extracted from experimental decay energy distribution $dR_\tau/ds$

$$\frac{dR_{i,j;V/A}}{ds} = \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \omega_\tau(s) \rho(s)$$
\[ |V_{us}| \text{ determination from FESR} \]

[ E. Gamiz, et al., 2003, 2005, Maltman et al. 2006 ]

- Inclusive differential \( \tau \) decay rate with weight \( w(s) \)

\[
R_{ij}^{\omega}(s_0) \equiv \int_{s_{th}}^{s_0} ds \frac{dR_{ij}}{ds} \frac{\omega(s/s_0)}{\omega_{\tau}(s/m_\tau^2)}
\]

- Take difference between up-down and up-strange channel

\[
\Delta R^{\omega} = \frac{R_{ud}^{\omega}}{|V_{ud}|^2} - \frac{R_{us}^{\omega}}{|V_{us}|^2}
\]

- \( |V_{ud}| \) and \( m_s \) as input, selecting \( s_0 = m_\tau^2, \omega = \omega_{\tau}(s/s_0) \)

\[
|V_{us}| = \left( \frac{R_{us}^{\omega}(s_0)}{R_{ud}(s_0)} \right) \frac{1}{|V_{ud}|^2} \left[ \frac{R_{us}(s_0)}{|V_{us}|^2} - [\Delta R^{\omega}(s_0)]^{pQCD} \right]
\]

- For \( s > s_0 \), fixed-order or contour-improved pQCD is used. OPE condensations at \( \text{dim}=4,6 \ldots \) are input/assumed. (a source of unaccounted uncertainties)
\( V_{us} \) extraction from FESR

- \(-3.1\sigma\) deviation from CKM unitarity
- Use Vacuum Saturation assumption estimate for OPE’s \( D=6 \) condensations, and \( D=8 \) is dropped.
- Tests of these assumptions with different BF, or by varying \( s_0 \) and/or \( \omega(s) \) suggest unaccounted systematic error. Further more, a simultaneous fit of the condensates along with \( |V_{us}| \), yielded significantly large and stable \( |V_{us}| \) [Antonelli et al. 2013, Hudspith et al. 2017, HFLAV 2017].
- Uncertainties in \( us \) spectral data at high energy dominate error.
- How to improve?

\[
\begin{align*}
&|V_{us}| = 0.2237 \pm 0.0010, \text{PDG 2016} \\
&|V_{us}| = 0.2254 \pm 0.0007, \text{PDG 2016} \\
&|V_{us}| = 0.2258 \pm 0.0009, \text{CKM unitarity, PDG 2016} \\
&|V_{us}| = 0.2186 \pm 0.0021, \tau \to s \text{ incl., HFLAV Spring 2017} \\
&|V_{us}| = 0.2236 \pm 0.0018, \tau \to K\nu / \tau \to \pi\nu, \text{HFLAV Spring 2017} \\
&|V_{us}| = 0.2216 \pm 0.0015, \tau \text{ average, HFLAV Spring 2017}
\end{align*}
\]
Our proposal: Combining FESR and Lattice QCD

- By combining both the first principle lattice QCD and experimental data, more reliable precise analysis could be carried out.

**Idea**: generalized dispersion relations using weight function with poles at space-like (Euclidean) $s = -Q^2 < 0$ region.

- One could control uncertainties from pQCD OPE, quark-hadron duality violation (contribution of hadron resonance above $m^2_{\tau}$) in theoretical estimation, as well as the higher hadron multiplicity data, which have larger error, in experimental data.

Fit for OPE term is no longer necessary.

- If we have a reliable estimate for $\Pi(s)$ in Euclidean (space-like) points, $s = -Q^2_k < 0$, we could extend the FESR with weight function $\omega_N(s)$ to have $N$ poles,

$$\int_{s_{th}}^{\infty} w_N(s) \rho(s) = \sum_{k=1}^{N} \text{Res}[w_N(s)\Pi(s)]_{s=-Q^2_k}$$

- $|s| \to \infty$ circle integral vanishes for $N_p \geq 3$. 
weight function \( w_N(s) \)

- Weight function

\[
w(s) = \prod_{k=1}^{N_p} \frac{1}{(s + Q_k^2)} = \sum a_k\frac{1}{s + Q_k^2}, \quad a_k = \sum_{j \neq k} \frac{1}{Q_k^2 - Q_j^2}
\]

\[
\Rightarrow \sum_k (Q_k)^M a_k = 0 \quad (M = 0, 1, \cdots, N_p - 2)
\]

- The residue constraints automatically subtracts \( \Pi^{(0,1)}(0) \) and \( s\Pi^{(1)}(0) \) terms.

