Modelling and implementation of the "6D" beam-beam interaction

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Outline

- **Introduction**
- **"6D" beam beam treatment**
  - Handling the crossing angles: "the boost"
  - Transverse "generalized kicks"
  - Description of the strong beam (Σ-matrix)
  - Handing linear coupling
  - Longitudinal kick
- **Implementation**
- **Testing:**
  - "Boost" and "Anti-boost"
  - Transverse kicks
  - Other derivatives of the electric potential
  - Σ-matrix propagation with linear coupling
  - Σ-matrix transformation to un-coupled frame
  - Constant charge slicing
  - Complete multi-slice interaction
- **Handling the denominators**

Introduction

Goal: review of the 6D beam-beam lens implemented in SixTrack

Tried to answer two main questions:

- **What is the code supposed to do?**
  → Mathematical derivation behind the implemented numerical model

- **Is the code doing what it is supposed to do?**
  → Verify the implementation of the above numerical model

The code simulates a **beam-beam interaction** using the **"Synchro Beam Mapping" technique** in the presence of:

- **Crossing angle** (ϕ)
- Arbitrary **crossing plane** (α)
- Optics at the IP described by a **general 4D correlation matrix** (Σ-matrix) → hour glass effect, elliptic beams, alphas, and linear coupling at the IP are included in the modeling

This makes the **mathematical derivation quite heavy**

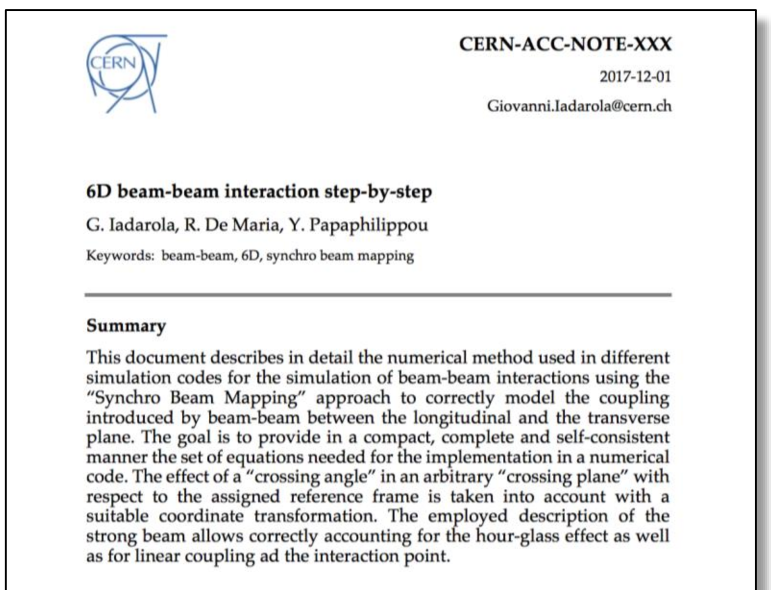
Implementation in Sixtrack in **largely based on**:

- *[1] A symplectic beam-beam interaction with energy change, by K. Hirata, H. W. Moshammer, F. Ruggiero, 1992*
- *[2] Don't be afraid of beam-beam interactions with a large crossing angle, by K. Hirata, 1993*
- *[3] 6D Beam-Beam Kick including Coupled Motion, by L.H.A. Leunissen, F. Schmidt, G. Ripken, 2001*

… but **important parts** (e.g. inverse boost, "optics de-coupling" including longitudinal derivatives) are **not reported in the papers nor anywhere else**, to our best knowledge…

Introduction

- Invested some time in **understanding and re-constructing the mathematical treatment** trying to use as little as possible the source code as a reference
  - → **Independent reconstruction** of the equations to verify the implementation in Sixtrack and to be used as a basis for a modern implementation (GPU compatible, for example)
  - → **Parts not available in literature** (mainly inverse Lorentz boost, and a large fraction of the coupling treatment) **had to be re-derived**
- Prepared a **document** including the full set of equation to enable a possible re-implementation (and avoid that somebody has to redo the same exercise in ten years :-)



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6D beam-beam interaction step-by-step

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Keywords: beam-beam, 6D, synchro beam mapping

Summary

This document describes in detail the numerical method used in different simulation codes for the simulation of beam-beam interactions using the "Synchro Beam Mapping" approach to correctly model the coupling introduced by beam-beam between the longitudinal and the transverse plane. The goal is to provide in a compact, complete and self-consistent manner the set of equations needed for the implementation in a numerical code. The effect of a "crossing angle" in an arbitrary "crossing plane" with respect to the assigned reference frame is taken into account with a suitable coordinate transformation. The employed description of the strong beam allows correctly accounting for the hour-glass effect as well as for linear coupling ad the interaction point.

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  - Constant charge slicing
  - Complete multi-slice interaction
- Handling the denominators

The goal

- We want to simulate a **beam-beam interaction** taking into account the **finite longitudinal size of the two beams**

- We are in the framework on the **weak-strong treatment**: we have a particle (of the weak-beam) that we are tracking. It interacts with a strong beam that is "rigid", i.e. unaffected by the weak beam

Hypotheses that need to be satisifed

We will use the **"synchro-beam mapping"** approach introduced by Hirata, Moshammer and Ruggiero [1]. To do so, the following **conditions need do be satisfied**:

- We work in **ultra-relativistic** approximation v=c for both beams
- The **strong beam is travelling backwards** $s_{strong}(t) = \sigma_{strong} + ct$
- **Px = Py = 0 for the strong beam**:
  - → The transverse electric field can be calculated solving a 2D Poisson problem
- The **angles of the test particle are small** so that we can assume that it travels at the speed of light along s: $s(t) = \sigma - ct$

- In the presence of a **crossing angle** a reference frame satisfying all the conditions above cannot be found by simple rotation in the lab frame, but this can be obtaining by applying also a **Lorentz boost in the crossing plane** as shown by Hirata in [2]

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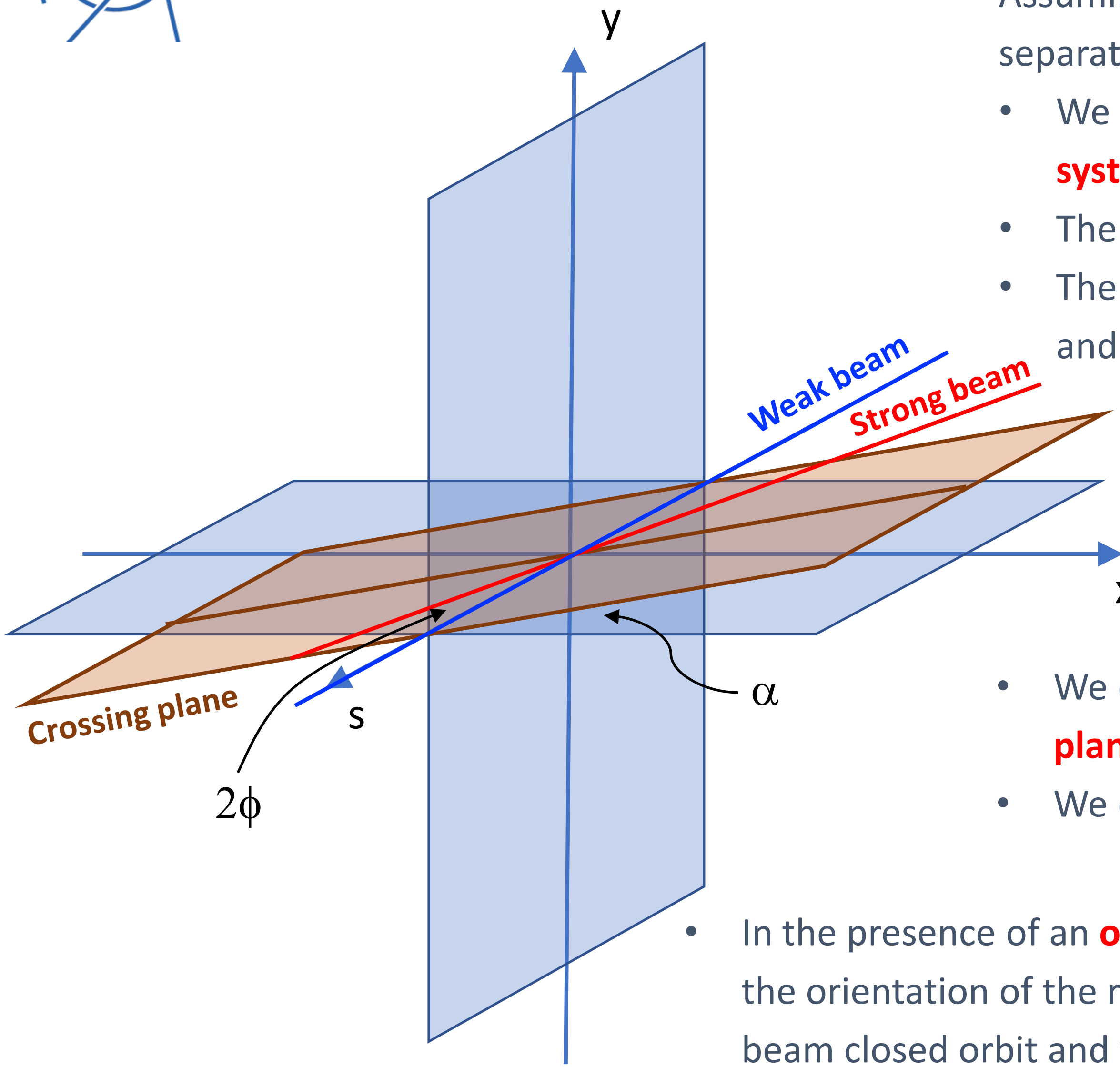
A dance of reference systems



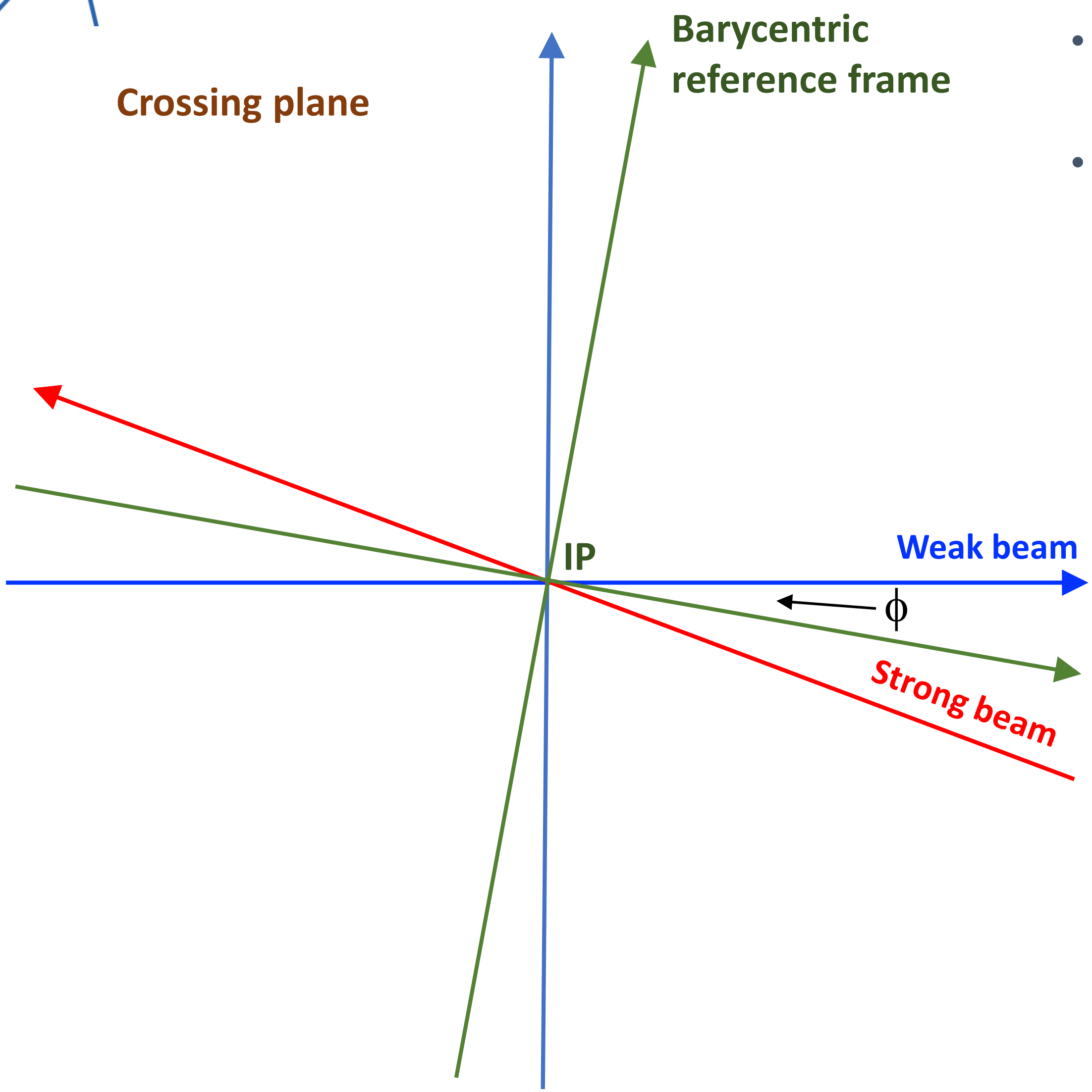
Assuming that the beams are colliding (no separation):

- We assume that we are in the **reference system of the weak beam**
- The Interaction Point **(IP) is at s=0**
- The **crossing plane** is defined by our *s*-axis and by the strong beam

- We call α the **angle between the crossing plane and the *x*-*s* plane**
- We call ϕ the **half crossing angle**

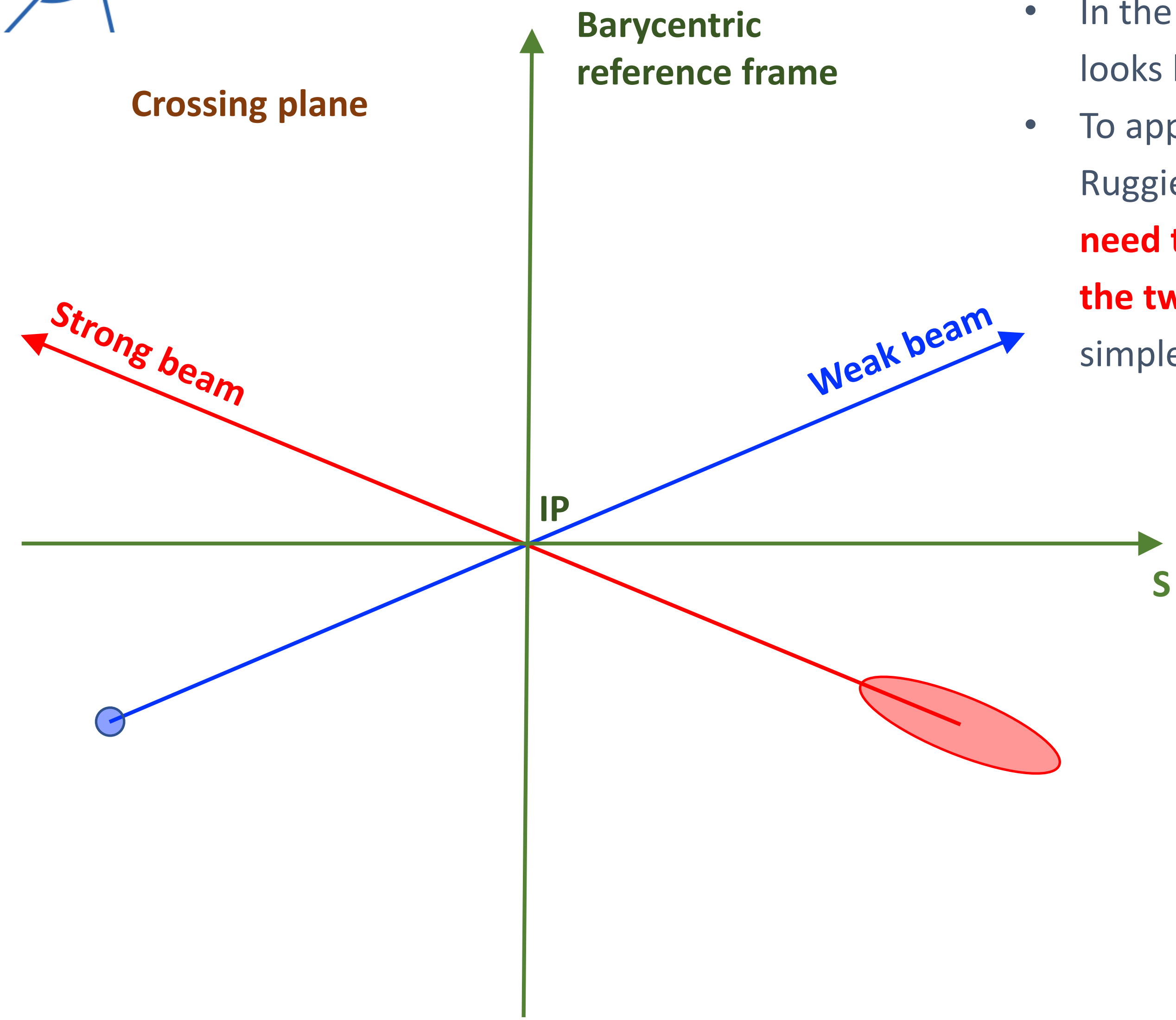
- In the presence of an **offset between the beam (separation)**, the orientation of the reference system is defined by the weak beam closed orbit and the system is centered at the IP location as defined for the strong beam → the strong beam passes always through the origin of the reference frame