- For experimental data, \( w(s) \sim 1/s^n, n \geq 3 \) suppresses
  
  > larger error from higher multiplicity final states at larger \( s < m_\tau^2 \)
  > uncertainties due to pQCD+OPE at \( m_\tau^2 < s \)

- For lattice, \( Q_k^2 \) should be not too small to avoid large stat. error, \( Q^2 \rightarrow 0 \) extrapolation, Finite Volume/Time error. Also not too larger compared to \( m_\tau^2 \) otherwise the higher energy, higher multiplicity, OPE region will be enhanced.
**Experimental $\tau$ data**

- K pole contributions, $\omega_N(m_K^2)\gamma_K$ : $\gamma_K$ either from $\Gamma(\tau \rightarrow K \nu)$ or $\Gamma(K_{\mu2})$.

  Exclusive determination, $\gamma_K = 2|V_{us}|^2f_K^2$ with lattice $f_K$:

  $|V_{us}| = \begin{cases} 
  0.2233(15)_{exp}(12)_{th} & \text{for } \gamma_K[\tau_K] = 0.0012061(167)_{exp}(13)_{IB} \\
  0.2260(3)_{exp}(12)_{th} & \text{for } \gamma_K[K_{\mu2}] = 0.0012347(29)_{exp}(22)_{IB} 
  \end{cases}$

- Belle, BarBar, ALEPH, strange dR/ds distribution data for exclusive, semi-inclusive modes, in the form of

  $$\rho(s) = \left(1 + 2\frac{s}{m_{\tau}^2}\right)\rho^{(1)}(s) + \rho^{(0)}(s)$$

- Unit normalized invariant hadron mass distribution $1/N \times dN(s)/ds$ and Branching Fraction. HFLAV $K\pi$ normalization. Alternate results using the ACLP normalization [Antonelli et al. 2013] are given in the Supplementary Material of arXiv:1803.07228.

- higher multiplicity final states, at larger $s$, have larger errors

- In each channels, correlation matrix for unit normalized distribution are used for available channels ($K\pi$) and added by the Branching Fraction in quadrature. Error from all channel are added in quadrature.
\[ \rho(s) \equiv |V_{us}|^2 \left[ \left( 1 + 2 \frac{s}{m_\tau^2} \right) \text{Im} \Pi^1(s) + \text{Im} \Pi^0(s) \right] \]

To compare with experiments, a conventional value of \( |V_{us}| = 0.2253 \) is used.

For K pole, we assume a delta function form, whose coefficient is obtained from the experimental value of \( \tau \to K \) or \( K \to \mu \) decay widths

\[ \delta(s - m_k^2)0.0012299(46) \sim 2f_k^2|V_{us}|^2 \]
• example: N=3, \( \{ Q_1^2, Q_2^2, Q_3^2 \} = \{ 0.1, 0.2, 0.3 \} \) [GeV^2]
• example: N=4, \( \{ Q_1^2, Q_2^2, Q_3^2, Q_4^2 \} = \{ 0.1, 0.2, 0.3, 0.4 \} \) [GeV^2]
• example: \( N=5 \), \( \{ Q_1^2, Q_2^2, Q_3^2, Q_4^2, Q_5^2 \} = \{0.1, 0.2, 0.3, 0.4, 0.5\} \) [GeV^2]
Lattice HVPs

- From Vector-Vector and Axial-Axial correlation function made of up and strange quarks $\Pi^V/A_{\nu\nu}(Q^2)$

- Local currents with appropriate renormalization factors, $Z_V, Z_A$ computed non-perturbatively, or conserved currents.