A dance of reference systems



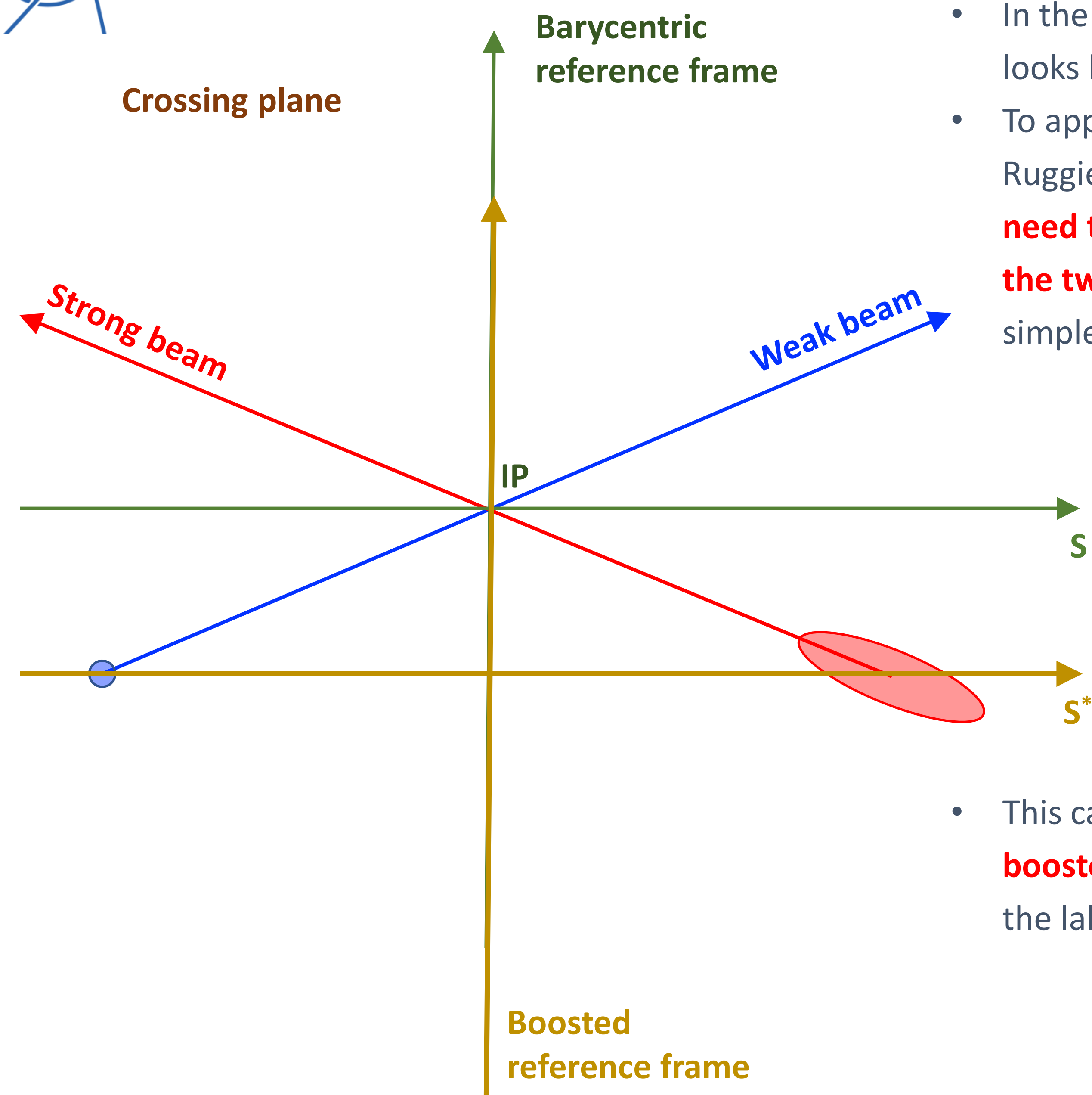
- We look at the problem in the crossing plane
- We introduce move to the **"barycentric" reference system** in which the weak and the strong beam are at $+\phi$ and $-\phi$ respectively

A dance of reference systems



- In the crossing plane the interaction looks like this…
- To apply the Hirata, Moshammer, Ruggiero treatment we practically **need to suppress the angle of for the two beams** (impossible by simple rotation)

A dance of reference systems

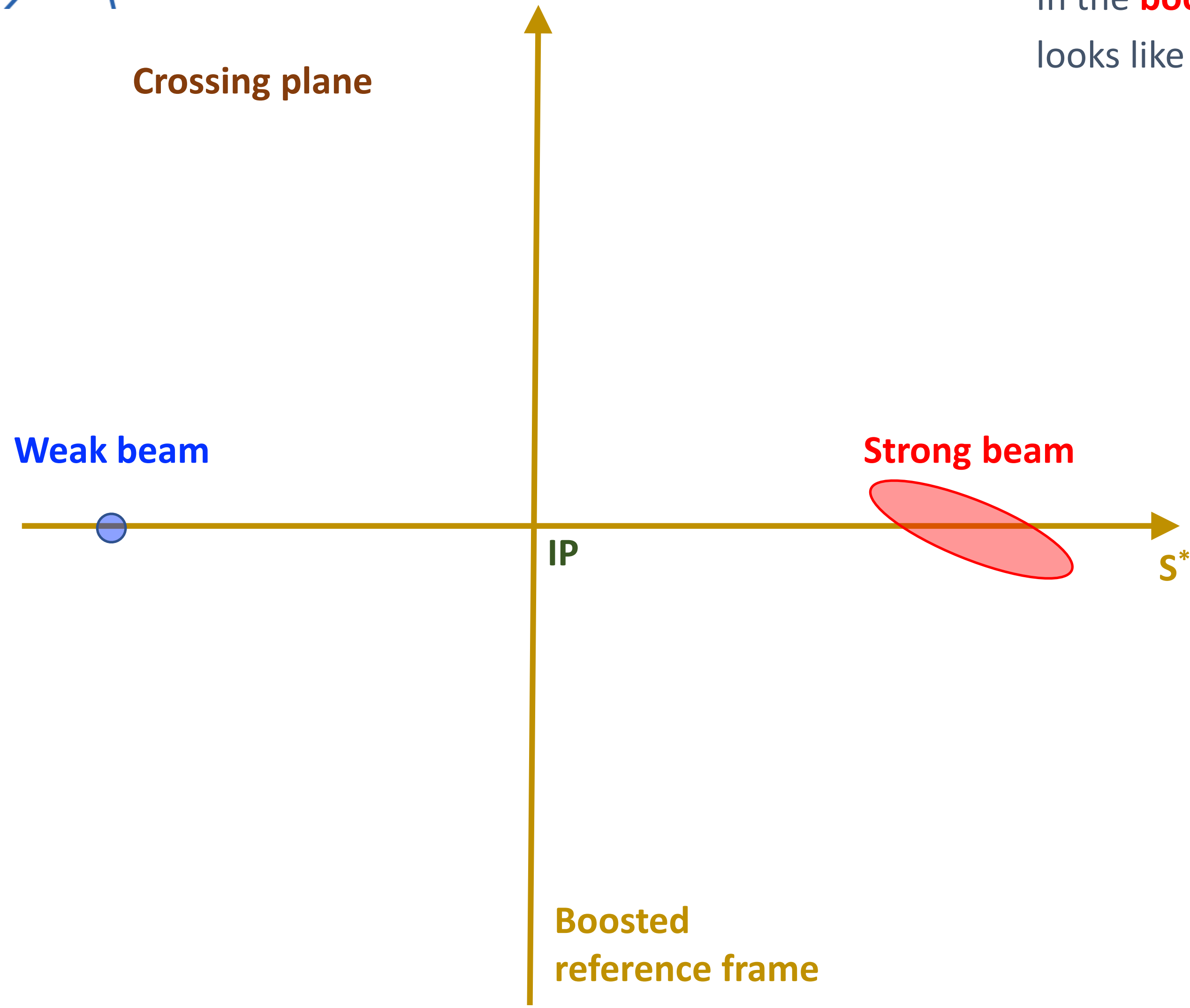


- In the crossing plane the interaction looks like this…
- To apply the Hirata, Moshammer, Ruggiero treatment we practically **need to suppress the angle of for the two beams** (impossible by simple rotation)

- This can be achieved by using a **boosted frame** that is moving w.r.t. the lab

A dance of reference systems

In the **boosted frame** the interaction looks like this



"Boost transformation" in formulas

This transformation is applied for positions:

$$\begin{pmatrix} \sigma^* \\ x^* \\ s^* \\ y^* \end{pmatrix} = A^{-1} R_{CP}{}^{-1} L_{boost} R_{CA} R_{CP} A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix}$$

- A is the matrix transforming the accelerator coordinates (Courant-Snyder) to Cartesian coordinates:

$$\begin{pmatrix} ct \\ X \\ Z \\ Y \end{pmatrix} = A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix}$$

- R_{CP} is the rotation matrix bringing the crossing plane in the X-Z plane:
- R_{CA} is the rotation matrix moving to the barycentric frame:

$$R_{CA} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{CP} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

- L_{boost} is the Lorentz boost in the direction of the rotated X-axis:

$$L_{boost} = \begin{pmatrix} 1/\cos\phi & -\tan\phi & 0 & 0 \\ -\tan\phi & 1/\cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

“Boost transformation” in formulas

This transformation is applied for momenta:

$$\begin{pmatrix} \delta^* \\ p_x^* \\ h^* \\ p_y^* \end{pmatrix} = B^{-1} R_{\mathrm{CP}}{}^{-1} L_{\mathrm{boost}} R_{\mathrm{CA}} R_{\mathrm{CP}} B \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix}$$

- B is the matrix transforming the accelerator coordinates (Courant-Snyder) to Cartesian coordinates:

$$\begin{pmatrix} E/c - p_0 \\ P_x \\ P_z - p_0 \\ P_y \end{pmatrix} = p_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix}$$

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“Boost transformation” in formulas

Not all particles with s=0 are fixed points of the transformation:

→ **A drift back to s=0** needs to be performed as we are tracking w.r.t. s and not w.r.t. time

$$p_z^* = \sqrt{(1+\delta^*)^2 - p_x^{*2} - p_y^{*2}}$$

We compute the angles:

$$h_x^* = \frac{\partial h^*}{\partial p_x^*} = \frac{p_x^*}{p_z^*}$$

$$h_y^* = \frac{\partial h^*}{\partial p_y^*} = \frac{p_y^*}{p_z^*}$$

$$h_\sigma^* = \frac{\partial h^*}{\partial \delta} = 1 - \frac{\delta^* + 1}{p_z^*}$$

We drift the particles to s = 0:

$$x^*(s^* = 0) = x^*(s) - h_x^* s$$

$$y^*(s^* = 0) = y^*(s) - h_y^* s$$

$$\delta^*(s^* = 0) = \delta^*(s) - h_\delta^* s$$

The entire procedure needs to be reverted after the interaction, see note.

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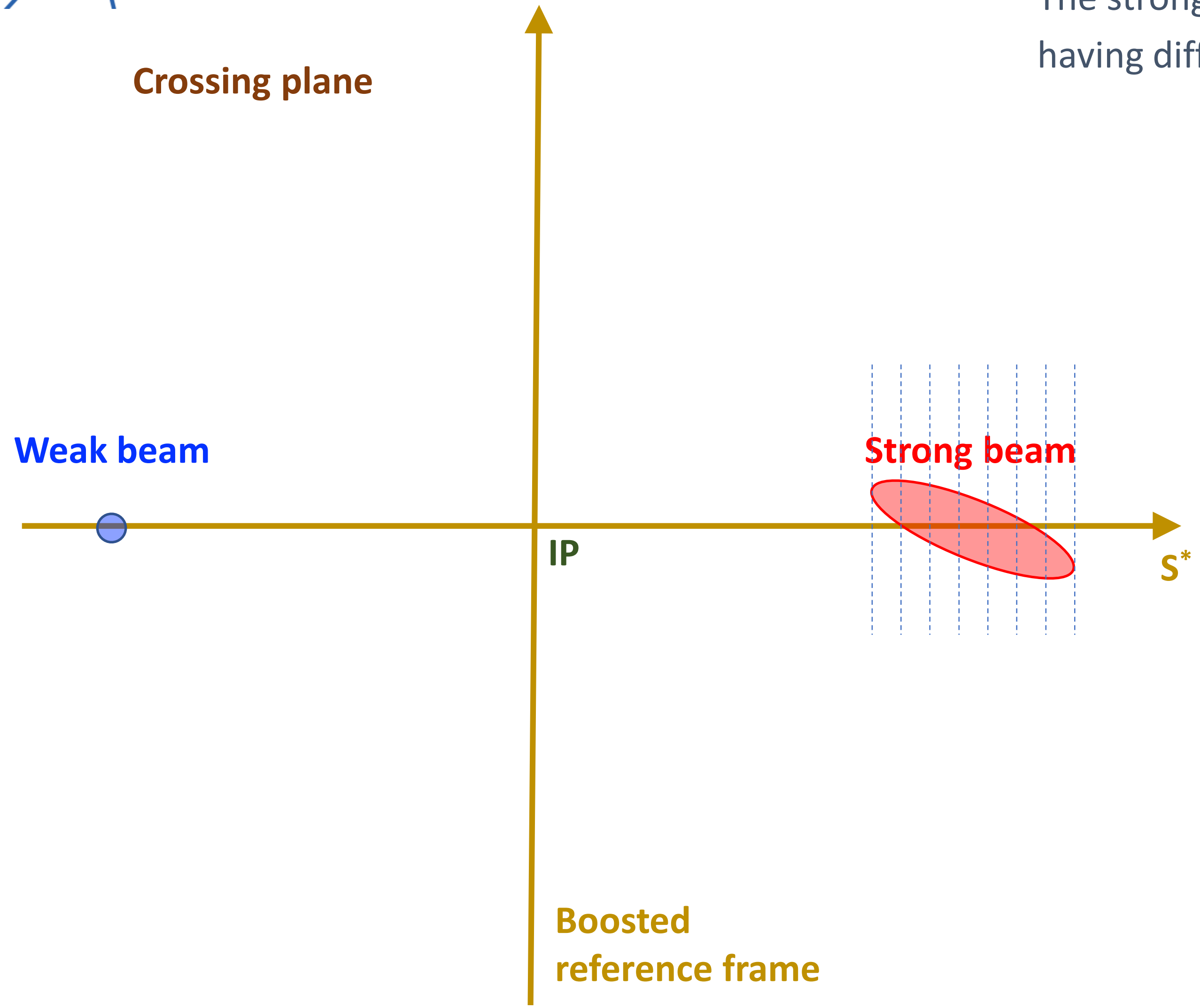
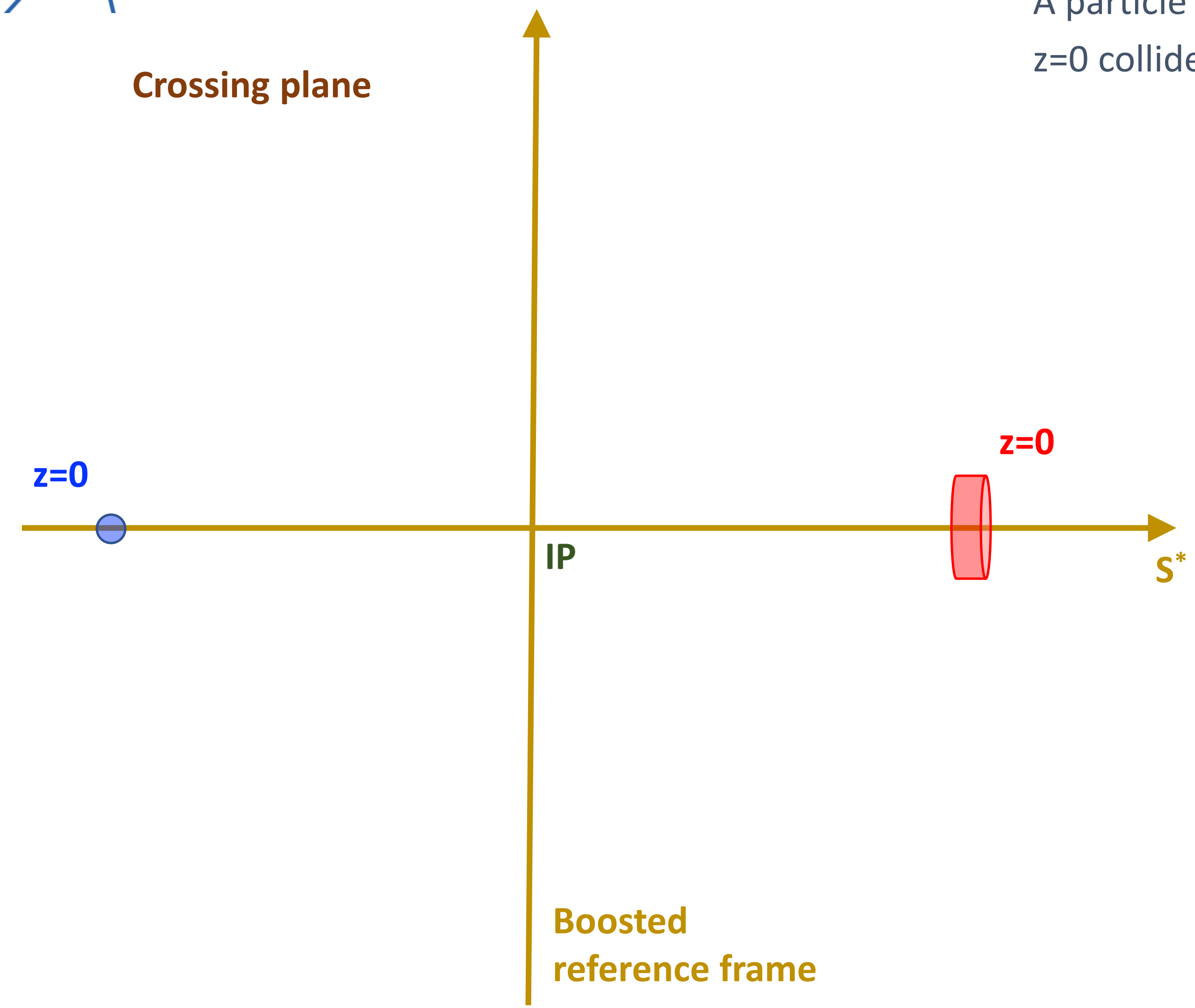
The synchro-beam method: transverse "generalized kicks"

The strong beam is cut in several **slices** having different transverse offset



The synchro-beam method: transverse “generalized kicks”

A particle with z=0 and a slice having z=0 collide at the IP



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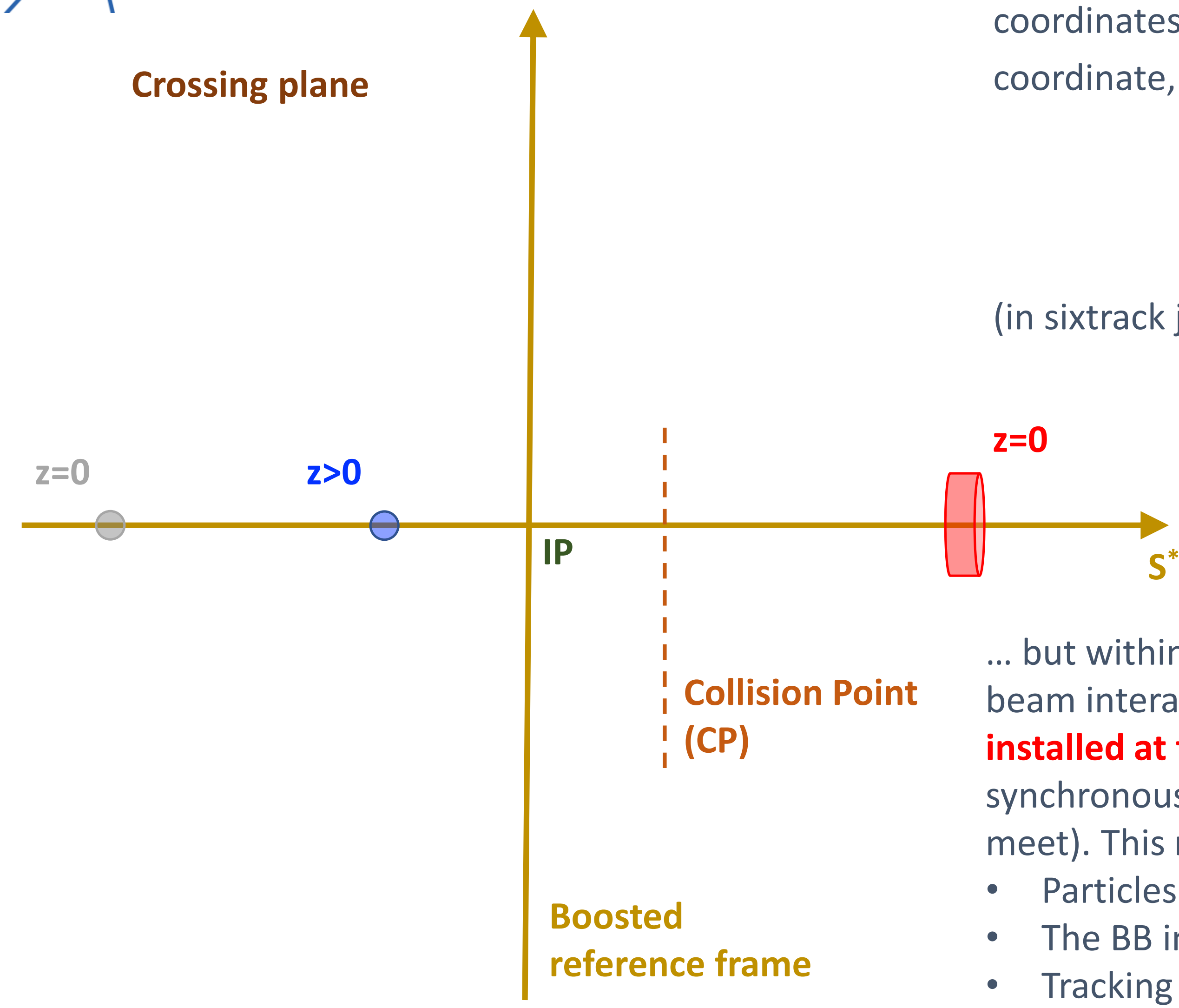


A particle and a slice with generic z coordinates will collide at a different s coordinate, **Collision Point - CP**, given by:

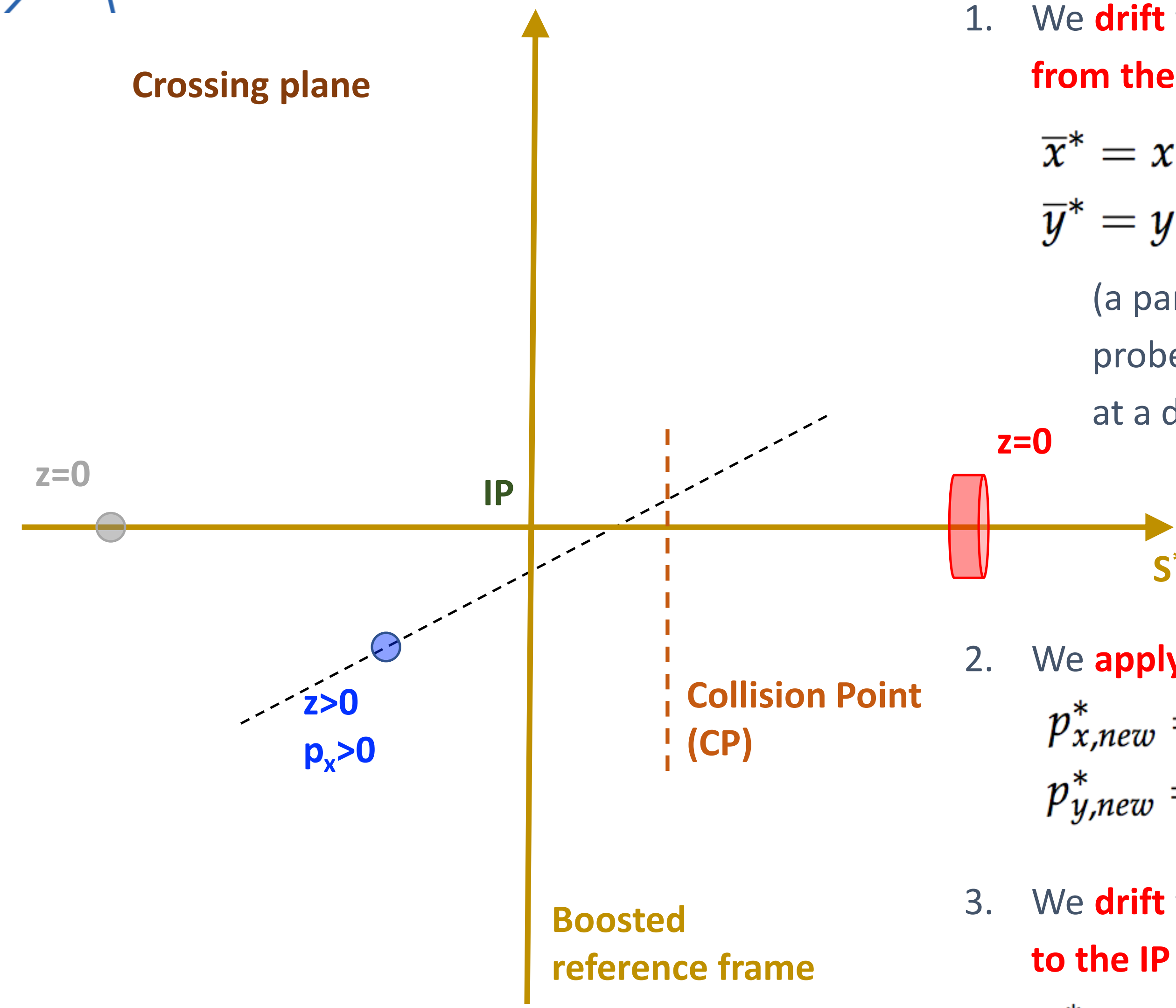
$$S = \frac{\sigma^* - \sigma^*_{\text{sl}}}{2}$$

(in sixtrack jargon z is called σ)

… but within the tracking code, the beam-beam interaction acts as a **thin element installed at the IP** (i.e. the s where the synchronous particles of the two beams meet). This means that:

- Particles are tracked to the IP
- The BB interaction is applied
- Tracking restarts from the IP
- The description of the strong beam is provided at the IP

The synchro-beam method: transverse "generalized kicks"



We proceed as follows:

1. We **drift** the slice and the weak particle **from the IP to the CP**

$$\bar{x}^* = x^* + p_x^* S - x_{\text{sl}}^* \quad \bar{y}^* = y^* + p_y^* S - y_{\text{sl}}^*$$

w.r.t. the slice centroid

(a particle and having an angle will probe the strong-beam electric field at a different transverse coordinates)

Transverse kicks need to be computed based on the shape of the strong beam…

2. We **apply the kick** a the CP:

$$p^*_{x,new} = p^*_x + F^*_x \quad p^*_{y,new} = p^*_y + F^*_y$$

3. We **drift** the particles **back from the CP to the IP** using the new angles:

$$x^*_{new} = x^* - SF^*_x \quad y^*_{new} = y^* - SF^*_y$$

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Optics of the strong beam: Σ matrix

- The shape of the strong beam is described by **4D correlation matrix (Σ-matrix)**

The **phase space distribution** can be written as:

$$f(\eta) = f_0 e^{-\eta^{\mathrm{T}}\Sigma^{-1}\eta} \quad \text{with} \quad \eta = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}$$

Points having same phase space density lie on hyper-elliptic manifolds defined by the equation:

$$\eta^{\mathrm{T}}\Sigma^{-1}\eta = \text{const.}$$

Σ contains all the information about the beam shape and divergence (including linear coupling) and **can be transported** from the IP to the CP (assuming that we are in a drift):

$$\begin{aligned}
\Sigma_{11}^{*} &= \Sigma_{11}^{*0} + 2\Sigma_{12}^{*0}S + \Sigma_{22}^{*0}S^2 \\
\Sigma_{33}^{*} &= \Sigma_{33}^{*0} + 2\Sigma_{34}^{*0}S + \Sigma_{44}^{*0}S^2 \\
\Sigma_{13}^{*} &= \Sigma_{13}^{*0} + \left(\Sigma_{14}^{*0} + \Sigma_{23}^{*0}\right) S + \Sigma_{24}^{*0}S^2 \\
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\Sigma_{44}^{*} &= \Sigma_{44}^{*0}
\end{aligned}$$

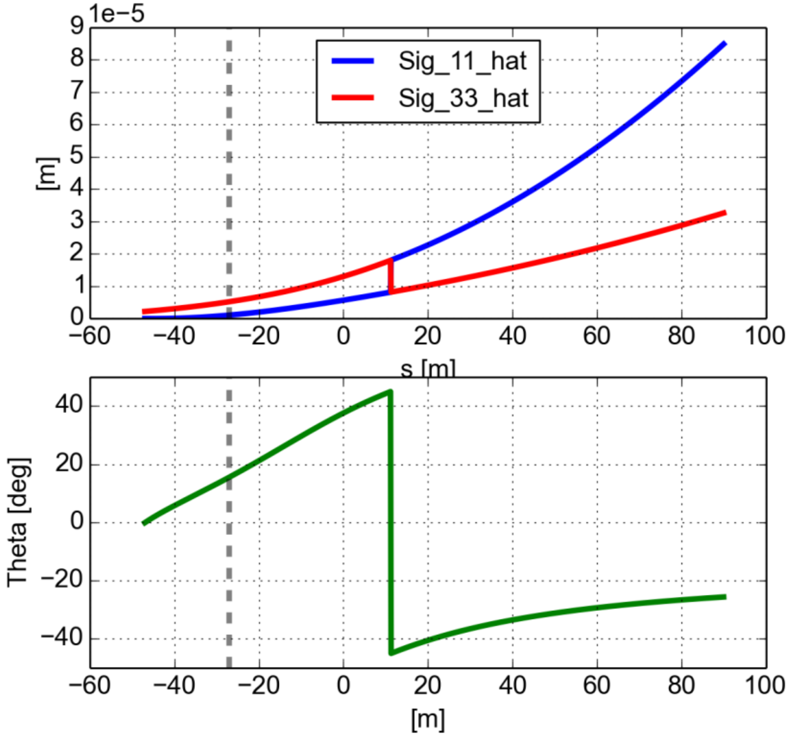
Convention:
1→x, 2→p_x, 3→y, 4→p_y

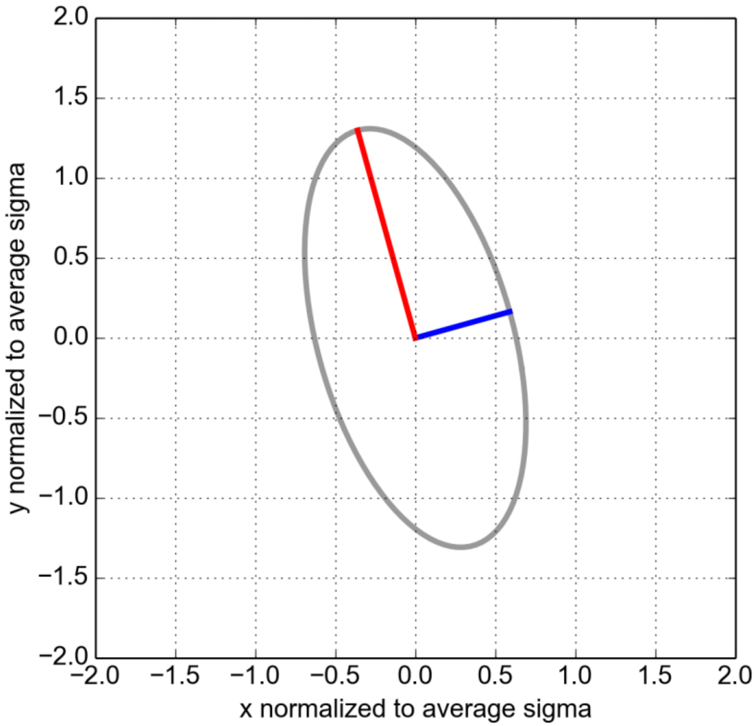
Linear coupling of the strong beam

In general, **linear coupling** of the strong beam can be present:

→ The **coupling angle** and the **beam sizes** in the decoupled frame can be obtained by **diagonalization** of the Σ-matrix

→ Coupling angle depends on the s-coordinate





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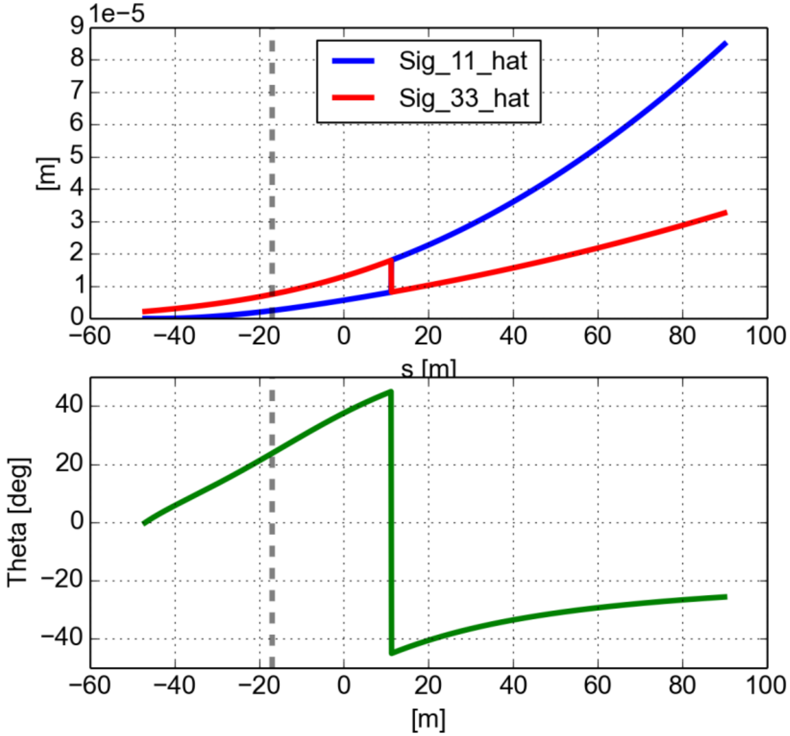

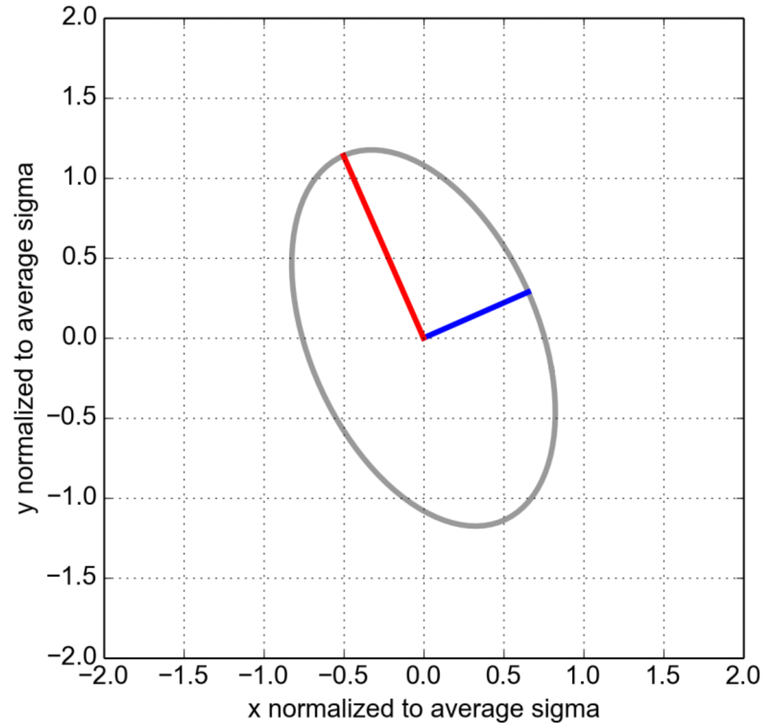

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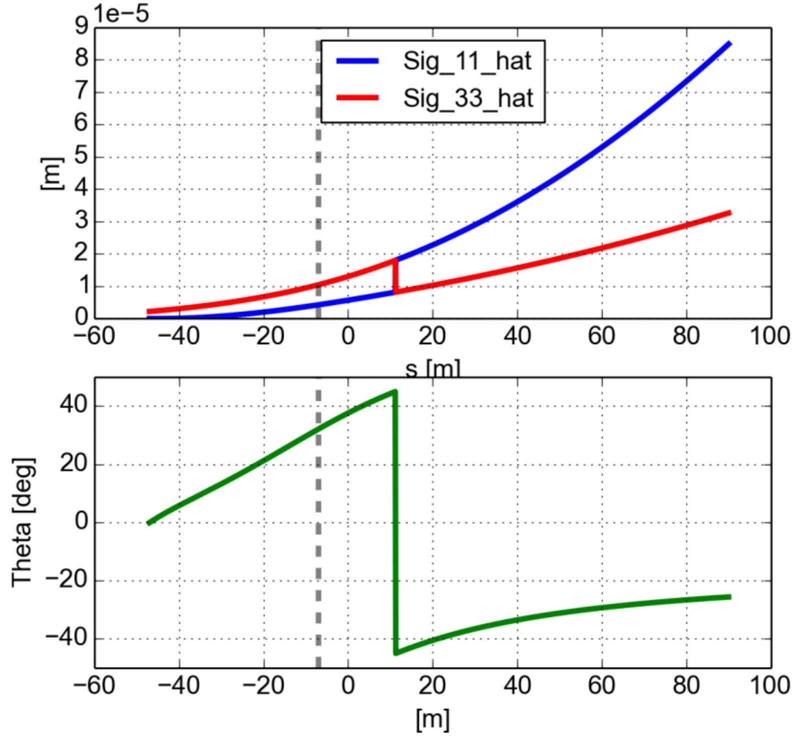