- Tensor $\Pi_{\mu\nu}$ are decomposed into spin 1 from $C^{zz}$ and spin 0 from $C^{tt}$ components, with tensor zero-mode subtraction [Blum 2001, Bernecker & Myer 2011, …] \[(g - 2)_\mu\] HVP calculations.

- Smaller $Q^2$ data has a larger statistical error, while larger $Q^2$ suffers discretization errors.
QCD ensemble and statistics

- Main analysis is on two ensemble, at almost physical quark masses ($M_{\pi} \approx 140$ MeV, $M_K \approx 499$ MeV), $V=(5 \text{ fm})^3$.

- Correct the residual up and strange quark mass error by partially quenched calculation.

- Consistent with results on other heavier / smaller ensemble, which are used to estimate size and direction of discretization errors.

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$48^3 \times 96$</td>
<td>1.7295(38)</td>
<td>139</td>
<td>499</td>
<td>88</td>
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<tr>
<td></td>
<td></td>
<td>135</td>
<td>496</td>
<td>5 (PQ-correction)</td>
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<tr>
<td>$64^3 \times 128$</td>
<td>2.359(7)</td>
<td>139</td>
<td>508</td>
<td>80</td>
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<td></td>
<td></td>
<td>135</td>
<td>496</td>
<td>5 (PQ-correction)</td>
</tr>
</tbody>
</table>
Tuning “inclusiveness”

- $N$: number of poles
- $C$ [GeV$^2$]: center of poles
- $\Delta$: spacings of poles (insensitive)

- $N = 4$, $\Delta = 0.05$ GeV$^2$ example
- $K, K\pi$ contribution are largest
- higher multiplicity modes and OPE strongly suppressed.

![Relative spectral contributions (N=4)]
**Error budget**

- Lattice stat errors are still significant (could be improved)
- Discretization error from $O(a\Lambda_{QCD})^4$
- Finite Volume error, estimated from ChPT in FV for $K\pi$ channel
- Isospin breaking effects, corrected from IB in $K_{l2}$ and $K \rightarrow \pi$ analysis [Antonelli, Cirigliano, Lusiani, Passemar, JHEP10(2013)070] ($\pi^0 - \eta$ mixing, EM, IB phase space)  
  Residual theory error: 0.2 %
- pQCD/OPE uncertainty 2% of pQCD contribution
- Experimental spectral function error is currently comparable.
## Error budget

<table>
<thead>
<tr>
<th>contribution</th>
<th>[N, C[GeV²]]</th>
<th>[3, 0.3]</th>
<th>[3, 1]</th>
<th>[4, 0.7]</th>
<th>[5, 0.9]</th>
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<tbody>
<tr>
<td><strong>theory</strong></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( f_K )</td>
<td>0.37</td>
<td>0.20</td>
<td>0.34</td>
<td>0.36</td>
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<tr>
<td>others, stat.</td>
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<td>0.19</td>
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<td>discretization</td>
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<td>0.80</td>
<td>0.25</td>
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<td>scale setting</td>
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<td>0.21</td>
<td>0.11</td>
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<td>FV</td>
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<td>0.04</td>
<td>0.13</td>
<td>0.18</td>
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<tr>
<td>pQCD</td>
<td>0.05</td>
<td>0.26</td>
<td>0.03</td>
<td>0.01</td>
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<tr>
<td><strong>total</strong></td>
<td>0.59</td>
<td>0.91</td>
<td>0.58</td>
<td>0.65</td>
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</tr>
<tr>
<td><strong>experiment</strong></td>
<td></td>
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<td>( K )</td>
<td>0.48</td>
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<tr>
<td>( K\pi )</td>
<td>0.20</td>
<td>0.32</td>
<td>0.23</td>
<td>0.22</td>
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</tr>
<tr>
<td>( K^-\pi^+\pi^- )</td>
<td>0.06</td>
<td>0.16</td>
<td>0.06</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>( \bar{K}^0\pi^-\pi^0 )</td>
<td>0.03</td>
<td>0.09</td>
<td>0.03</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>residual</td>
<td>0.41</td>
<td>1.35</td>
<td>0.41</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td><strong>total</strong></td>
<td>0.66</td>
<td>1.43</td>
<td>0.65</td>
<td>0.59</td>
<td></td>
</tr>
<tr>
<td><strong>Combined total</strong></td>
<td>0.88</td>
<td>1.70</td>
<td>0.87</td>
<td>0.88</td>
<td></td>
</tr>
</tbody>
</table>
Errors Breakup $|V_{us}|$