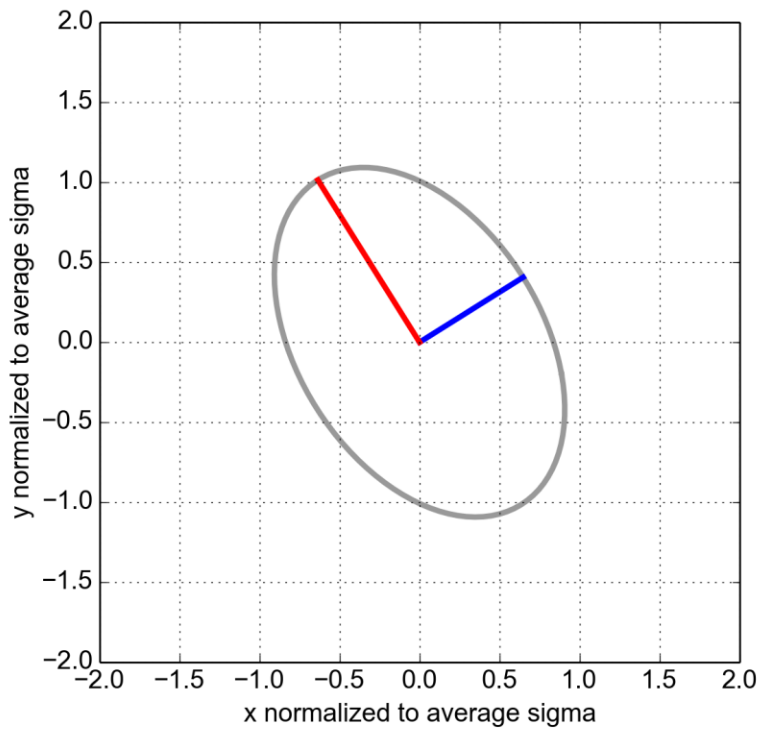

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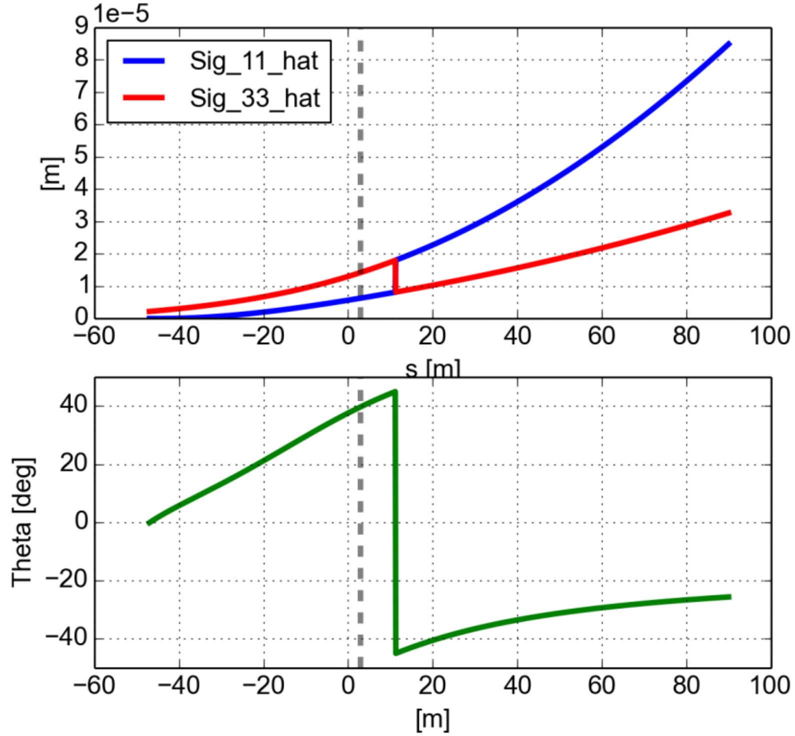

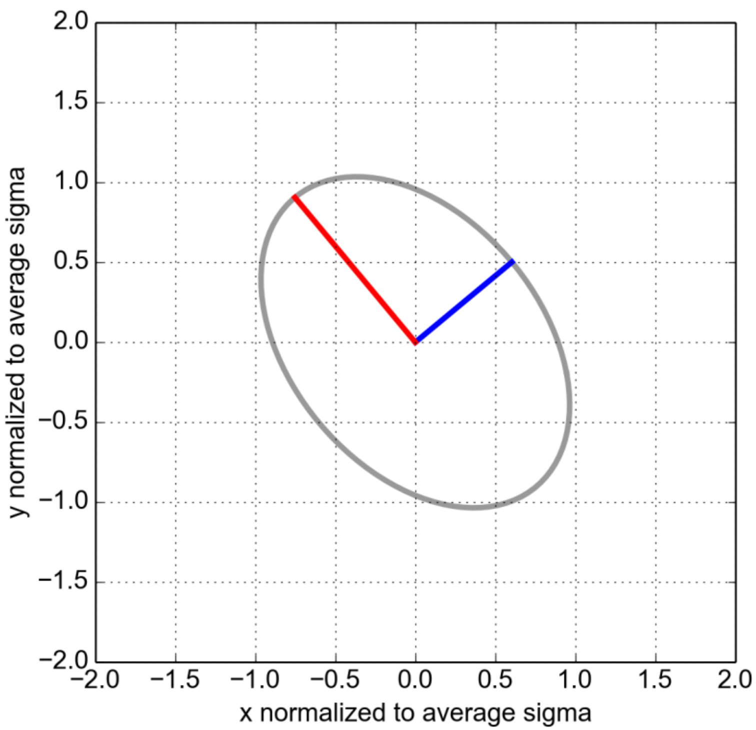

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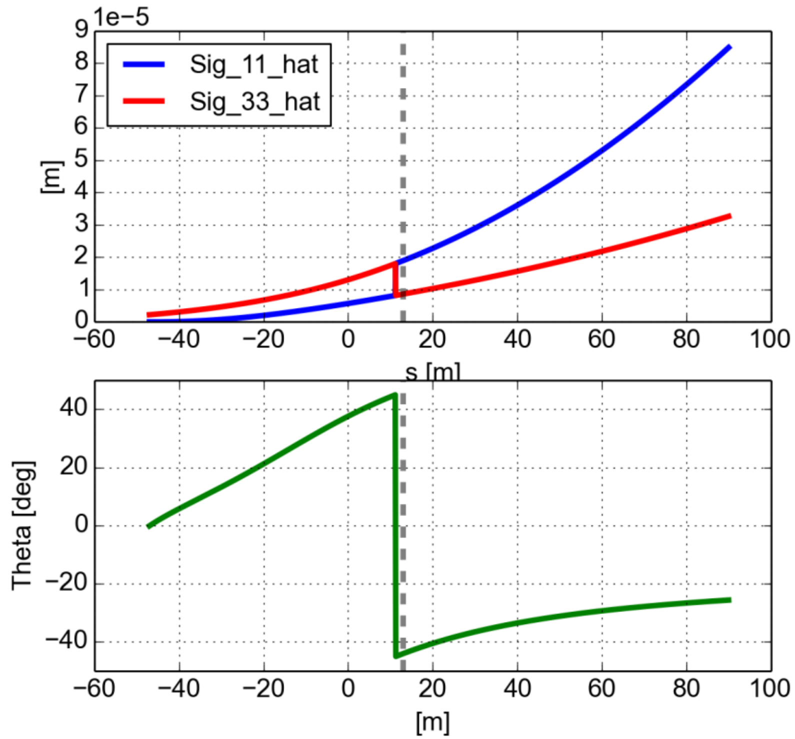

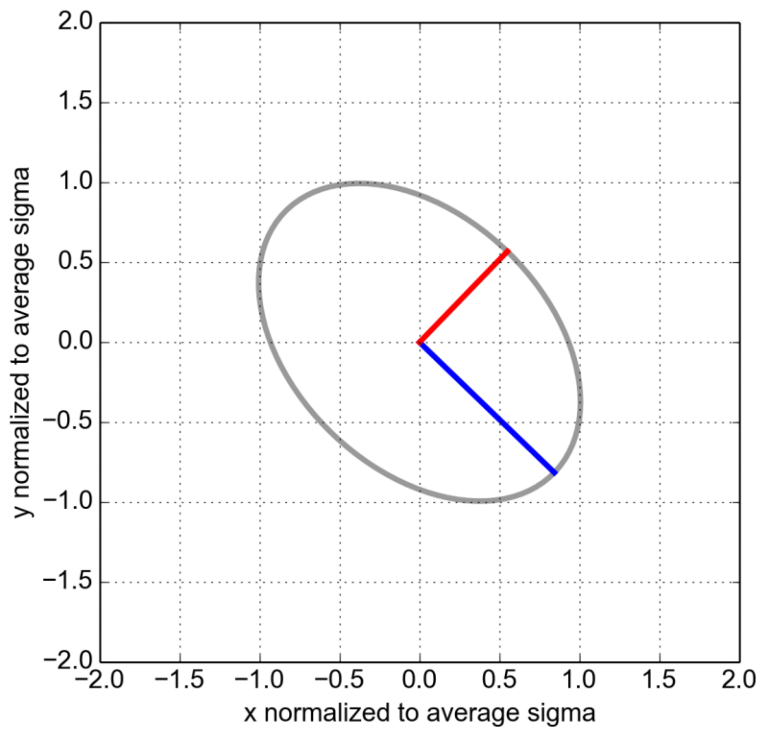

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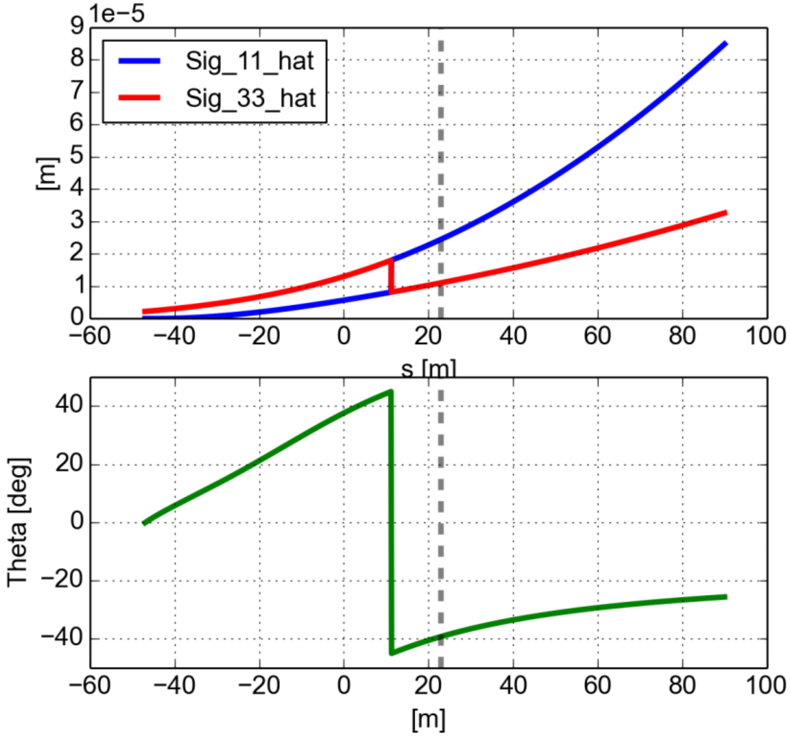

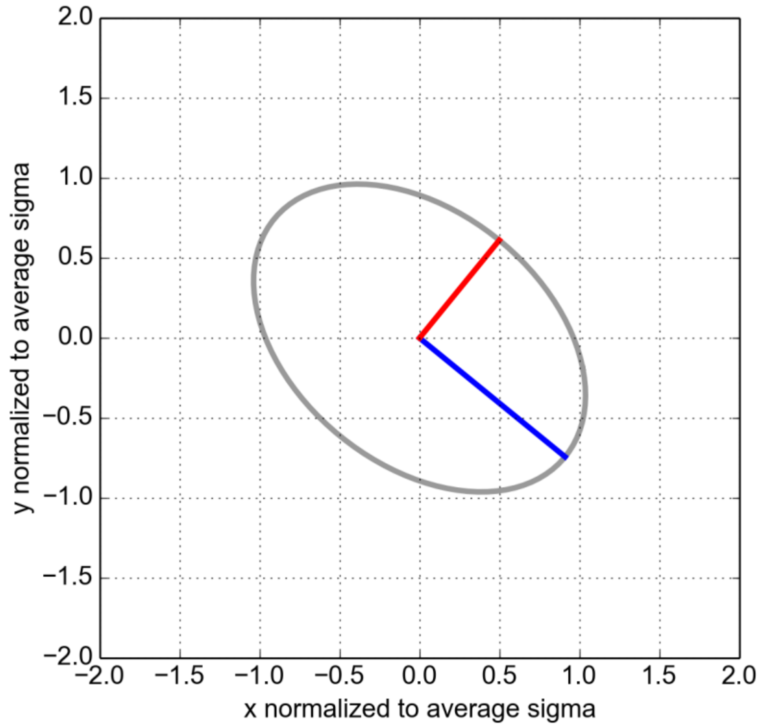

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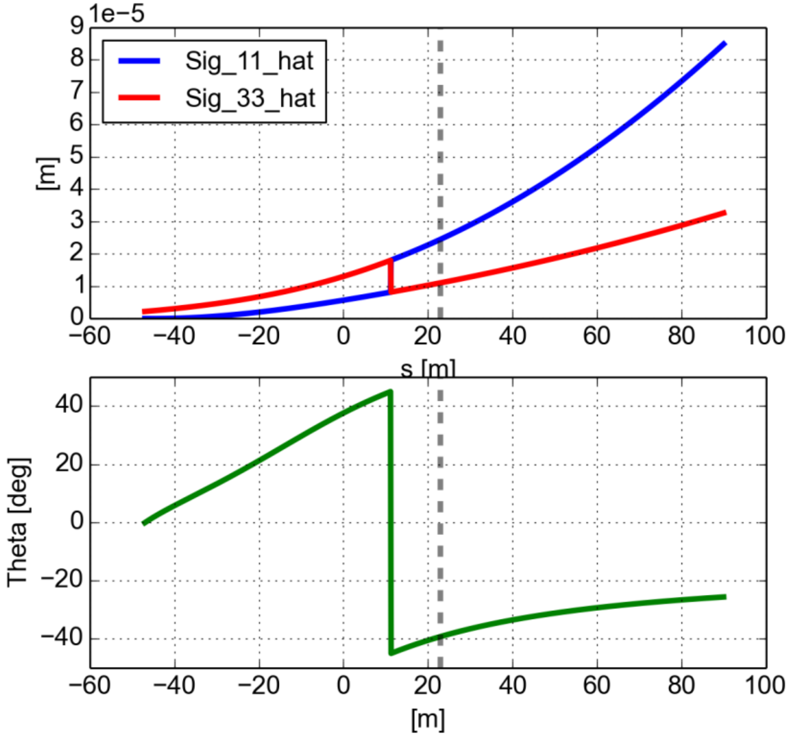

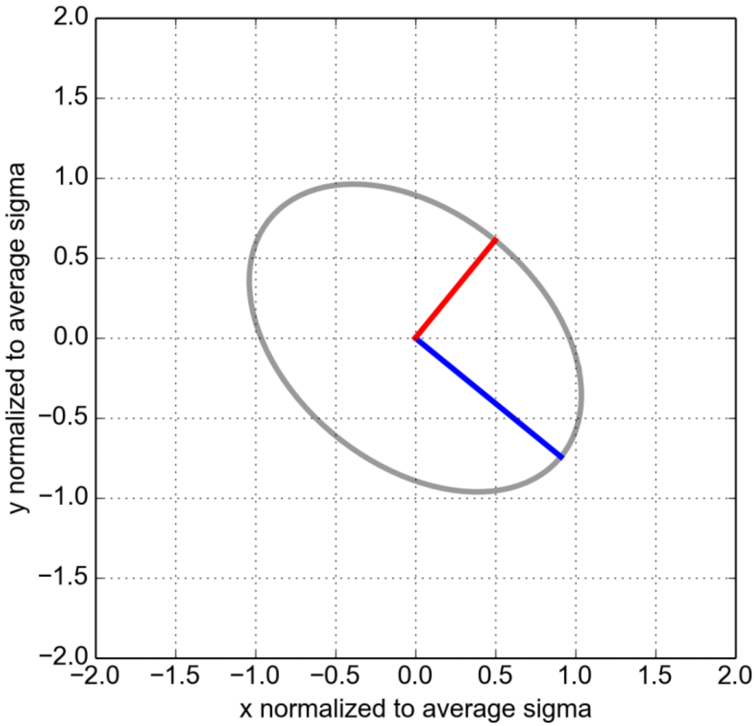

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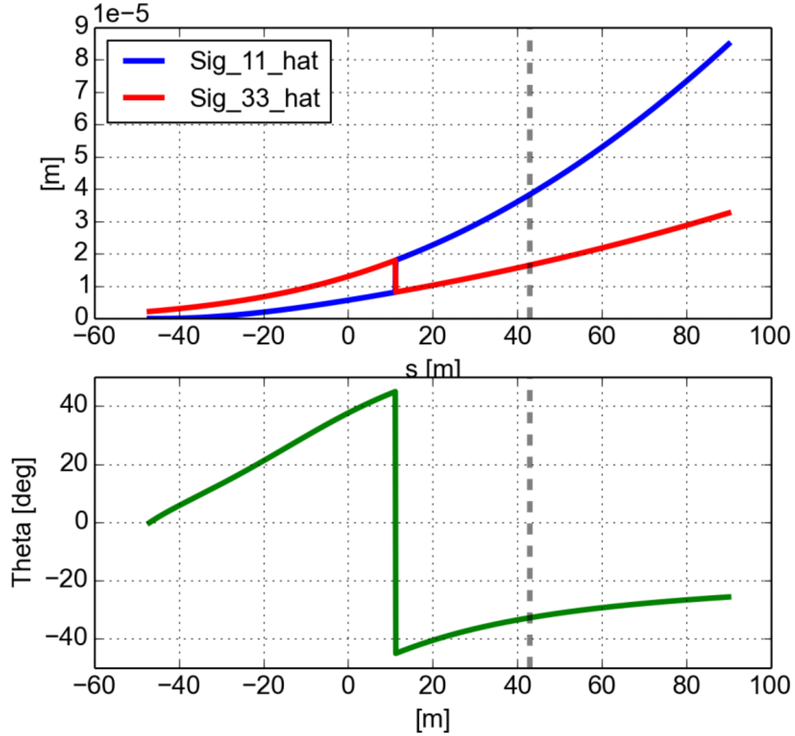

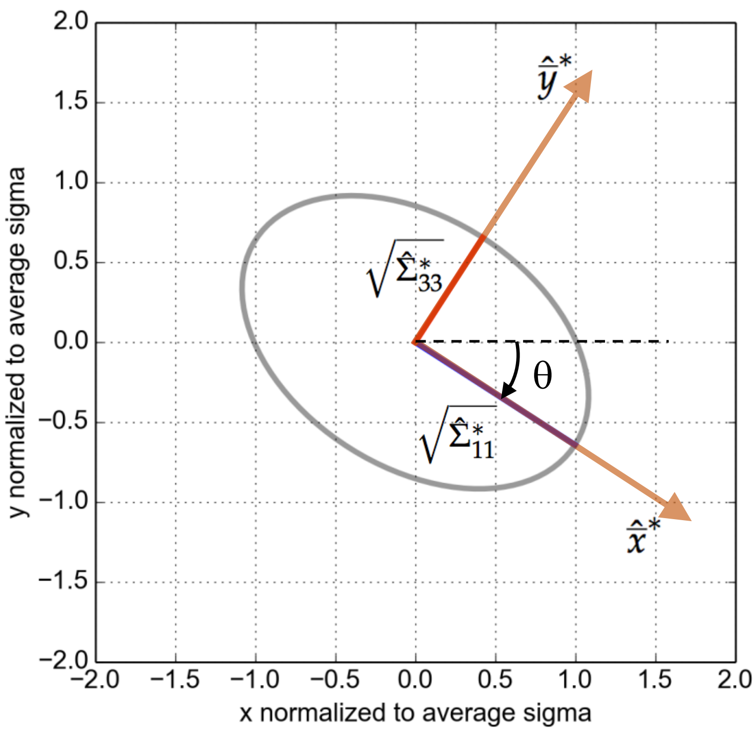

Worked on simplifying the notation in this part:

$$R(S) = \Sigma_{11}^{*} - \Sigma_{33}^{*}$$
$$W(S) = \Sigma_{11}^{*} + \Sigma_{33}^{*}$$
$$T(S) = R^2 + 4{\Sigma_{13}^{*}}^2$$

Semi-axes in the decoupled frame:

$$\hat{\Sigma}_{11}^{*} = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}_{33}^{*} = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right)$$

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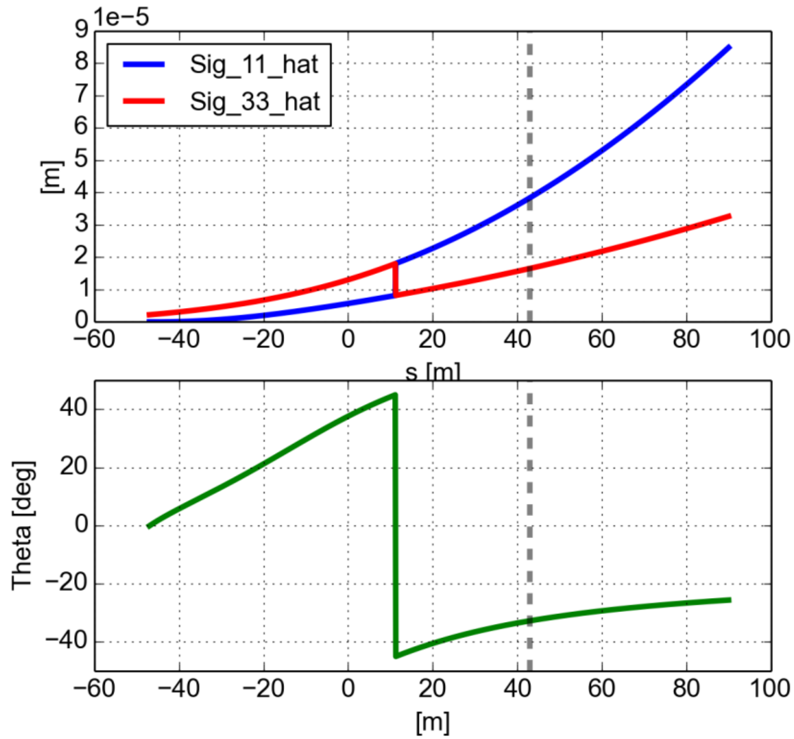
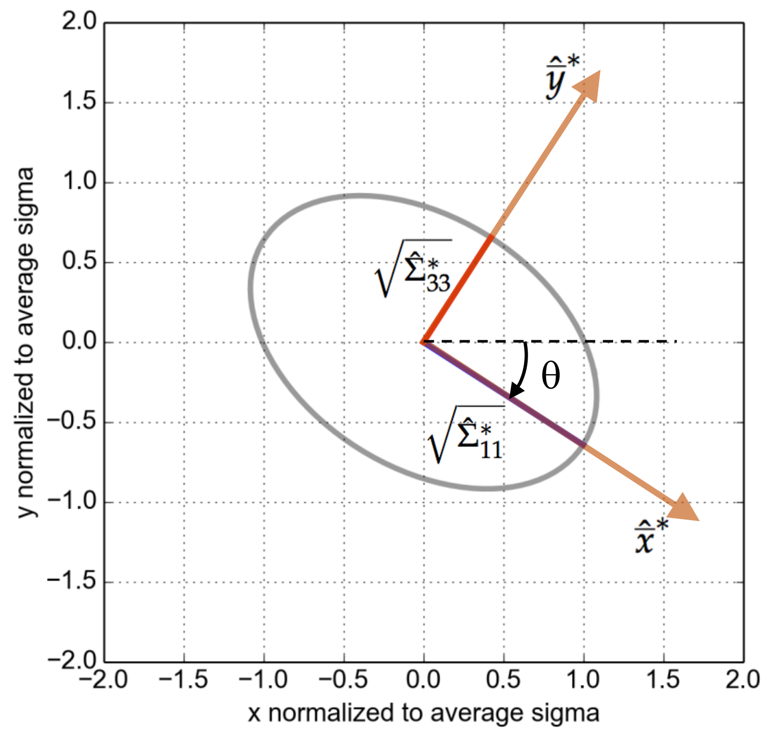
→ Coupling angle depends on the s-coordinate





Worked on simplifying the notation in this part:

$$R\,(S) = \Sigma^*_{11} - \Sigma^*_{33}$$
$$W\,(S) = \Sigma^*_{11} + \Sigma^*_{33}$$
$$T\,(S) = R^2 + 4{\Sigma^*_{13}}^2$$

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\cos\theta = \sqrt{\frac{1}{2}\left(1 + \cos 2\theta\right)}$$

$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1 - \cos 2\theta\right)}$$

Linear coupling of the strong beam

Once the coupling angle and the beam sizes in the decoupled plain are known, we proceed as follows:

1. We calculate the particle coordinates in the **decoupled frame** at the **CP**:

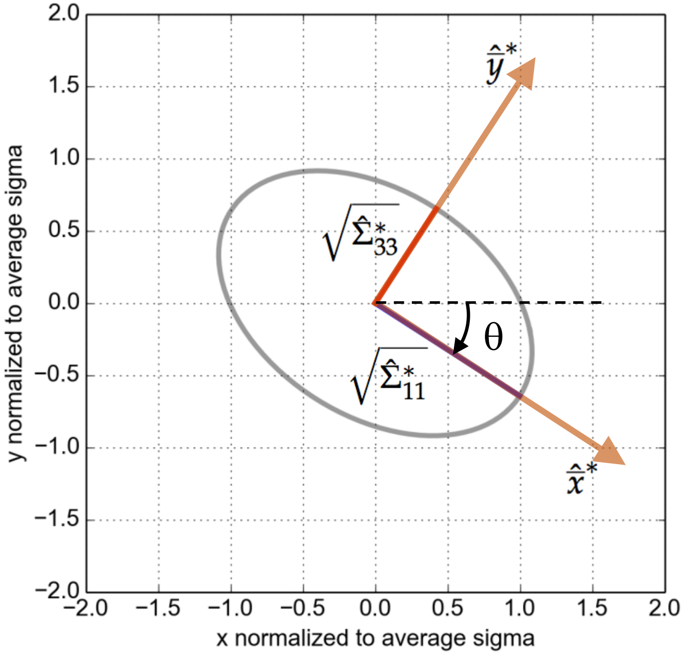
$$\hat{x}^* = \overline{x}^* \cos\theta + \overline{y}^* \sin\theta$$
$$\hat{y}^* = -\overline{x}^* \sin\theta + \overline{y}^* \cos\theta$$

2. We calculate the **kick** from the slide in the decoupled reference frame:

$$\hat{F}_x^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$
$$\hat{F}_y^* = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

where $\hat{U}^*$ is the electric potential

$$K_{sl} = \frac{N_{sl} q_{sl} q_0}{P_0 c}$$



For Gaussian (uncoupled) beams, closed forms exist to evaluate these quantities.

For a bi-Gaussian beam (elliptic) [2]:

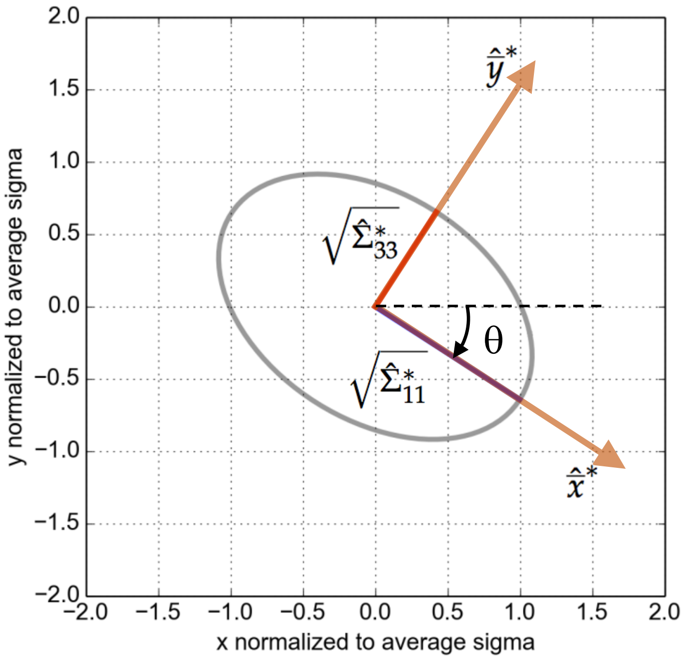
Bassetti-Erskine

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}} \mathrm{Im}\left[w\left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}} \mathrm{Re}\left[w\left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

Linear coupling of the strong beam

Once the coupling angle and the beam sizes in the decoupled plain are known, we proceed as follows:



1. We calculate the particle coordinates in the **decoupled frame** at the **CP**:

$$\hat{x}^* = \overline{x}^* \cos\theta + \overline{y}^* \sin\theta$$
$$\hat{y}^* = -\overline{x}^* \sin\theta + \overline{y}^* \cos\theta$$

2. We calculate the **kick** from the slide in the decoupled reference frame:

$$\hat{F}^*_x = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{x}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}^*_{11}, \hat{\Sigma}^*_{33}\right)$$
$$\hat{F}^*_y = -K_{sl} \frac{\partial \hat{U}^*}{\partial \hat{y}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}^*_{11}, \hat{\Sigma}^*_{33}\right)$$

where $\hat{U}^*$ is the electric potential

$$K_{sl} = \frac{N_{sl} q_{sl} q_0}{P_0 c}$$

For Gaussian (uncoupled) beams, closed forms exist to evaluate these quantities.

3. We **rotate the kicks** to de coupled reference frame

$$F^*_x = \hat{F}^*_x \cos\theta - \hat{F}^*_y \sin\theta$$
$$F^*_y = \hat{F}^*_x \sin\theta + \hat{F}^*_y \cos\theta$$

4. We **apply the kicks** to the transverse momenta and **drift back** to the **IP** (as explained before)

$$p^*_{x,new} = p^*_x + F^*_x \qquad x^*_{new} = x^* - SF^*_x$$
$$p^*_{y,new} = p^*_y + F^*_y \qquad y^*_{new} = y^* - SF^*_y$$

Outline

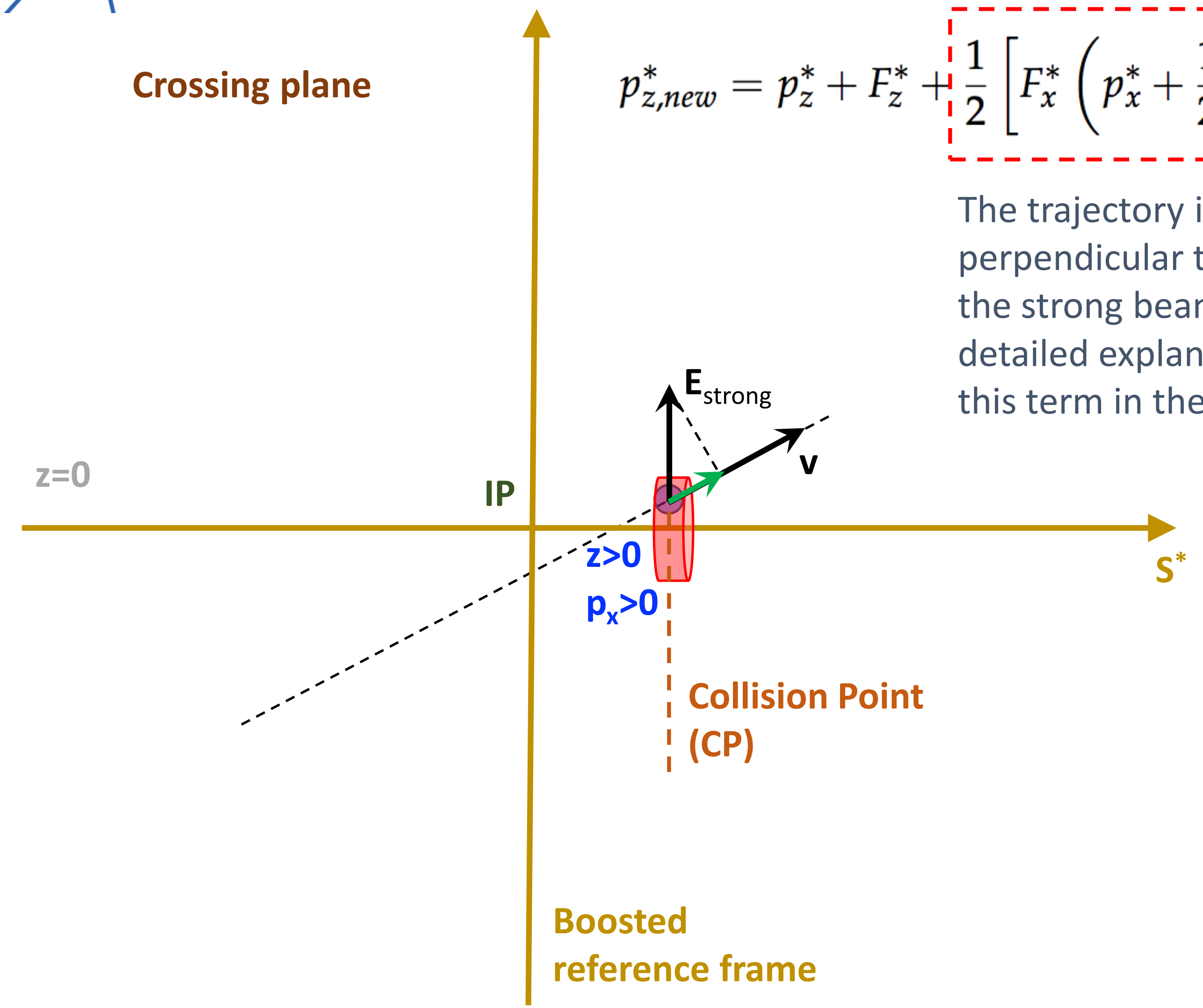
- Introduction
- **“6D” beam beam treatment**
  - Handling the crossing angles: “the boost”
  - Transverse “generalized kicks”
  - Description of the strong beam (Σ-matrix)
  - Handing linear coupling
  - Longitudinal kick
- Implementation
- Testing:
  - “Boost” and “Anti-boost”
  - Transverse kicks
  - Other derivatives of the electric potential
  - Σ-matrix propagation with linear coupling
  - Σ-matrix transformation to un-coupled frame
  - Constant charge slicing
  - Complete multi-slice interaction
- Handling the denominators

Energy change: effect of the angle

The longitudinal kick has **two components**:

$$p^*_{z,new} = p^*_z + F^*_z + \frac{1}{2}\left[F^*_x\left(p^*_x + \frac{1}{2}F^*_x\right) + F^*_y\left(p^*_y + \frac{1}{2}F^*_y\right)\right]$$

The trajectory is, in general, not perpendicular to the transverse fields of the strong beam (see Hirata [1] for detailed explanation) → this introduces this term in the longitudinal kick



Energy change: grad-phi effect

The longitudinal kick has **two components**:

$$p^*_{z,new} = p^*_z + F^*_z + \frac{1}{2}\left[F^*_x\left(p^*_x + \frac{1}{2}F^*_x\right) + F^*_y\left(p^*_y + \frac{1}{2}F^*_y\right)\right]$$

Another component of the longitudinal kick arises from the fact that the transverse **shape of the strong beam is changing along z** (hour-glass effect, “rotating” coupling angle)

→ The electric potential depends on z

→ The gradient of the electric potential (i.e. the electric field) has a z component

→ There is a z-kick, i.e. again a change in the particle energy

We need to evaluate the **derivative w.r.t. z** (or σ, or small-s) **of the electric potential**

As we have written down most of the involved quantities as a function of the coordinate of the CP (capital-S) we just notice that:

$$S = \frac{\sigma^* - \sigma^*_{\text{sl}}}{2} \Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial S} \Rightarrow F^*_z = \frac{1}{2}\frac{\partial}{\partial S}\left[\hat{U}^*\left(\hat{\bar{x}}^*\left(\theta(S)\right), \hat{\bar{y}}^*\left(\theta(S)\right), \hat{\Sigma}^*_{11}(S), \hat{\Sigma}^*_{33}(S)\right)\right]$$

(in sixtrack jargon

z is called σ)

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\frac{\partial}{\partial S}\left[\hat{U}^*\left(\hat{\bar{x}}^*\left(\theta(S)\right), \hat{\bar{y}}^*\left(\theta(S)\right), \hat{\Sigma}_{11}^*(S), \hat{\Sigma}_{33}^*(S)\right)\right]$$

Derivative rule for nested functions:

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^*\frac{\partial}{\partial S}\left[\hat{\bar{x}}^*\left(\theta(S)\right)\right] + \hat{F}_y^*\frac{\partial}{\partial S}\left[\hat{\bar{y}}^*\left(\theta(S)\right)\right] + \hat{G}_x^*\frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^*\frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

We need to evaluate these eight terms…

where:

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\bar{y}}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right)$$

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{x}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{y}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{x}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \quad \hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{y}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \quad \hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right)$$

For these four terms a closed forms exist for transverse Gaussian beams

For a bi-Gaussian beam (elliptic) [2]:

Bassetti-Erskine

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Im}\left[w\left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Re}\left[w\left(\frac{\hat{x}^* + i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{x}^*\hat{E}_x^* + \hat{y}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}\exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{x}^*\hat{E}_x^* + \hat{y}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}\exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

where w is the Faddeeva function.