$N = 4$ case
\begin{itemize}
  \item $N = 3, 4, 5$. Full error. Horizontal dots is exclusive $\tau \to K$ detemination using $f_K$.
  \item All estimated systematic erros included
\end{itemize}
$|V_{us}|$  

$K_{13}, \text{PDG 2016}$

$\Gamma[K_{\mu2}]$

3-family unitarity, HT14 $|V_{ud}|$

$\tau$ FB FESR, HFAG17  
(problematic conventional implementation)

$\tau$ FB FESR, HLMZ17  
(new implementation)

$\tau$, lattice [N=3, C=0.3 GeV$^2$]

$\tau$, lattice [N=4, C=0.7 GeV$^2$]

$\tau$, lattice [N=5, C=0.9 GeV$^2$]

- filled square: K pole from $\gamma_K[\tau_K]$  
empty square: K pole from $\gamma_K[K_{\mu2}]$

HLMZ17 uses the fitted OPE condensations for D $>$ 4 varying $s_0$ and $\omega(s)$. 

Taku Izubuchi, CKM2018, Heidelberg, Germany, September 18, 2018

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• BFs are updated using those reported at Charm 2018 by A. Lusiani

• HLMZ17 is also updated
## Error budget (updated BF at Charm 2018 by A. Lusiani)

<table>
<thead>
<tr>
<th>contribution</th>
<th>relative error (%)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>[N, C[GeV²]]</td>
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<tr>
<td>theory</td>
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<tr>
<td>pQCD</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td></td>
</tr>
<tr>
<td>experiment</td>
<td>$K$</td>
</tr>
<tr>
<td></td>
<td>$K\pi$</td>
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<tr>
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<td>$K^{-}\pi^+\pi^-$</td>
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<tr>
<td></td>
<td>$\bar{K}^0\pi^-\pi^0$</td>
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<tr>
<td>residual</td>
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<tr>
<td>total</td>
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</tr>
<tr>
<td>Combined</td>
<td>total</td>
</tr>
</tbody>
</table>

- Exp error reduces about 10%. Now theory and experimental errors are same for the preferred $N = 4$ value.
Discussions and Conclusions

- By introducing a weight function with poles at space-like momentum region with Lattice QCD’s first principle calculation replacing OPE, we propose a new type of $|V_{us}|$ determination. Systematical robust cross check by changing weight function $N, C$ (controlling ‘‘inclusiveness’’). Contribution from higher multiplicity could be suppressed.

- Our results including all sys error from $\tau$ inclusive decay

$$|V_{us}| = \begin{cases} 
0.2228(15)_{exp}(13)_{th}[0.9\%], & \text{for } \gamma_K[\tau_K] \\
0.2245(11)_{exp}(13)_{th}[0.8\%], & \text{for } \gamma_K[K_{\mu 2}] 
\end{cases}$$

for $N = 4, C = 0.7 [\text{GeV}^2]$, vs $V_{us} = 0.2258(9)[0.4\%]$ from PDG17 (c.f. $V_{us} = 22333(60)[0.3\%]$ FNAL/MILC 18 [A. Kronfeld’s plenary talk] )

- By using updated BF for $K, K\pi$ [A. Lusiani CHARM 2018],

$$|V_{us}| = \begin{cases} 
0.2242(13)_{exp}(13)_{th}[0.8\%], & \text{for } \gamma_K[\tau_K] \\
0.2256(10)_{exp}(13)_{th}[0.7\%] & \text{for } \gamma_K[K_{\mu 2}] 
\end{cases}$$

- Future improvements of this analysis will provide further information on new physics, which may be sensitive to $\tau$ inclusive decays. The combined experimental uncertainty can be further reduced through improvements to the experimental $\tau \to K, K\pi$ BF$s$, while the largest of the Lattice error is due to statistics, easy to improve.