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{x}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{y}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{x}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \quad \hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{y}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \quad \hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{x}^*, \hat{y}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right)$$

For these four terms a closed forms exist for transverse Gaussian beams

For a round beam, i.e. $\hat{\Sigma}_{11}^* = \hat{\Sigma}_{33}^* = \hat{\Sigma}^*$:

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\pi\epsilon_0}\left[1 - \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right)\right]\frac{x}{(\hat{x}^*)^2 + (\hat{y}^*)^2}$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\pi\epsilon_0}\left[1 - \exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right)\right]\frac{y}{(\hat{x}^*)^2 + (\hat{y}^*)^2}$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = \frac{1}{2\left[(\hat{x}^*)^2 + (\hat{y}^*)^2\right]}\left[\hat{y}^*\hat{E}_y^* - \hat{x}^*\hat{E}_x^* + \frac{1}{2\pi\epsilon_0}\frac{\left(\hat{x}^*\right)^2}{\hat{\Sigma}^*}\exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right)\right]$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left[(\hat{x}^*)^2 + (\hat{y}^*)^2\right]}\left[\hat{x}^*\hat{E}_x^* - \hat{y}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\frac{\left(\hat{y}^*\right)^2}{\hat{\Sigma}^*}\exp\left(-\frac{(\hat{x}^*)^2 + (\hat{y}^*)^2}{2\hat{\Sigma}^*}\right)\right]$$

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{\bar{x}}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{\bar{y}}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

$$\hat{\bar{x}}^* = \bar{x}^* \cos\theta + \bar{y}^* \sin\theta$$
$$\hat{\bar{y}}^* = -\bar{x}^* \sin\theta + \bar{y}^* \cos\theta$$

$$\frac{\partial}{\partial S}\left[\hat{\bar{x}}^*\left(\theta(S)\right)\right] = \bar{x}^* \frac{\partial}{\partial S}\left[\cos\theta\right] + \bar{y}^* \frac{\partial}{\partial S}\left[\sin\theta\right]$$
$$\frac{\partial}{\partial S}\left[\hat{\bar{y}}^*\left(\theta(S)\right)\right] = -\bar{x}^* \frac{\partial}{\partial S}\left[\sin\theta\right] + \bar{y}^* \frac{\partial}{\partial S}\left[\cos\theta\right]$$

With some some goniometric trick

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

We just need to evaluate $\frac{\partial}{\partial S}\cos 2\theta$

Before we had written: $\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

with

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$
$$T(S) = R^2 + 4{\Sigma_{13}^*}^2$$

where we need to evaluate the derivatives of R, T and W…

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{\bar{x}}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{\bar{y}}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

Derivatives of R, T and W

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$
$$T(S) = R^2 + 4{\Sigma_{13}^*}^2$$

$$\Sigma_{11}^* = \Sigma_{11}^{*0} + 2\Sigma_{12}^{*0} S + \Sigma_{22}^{*0} S^2$$
$$\Sigma_{33}^* = \Sigma_{33}^{*0} + 2\Sigma_{34}^{*0} S + \Sigma_{44}^{*0} S^2$$
$$\Sigma_{13}^* = \Sigma_{13}^{*0} + \left(\Sigma_{14}^{*0} + \Sigma_{23}^{*0}\right) S + \Sigma_{24}^{*0} S^2$$

$$\frac{\partial R}{\partial S} = 2\left(\Sigma_{12}^0 - \Sigma_{34}^0\right) + 2S\left(\Sigma_{22}^0 - \Sigma_{44}^0\right)$$
$$\frac{\partial W}{\partial S} = 2\left(\Sigma_{12}^0 + \Sigma_{34}^0\right) + 2S\left(\Sigma_{22}^0 + \Sigma_{44}^0\right)$$
$$\frac{\partial \Sigma_{13}^*}{\partial S} = \Sigma_{14}^0 + \Sigma_{23}^0 + 2\Sigma_{24}^0 S$$
$$\frac{\partial T}{\partial S} = 2R\frac{\partial R}{\partial S} + 8\Sigma_{13}^* \frac{\partial \Sigma_{13}^*}{\partial S}$$

With s

gonio

Before we had written: $\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

with

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$
$$T(S) = R^2 + 4{\Sigma_{13}^*}^2$$

where we need to evaluate the derivatives of R, T and W…

Energy change: grad-phi effect

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{x}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{y}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right)$$

$$\hat{\Sigma}_{11}^* = \frac{1}{2}\left(W + \text{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}_{33}^* = \frac{1}{2}\left(W - \text{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \text{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \text{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

Again what we need to know are the derivatives of R, T and W, which were already shown in the previous slides

Derivatives of R, T and W

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$
$$T(S) = R^2 + 4{\Sigma_{13}^*}^2$$

$$\Sigma_{11}^* = \Sigma_{11}^{*0} + 2\Sigma_{12}^{*0}S + \Sigma_{22}^{*0}S^2$$
$$\Sigma_{33}^* = \Sigma_{33}^{*0} + 2\Sigma_{34}^{*0}S + \Sigma_{44}^{*0}S^2$$
$$\Sigma_{13}^* = \Sigma_{13}^{*0} + \left(\Sigma_{14}^{*0} + \Sigma_{23}^{*0}\right)S + \Sigma_{24}^{*0}S^2$$

$$\frac{\partial R}{\partial S} = 2\left(\Sigma_{12}^0 - \Sigma_{34}^0\right) + 2S\left(\Sigma_{22}^0 - \Sigma_{44}^0\right)$$
$$\frac{\partial W}{\partial S} = 2\left(\Sigma_{12}^0 + \Sigma_{34}^0\right) + 2S\left(\Sigma_{22}^0 + \Sigma_{44}^0\right)$$
$$\frac{\partial \Sigma_{13}^*}{\partial S} = \Sigma_{14}^0 + \Sigma_{23}^0 + 2\Sigma_{24}^0 S$$
$$\frac{\partial T}{\partial S} = 2R\frac{\partial R}{\partial S} + 8\Sigma_{13}^*\frac{\partial \Sigma_{13}^*}{\partial S}$$

We have all the pieces, but on the way **we introduced some denominators** which can become zero! → we will deal with it later…

$$R(S) = \Sigma^*_{11} - \Sigma^*_{33}$$
$$W(S) = \Sigma^*_{11} + \Sigma^*_{33}$$
$$T(S) = R^2 + 4{\Sigma^*_{13}}^2$$

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

$$\cos\theta = \sqrt{\frac{1}{2}\left(1 + \cos 2\theta\right)}$$
$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1 - \cos 2\theta\right)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Outline

- Introduction
- "6D" beam beam treatment
  - Handling the crossing angles: "the boost"
  - Transverse "generalized kicks"
  - Description of the strong beam (Σ-matrix)
  - Handing linear coupling
  - Longitudinal kick
- **Implementation**
- Testing:
  - "Boost" and "Anti-boost"
  - Transverse kicks
  - Other derivatives of the electric potential
  - Σ-matrix propagation with linear coupling
  - Σ-matrix transformation to un-coupled frame
  - Constant charge slicing
  - Complete multi-slice interaction
- Handling the denominators

The algorithm in one slide

Initialization stage:

- Prepare **coefficients** for **Lorentz boost**
- **Slice** strong bunch
  - Compute slice charges and centroid coordinates
- **Boost strong beam** slices
  - Boost centroid coordinates
  - Boost Σ-matrix
- Store all information in a **data block**

Tracking routine:

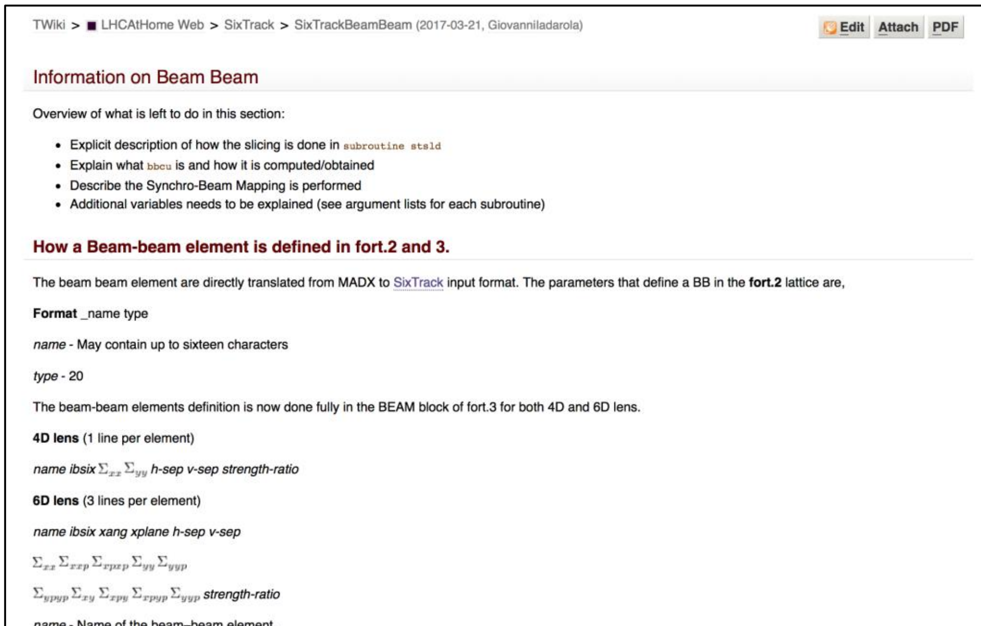
- **Boost** coordinates of the **weak beam particle**
- Compute S coordinate of the **collision point** (CP)
- **Transport strong beam** optics from the IP to the CP:
  - Transport sigma matrix to the CP
  - Compute coupling angle and beam sizes in the decoupled plane
  - Compute auxiliary quantities for the calculation of the longitudinal kick
- Compute **transverse kicks**
  - Transform coordinates of the weak beam particles to the un-coupled frame
  - Compute transverse forces in the un-coupled frame
  - Transform transverse kicks to the coupled frame
  - Apply transverse kicks in the coupled frame (change p_x, p_y)
  - Transport transverse kick from the CP to the IP and change particle positions (x,y) accordingly
- Compute and apply the **longitudinal kick**
- **Anti-boost** coordinates of the weak beam particles

SixTrack implementation

Very hard to read and to debug, it can be kept alive… but definitely not ideal

```
      ...
      if(ibbc1.eq.1) then
        dum(8)=two*((bcu(ibb,4)-bcu(ibb,9))+                       &!hr06
&(bcu(ibb,6)-bcu(ibb,10))*sp)                                       !hr06
        dum(9)=(bcu(ibb,5)+bcu(ibb,7))+(two*bcu(ibb,8))*sp          !hr06
        dum(10)=(((dum(4)*dum(8)+(four*dum(3))*dum(9))/            &!hr06
&dum(5))/dum(5))/dum(5)                                             !hr06
        dum(11)=sfac*(dum(8)/dum(5)-dum(4)*dum(10))
        dum(12)=(bcu(ibb,4)+bcu(ibb,9))+(bcu(ibb,6)+bcu(ibb,10))*sp !hr06
 dum(13)=(sfac*((dum(4)*dum(8))*half+(two*dum(3))*dum(9)))/dum(5)   !hr06
        if(abs(costh).gt.pieni) then
          costhp=(dum(11)/four)/costh                               !hr06
        else
          costhp=zero
        endif
        if(abs(sinth).gt.pieni) then
          sinthp=((-1d0*dum(11))/four)/sinth                        !hr06
        else
          sinthp=zero
        endif
        track(6,i)=track(6,i)-                                     &!hr06
&((((bbfx*(costhp*sepx0+sinthp*sepy0)+                             &!hr06
&bbfy*(costhp*sepy0-sinthp*sepx0))+                                &!hr06
&bbgx*(dum(12)+dum(13)))+bbgy*(dum(12)-dum(13)))/                  &!hr06
&cphi)*half                                                         !hr06
        bbf0=bbfx
        bbfx=bbf0*costh-bbfy*sinth
        bbfy=bbf0*sinth+bbfy*costh
      else
        track(6,i)=track(6,i)-                                     &
&(bbgx*(bcu(ibb,4)+bcu(ibb,6)*sp)+                                 &
&bbgy*(bcu(ibb,9)+bcu(ibb,10)*sp))/cphi
      endif
      track(6,i)=track(6,i)-(bbfx*(track(2,i)-bbfx*half)+          &
&bbfy*(track(4,i)-bbfy*half))*half
      track(1,i)=track(1,i)+s*bbfx
      track(2,i)=track(2,i)-bbfx
      track(3,i)=track(3,i)+s*bbfy
      track(4,i)=track(4,i)-bbfy
```

- Started from **previous work** done by J. Barranco
  - Identified and described the **interface of the main functional blocks**
  - Built tables with the descriptions of the cumbersome **notation** used in the code



TWiki > LHCAtHome Web > SixTrack > SixTrackBeamBeam (2017-03-21, Giovanniladarola) Edit Attach PDF

Information on Beam Beam

Overview of what is left to do in this section:

- Explicit description of how the slicing is done in `subroutine stsld`
- Explain what `bbcu` is and how it is computed/obtained
- Describe the Synchro-Beam Mapping is performed
- Additional variables needs to be explained (see argument lists for each subroutine)

How a Beam-beam element is defined in fort.2 and 3.

The beam beam element are directly translated from MADX to SixTrack input format. The parameters that define a BB in the **fort.2** lattice are,

Format _name type

name - May contain up to sixteen characters

type - 20

The beam-beam elements definition is now done fully in the BEAM block of fort.3 for both 4D and 6D lens.

4D lens (1 line per element)

name ibsix Σ_{xx} Σ_{yy} *h-sep v-sep strength-ratio*

6D lens (3 lines per element)

name ibsix xang xplane h-sep v-sep

Σ_{xx} Σ_{xxp} Σ_{xpxp} Σ_{yy} Σ_{yyp}

Σ_{ypyp} Σ_{xy} Σ_{xpy} Σ_{xpyp} Σ_{yyp} *strength-ratio*

name - Name of the beam–beam element.

- Moved to the understanding and testing of the source code…

Library implementation

It quickly became evident that the only viable way of checking the SixTrack code was to build an **independent implementation to compare against**. Done keeping in mind:

- **Readability, modularity**, possibility to **interface with other codes** (PyHEADTAIL, SixTrackLib)
- Compatibility with **GPU**

```
// Boost coordinates of the weak beam
BB6D_boost(&(bb6ddata->parboost), &x_star, &px_star, &y_star, &py_star,
            &sigma_star, &delta_star);

// Synchro beam
for (i_slice=0; i_slice<N_slices; i_slice++)
{
    double sigma_slice_star = sigma_slices_star[i_slice];
    double x_slice_star = x_slices_star[i_slice];
    double y_slice_star = y_slices_star[i_slice];

    //Compute force scaling factor
    double Ksl = N_part_per_slice[i_slice]*bb6ddata->q_part*q0/(p0*C_LIGHT);

    //Identify the Collision Point (CP)
    double S = 0.5*(sigma_star - sigma_slice_star);

    // Propagate sigma matrix
    double Sig_11_hat_star, Sig_33_hat_star, costheta, sintheta;
    double dS_Sig_11_hat_star, dS_Sig_33_hat_star, dS_costheta, dS_sintheta;

    // Get strong beam shape at the CP
    BB6D_propagate_Sigma_matrix(&(bb6ddata->Sigmas_0_star),
        S, bb6ddata->threshold_singular, 1,
        &Sig_11_hat_star, &Sig_33_hat_star,
        &costheta, &sintheta,
        &dS_Sig_11_hat_star, &dS_Sig_33_hat_star,
        &dS_costheta, &dS_sintheta);

    // Evaluate transverse coordinates of the weake baem w.r.t. the strong beam centroid
    double x_bar_star = x_star + px_star*S - x_slice_star;
    double y_bar_star = y_star + py_star*S - y_slice_star;

    // Move to the uncoupled reference frame
    double x_bar_hat_star = x_bar_star*costheta +y_bar_star*sintheta;
    double y_bar_hat_star = -x_bar_star*sintheta +y_bar_star*costheta;

    // Compute derivatives of the transformation
    double dS_x_bar_hat_star = x_bar_star*dS_costheta +y_bar_star*dS_sintheta;
    double dS_y_bar_hat_star = -x_bar_star*dS_sintheta +y_bar_star*dS_costheta;
```

```
    // Compute derivatives of the transformation
    double dS_x_bar_hat_star = x_bar_star*dS_costheta +y_bar_star*dS_sintheta;
    double dS_y_bar_hat_star = -x_bar_star*dS_sintheta +y_bar_star*dS_costheta;

    // Get transverse fieds
    double Ex, Ey, Gx, Gy;
    get_Ex_Ey_Gx_Gy_gauss(x_bar_hat_star, y_bar_hat_star,
        sqrt(Sig_11_hat_star), sqrt(Sig_33_hat_star), bb6ddata->min_sigma_diff,
        &Ex, &Ey, &Gx, &Gy);

    // Compute kicks
    double Fx_hat_star = Ksl*Ex;
    double Fy_hat_star = Ksl*Ey;
    double Gx_hat_star = Ksl*Gx;
    double Gy_hat_star = Ksl*Gy;

    // Move kisks to coupled reference frame
    double Fx_star = Fx_hat_star*costheta - Fy_hat_star*sintheta;
    double Fy_star = Fx_hat_star*sintheta + Fy_hat_star*costheta;

    // Compute longitudinal kick
    double Fz_star = 0.5*(Fx_hat_star*dS_x_bar_hat_star  + Fy_hat_star*dS_y_bar_hat_star+
                    Gx_hat_star*dS_Sig_11_hat_star + Gy_hat_star*dS_Sig_33_hat_star);

    // Apply the kicks (Hirata's synchro-beam)
    delta_star = delta_star + Fz_star+0.5*(
                Fx_star*(px_star+0.5*Fx_star)+
                Fy_star*(py_star+0.5*Fy_star));
    x_star = x_star - S*Fx_star;
    px_star = px_star + Fx_star;
    y_star = y_star - S*Fy_star;
    py_star = py_star + Fy_star;

}
// Inverse boost on the coordinates of the weak beam
BB6D_inv_boost(&(bb6ddata->parboost), &x_star, &px_star, &y_star, &py_star,
            &sigma_star, &delta_star);
```

Outline

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  - Constant charge slicing
  - Complete multi-slice interaction
- **Handling the denominators**

Validation tests

- Very difficult to identify problems by using the full tracking simulations
  - Need to test the single routine **"on the bench"**

- **Procedure** being performed for each functional block
  - Built a **C/python implementation** from the equations in the document
  - Extracted the **corresponding sixtrack source code** and compiled as of a stand-alone python module (f2py)
  - **"Stress test"** performed on the two: consistency checks, comparison against each other

Validation tests

Module	Tests performed	Outcome
Boost/anti-boost	• Comparison Sixtrack vs C/python routine • Checked that the two cancel each other	• **Bug** identified and **corrected**
Beam-beam forces (with potential derivatives w.r.t. sigmas)	• Comparison sixtrack vs C/python routine • Force compared against Finite Difference Poisson solved (PyPIC) • Other derivatives compared against numerical integration/derivation	• **All checks passed**
Beam shape propagation and coupling treatment	• Comparison Sixtrack vs C/python routine • Comparison against MAD for a coupled beam line • Crosscheck with numerical derivation	• **Bug** identified and **corrected** • **Vanishing denominators** not treated correctly → **correct treatment developed and implemented** in the library, **to be ported in SixTrack**
Slicing	• Check against independent implementation	• Passed but precision is quite poor (1e-3)
Computation of the kicks	• Check against independent implementation	• **All checks passed**

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- Implementation
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  - Constant charge slicing
  - Complete multi-slice interaction
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Boost /anti-boost

- Boost and anti-boost **should cancel each other exactly**
- **“Bench-test” cases:** large crossing angle, test particle very off momentum and large px, py
- **Test passed for the library**
- **Problem identified** in the **Sixtrack implementation**

Error after boost + anti-boost

Python test routine		SixTrack routine	
x	4.3e-19	**x**	6.5e-19
px	0.0	**px**	0.065
y	4.3e-19	**y**	4.3e-19
py	3.e3-17	**py**	0.027
sigma	0.0	**sigma**	0.0
delta	1e-16	**delta**	2.0e-17

Boost /anti-boost

Discrepancy found between in the anti-boost between derived equations and SixTrack source code:

$$p_x = p_x^* \cos\phi + h \cos\alpha \tan\phi \quad (95)$$

$$p_y = p_y^* \cos\phi + h \sin\alpha \tan\phi \quad (96)$$

```
TRACK(2)=(TRACK(2)+CALPHA*SPHI*H1)*CPHI
TRACK(4)=(TRACK(4)+SALPHA*SPHI*H1)*CPHI
```

The lines should be:

```
TRACK(2)=(TRACK(2)*CPHI+CALPHA*TPHI*H1)
TRACK(4)=(TRACK(4)*CPHI+SALPHA*TPHI*H1)
```

- Digging a bit we found out that the issue was already present in Hirata's code from 1996, on which the Sixtrack implementation is based

Boost /anti-boost

- **Correction implemented in SixTrack**

Error after boost + anti-boost

Python test routine		SixTrack routine		SixTrack corrected	
x	4.3e-19	x	6.5e-19	x	6.5e-19
px	0.0	px	0.065	px	5.55e-17
y	4.3e-19	y	4.3e-19	y	4.3e-19
py	3.e3-17	py	0.027	py	0.1e-19
sigma	0.0	sigma	0.0	sigma	0.0
delta	1e-16	delta	2.0e-17	delta	2.0e-17

Boost /anti-boost

- Problem confirmed by Riccardo simulating a beam-beam interaction with **zero intensity in the strong beam**

Original implementation

Coordinates before interaction

dump_ip.dat

# ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack
1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03		
2	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03		
3	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02		
4	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02		
5	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02		
6	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02		
7	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02		
8	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02		
9	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02		
10	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02		
1	2	0.00000	1.308501246E-01	8.514045445E-03	-9.961266845E-03	3.153912424E-04		
2	2	0.00000	1.308501246E-01	8.514045445E-03	-9.961266845E-03	3.153912424E-04		
3	2	0.00000	1.041820622E-01	-1.200951762E-02	-8.217894405E-02	2.601833146E-03		
4	2	0.00000	1.041820622E-01	-1.200951762E-02	-8.217894405E-02	2.601833146E-03		
5	2	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888381596E-03		
6	2	0.00000	7.751399978E-02	-3.253308068E-02	-1.543977321E-01	4.888381596E-03		
7	2	0.00000	5.084593752E-02	-5.305664373E-02	-2.266176309E-01	7.175036594E-03		
8	2	0.00000	5.084593752E-02	-5.305664373E-02	-2.266176309E-01	7.175036594E-03		
9	2	0.00000	2.417787538E-02	-7.358020677E-02	-2.988386405E-01	9.461798139E-03		
10	2	0.00000	2.417787538E-02	-7.358020677E-02	-2.988386405E-01	9.461798139E-03		

Coordinates after interaction

dump_bb.dat

# ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack
1	1	0.00000	1.444989354E-01	1.217984945E-02	2.341007330E-02	-1.973250177E-03		
2	1	0.00000	1.444989354E-01	1.217984945E-02	2.341007330E-02	-1.973250177E-03		
3	1	0.00000	2.169989354E-01	1.829089158E-02	1.931331047E-01	-1.627927274E-02		
4	1	0.00000	2.169989354E-01	1.829089158E-02	1.931331047E-01	-1.627927274E-02		
5	1	0.00000	2.894989354E-01	2.440193367E-02	3.628561362E-01	-3.058532567E-02		
6	1	0.00000	2.894989354E-01	2.440193367E-02	3.628561362E-01	-3.058532567E-02		
7	1	0.00000	3.619989354E-01	3.051297574E-02	5.325791676E-01	-4.489140898E-02		
8	1	0.00000	3.619989354E-01	3.051297574E-02	5.325791676E-01	-4.489140898E-02		
9	1	0.00000	4.344989354E-01	3.662401777E-02	7.023021991E-01	-5.919752266E-02		
10	1	0.00000	4.344989354E-01	3.662401777E-02	7.023021991E-01	-5.919752266E-02		
1	2	0.00000	1.308501246E-01	8.514045441E-03	-9.961266845E-03	3.153866850E-04		
2	2	0.00000	1.308501246E-01	8.514045441E-03	-9.961266845E-03	3.153866850E-04		
3	2	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601823666E-03		
4	2	0.00000	1.041820622E-01	-1.200951763E-02	-8.217894405E-02	2.601823666E-03		
5	2	0.00000	7.751399978E-02	-3.253308074E-02	-1.543977321E-01	4.888313646E-03		
6	2	0.00000	7.751399978E-02	-3.253308074E-02	-1.543977321E-01	4.888313646E-03		
7	2	0.00000	5.084593752E-02	-5.305664388E-02	-2.266176309E-01	7.174856626E-03		
8	2	0.00000	5.084593752E-02	-5.305664388E-02	-2.266176309E-01	7.174856626E-03		
9	2	0.00000	2.417787538E-02	-7.358020705E-02	-2.988386405E-01	9.461452606E-03		
10	2	0.00000	2.417787538E-02	-7.358020705E-02	-2.988386405E-01	9.461452606E-03		

Corrected implementation

Coordinates before interaction

dump_ip.dat

ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack
1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03	1	
2	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03	1	
3	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02	1	
4	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02	1	
5	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02	1	
6	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02	1	
7	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02	1	
8	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02	1	
9	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02	1	
10	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02	1	
1	2	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04	9	
2	2	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04	9	
3	2	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03	9	
4	2	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03	9	
5	2	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03	9	
6	2	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03	9	
7	2	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03	9	
8	2	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03	9	
9	2	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03	9	
10	2	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03	9	

Coordinates after interaction

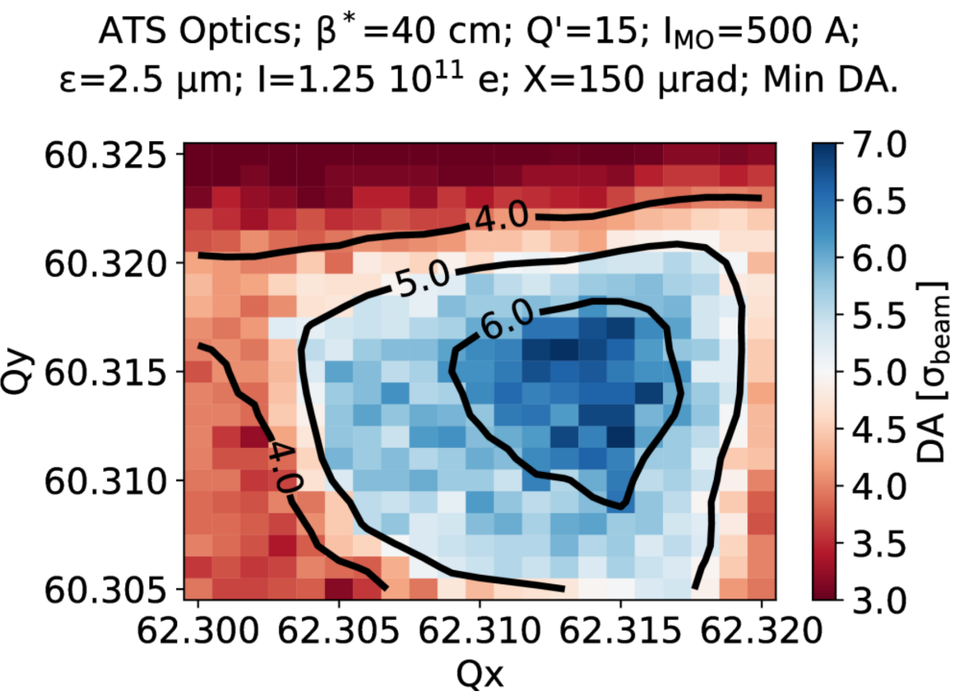
dump_bb.dat

ID	turn	s[m]	x[mm]	xp[mrad]	y[mm]	yp[mrad]	dE/E[1]	ktrack
1	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03	1	
2	1	0.00000	1.444989354E-01	1.217984946E-02	2.341007330E-02	-1.973240618E-03	1	
3	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02	1	
4	1	0.00000	2.169989354E-01	1.829089161E-02	1.931331047E-01	-1.627923509E-02	1	
5	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02	1	
6	1	0.00000	2.894989354E-01	2.440193375E-02	3.628561362E-01	-3.058522956E-02	1	
7	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02	1	
8	1	0.00000	3.619989354E-01	3.051297588E-02	5.325791676E-01	-4.489122400E-02	1	
9	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02	1	
10	1	0.00000	4.344989354E-01	3.662401801E-02	7.023021991E-01	-5.919721844E-02	1	
1	2	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04	9	
2	2	0.00000	1.308501247E-01	8.514045444E-03	-9.960917299E-03	3.153577120E-04	9	
3	2	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03	9	
4	2	0.00000	1.041820623E-01	-1.200951763E-02	-8.217756745E-02	2.601701095E-03	9	
5	2	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03	9	
6	2	0.00000	7.751400004E-02	-3.253308069E-02	-1.543942171E-01	4.888044424E-03	9	
7	2	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03	9	
8	2	0.00000	5.084593802E-02	-5.305664374E-02	-2.266108663E-01	7.174387701E-03	9	
9	2	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03	9	
10	2	0.00000	2.417787621E-02	-7.358020679E-02	-2.988275150E-01	9.460730924E-03	9	

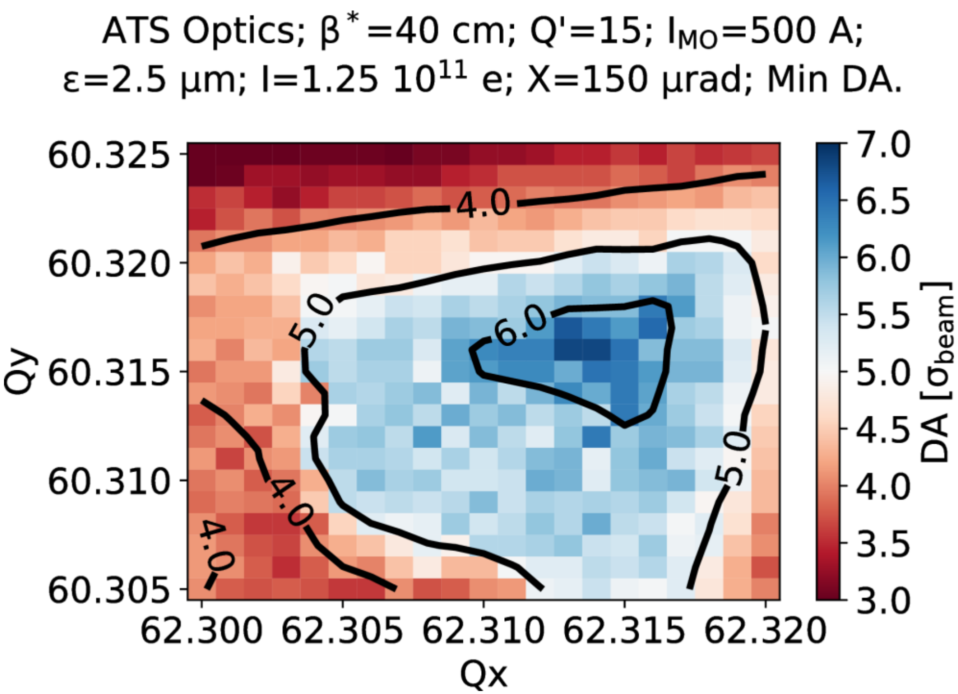
Boost /anti-boost

- Impact on **realistic simulation study** assessed by Dario
- Tune scans comparison with 2017 ATS optics show no dramatic change, but slightly worse DA

Old version



Corrected version



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Transverse kicks for a Gaussian beam

Transverse field for a Gaussian beam (Bassetti-Erskine)

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{x}^*}\left(\hat{x}^*,\hat{y}^*,\hat{\Sigma}_{11}^*,\hat{\Sigma}_{33}^*\right)$$

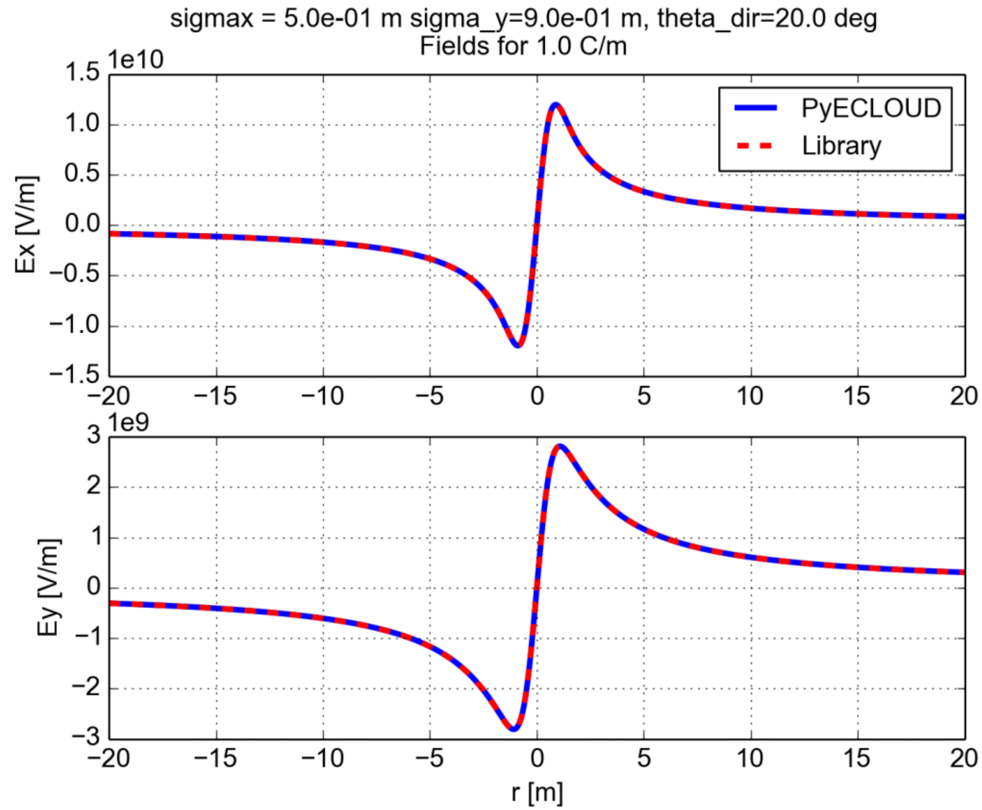
$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{y}^*}\left(\hat{x}^*,\hat{y}^*,\hat{\Sigma}_{11}^*,\hat{\Sigma}_{33}^*\right)$$

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\mathrm{Im}\left[w\left(\frac{\hat{x}^*+i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\right)-\exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*}-\frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right)w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}+i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\right)\right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{x}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\mathrm{Re}\left[w\left(\frac{\hat{x}^*+i\hat{y}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\right)-\exp\left(-\frac{(\hat{x}^*)^2}{2\hat{\Sigma}_{11}^*}-\frac{(\hat{y}^*)^2}{2\hat{\Sigma}_{33}^*}\right)w\left(\frac{\hat{x}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}+i\hat{y}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^*-\hat{\Sigma}_{33}^*\right)}}\right)\right]$$

Library tested against Poisson solver of PyECLOUD

(test repeated for tall, fat and round beams)



Transverse kicks for a Gaussian beam

Transverse field for a Gaussian beam (Bassetti-Erskine)

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

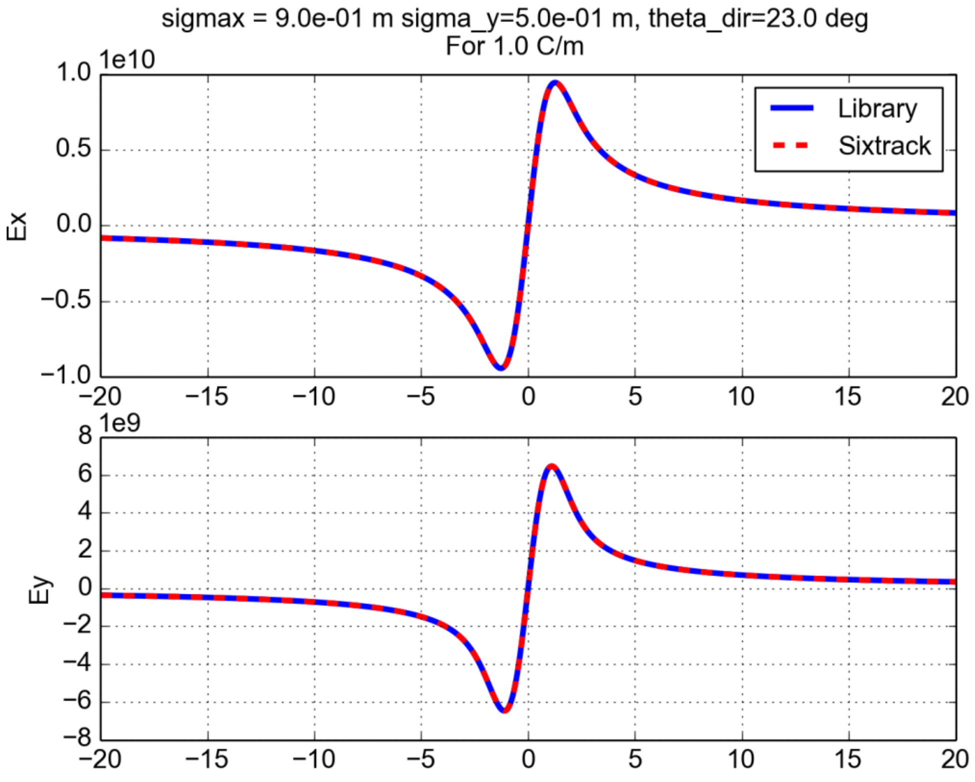
$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\bar{y}}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right)$$

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Im}\left[w\left(\frac{\hat{\bar{x}}^* + i\hat{\bar{y}}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{\bar{x}}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{\bar{y}}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Re}\left[w\left(\frac{\hat{\bar{x}}^* + i\hat{\bar{y}}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{\bar{x}}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{\bar{y}}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right]$$

SixTrack tested against library

(test repeated for tall, fat and round beams)



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  - Complete multi-slice interaction
- Handling the denominators

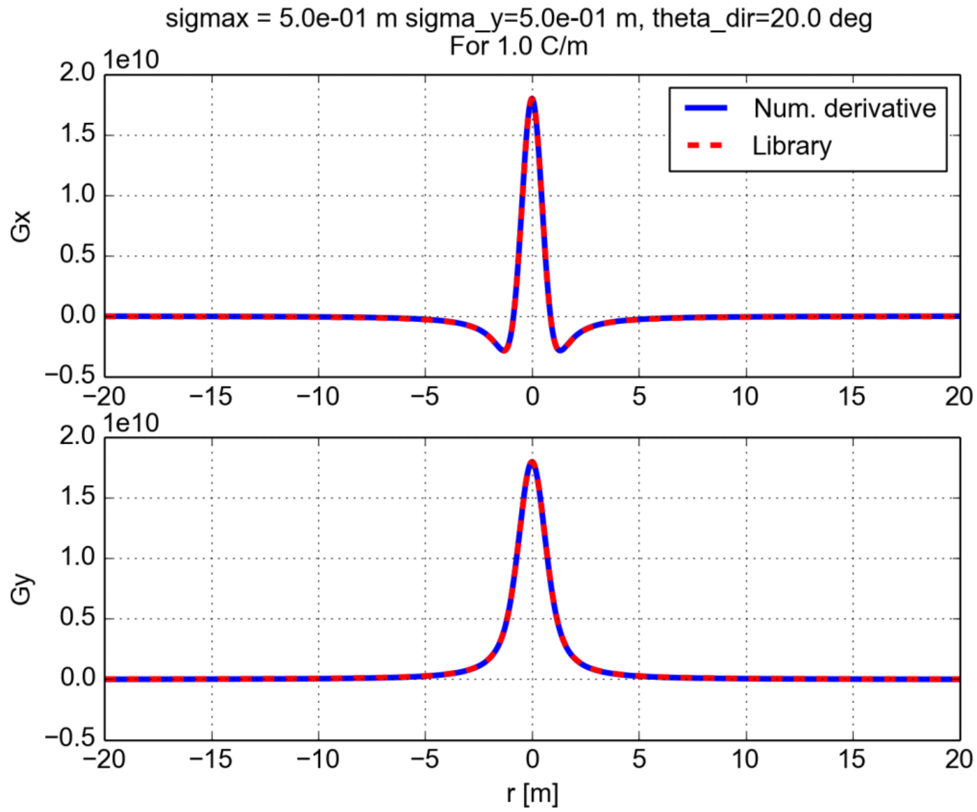
Other derivatives of the electric potential

$$\hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

$$\hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

Library tested against numerical derivative

(test repeated for tall, fat and round beams)



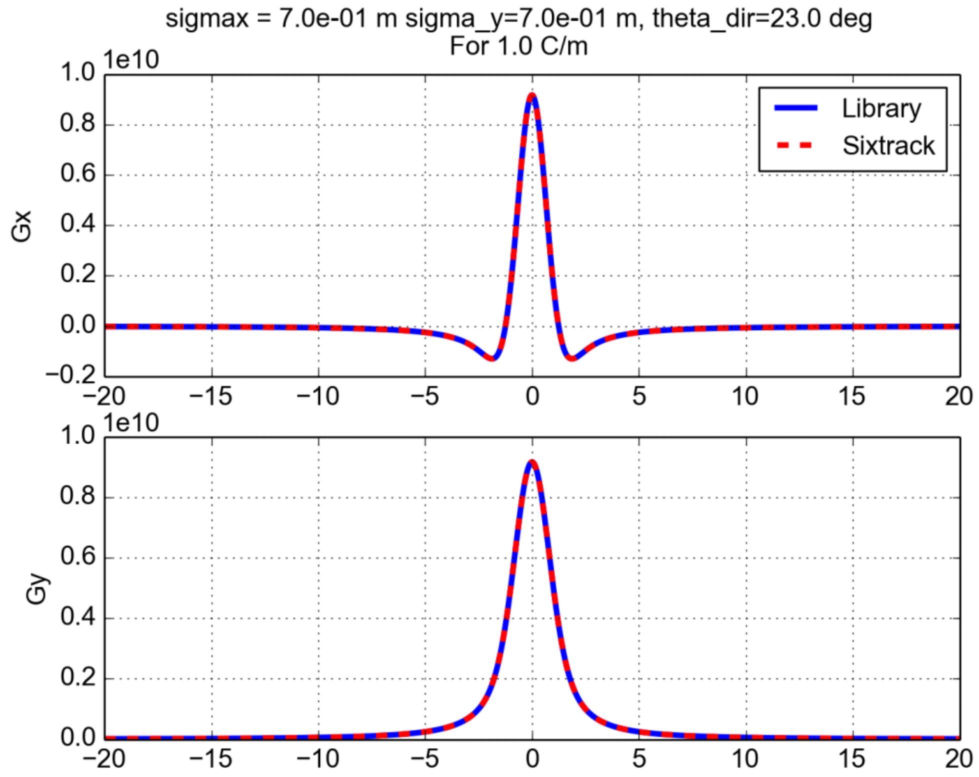
Other derivatives of the electric potential

$$\hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

$$\hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{\bar{x}}^*, \hat{\bar{y}}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right) \qquad \hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\}$$

SixTrack tested against library

(test repeated for tall, fat and round beams)



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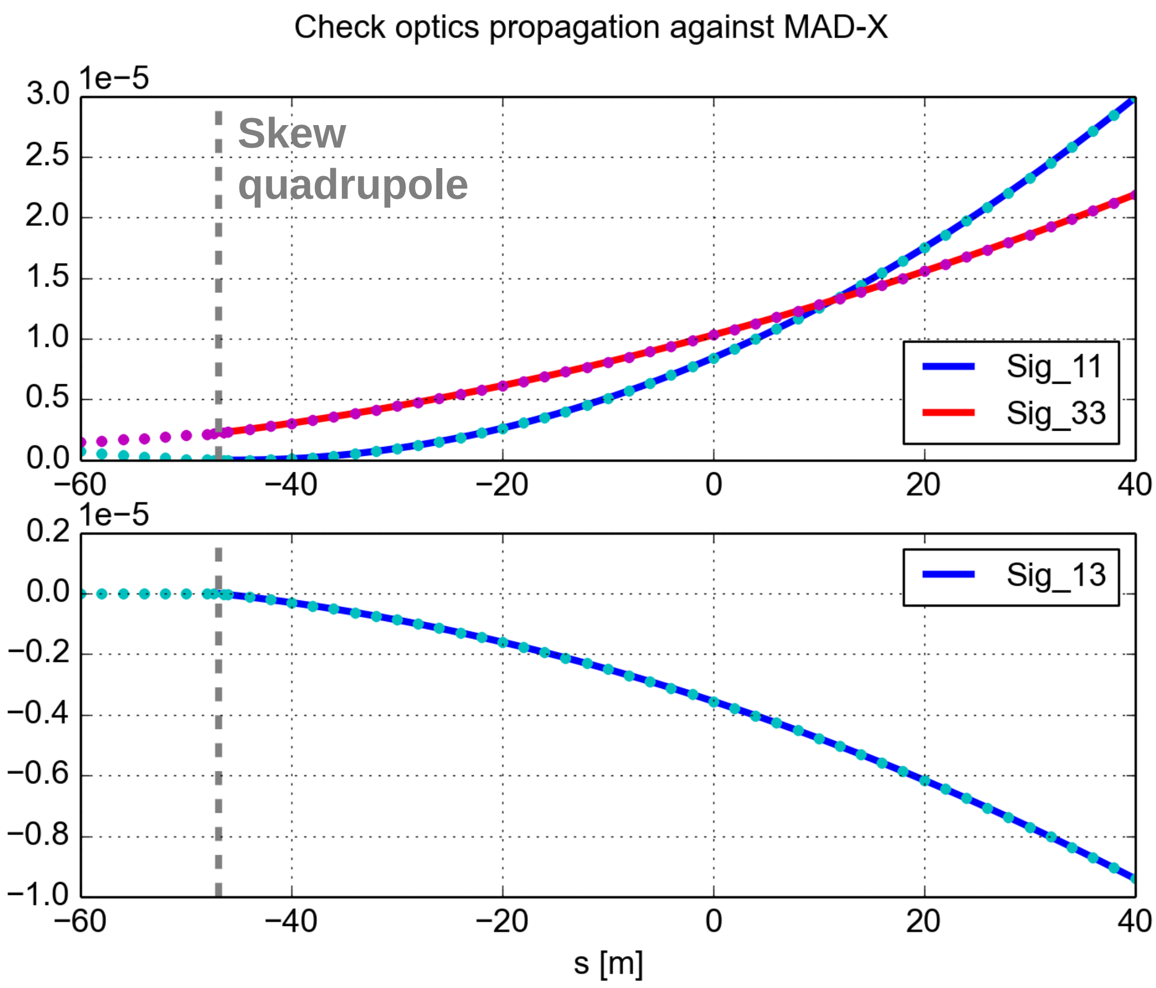
Σ-matrix propagation with linear coupling

Library tested against MAD-X:

- Built a simple line with a strong skew quadrupole
- Entering with a de-coupled beam
- Saves Σ-matrix at regularly spaced markers for comparison against library



Dots: MAD-X
Lines: library

Σ-matrix propagation with linear coupling

Library tested against MAD-X:

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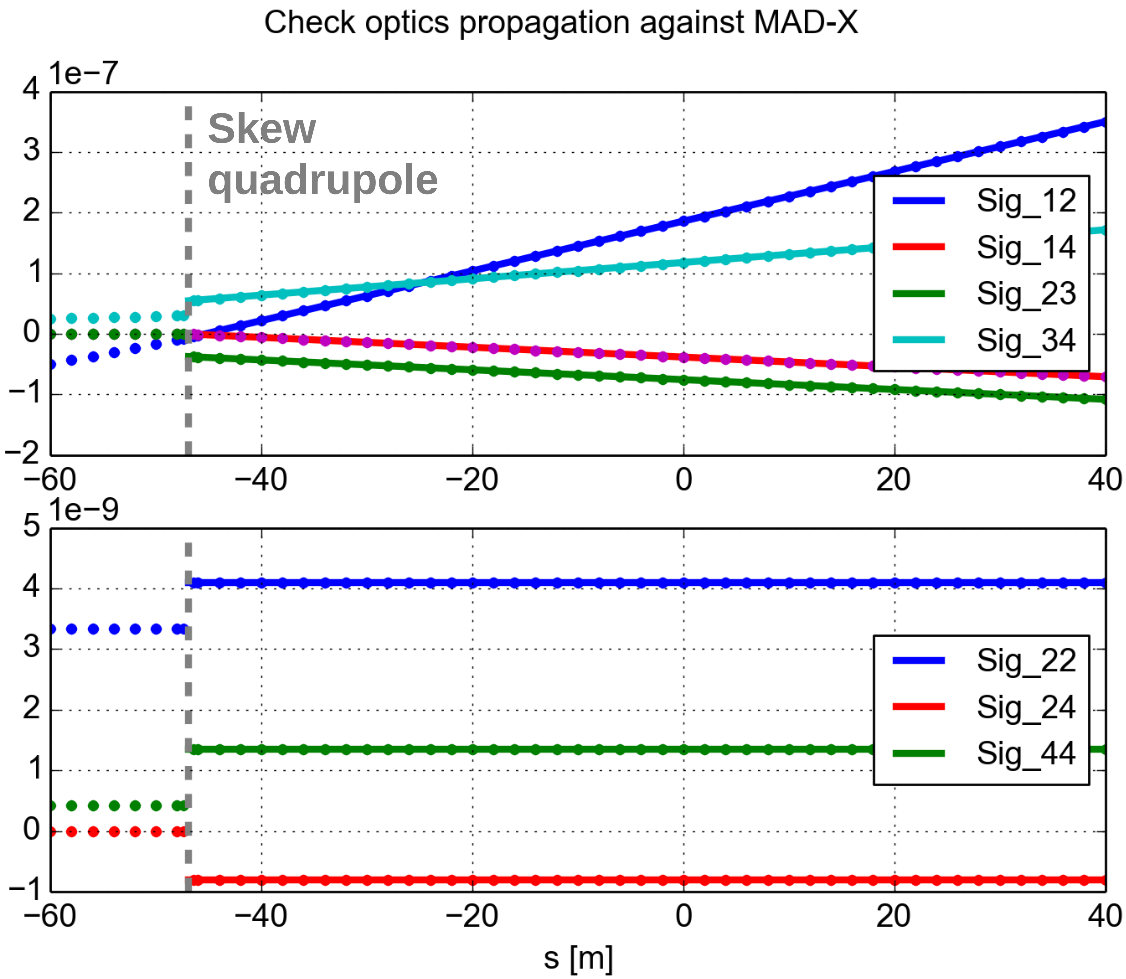

Dots: MAD-X
Lines: library

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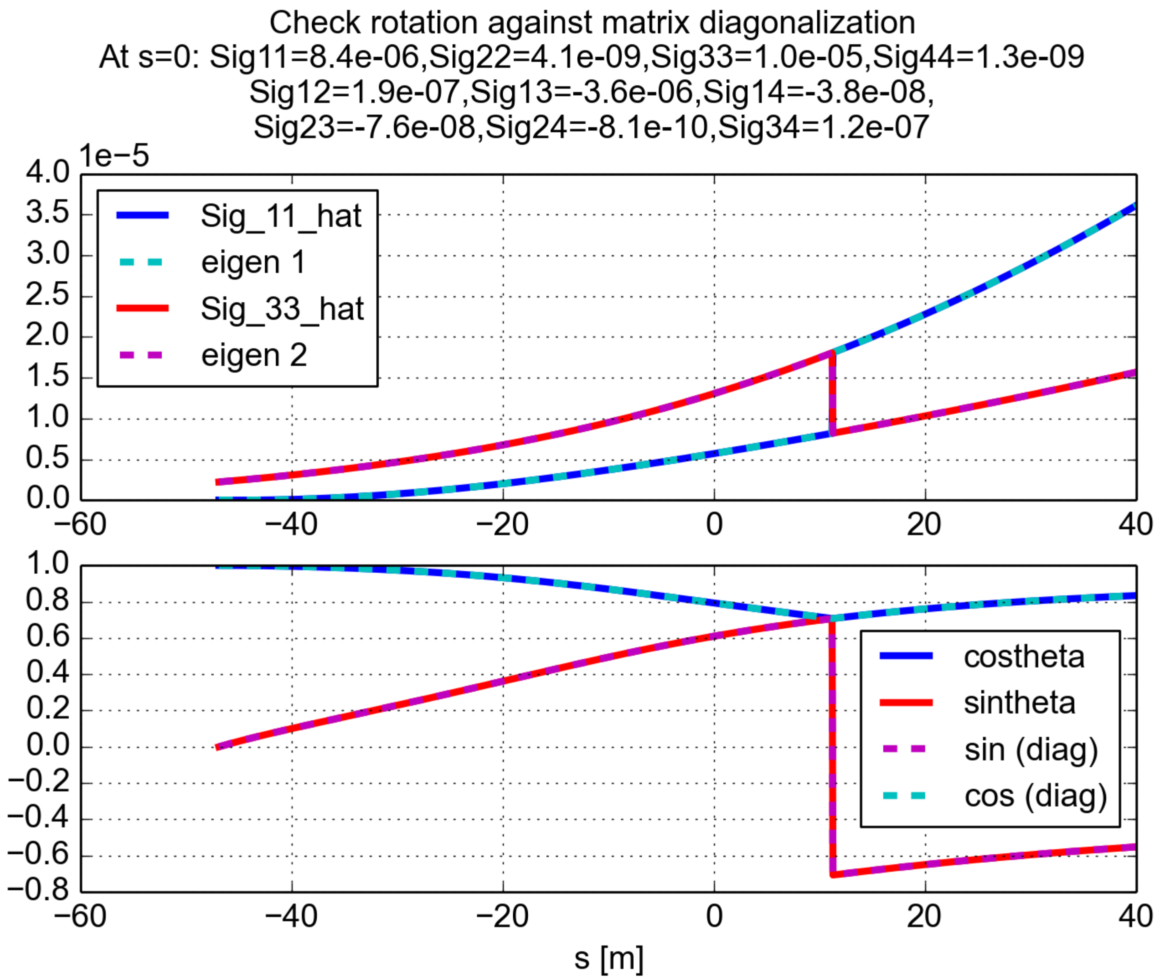
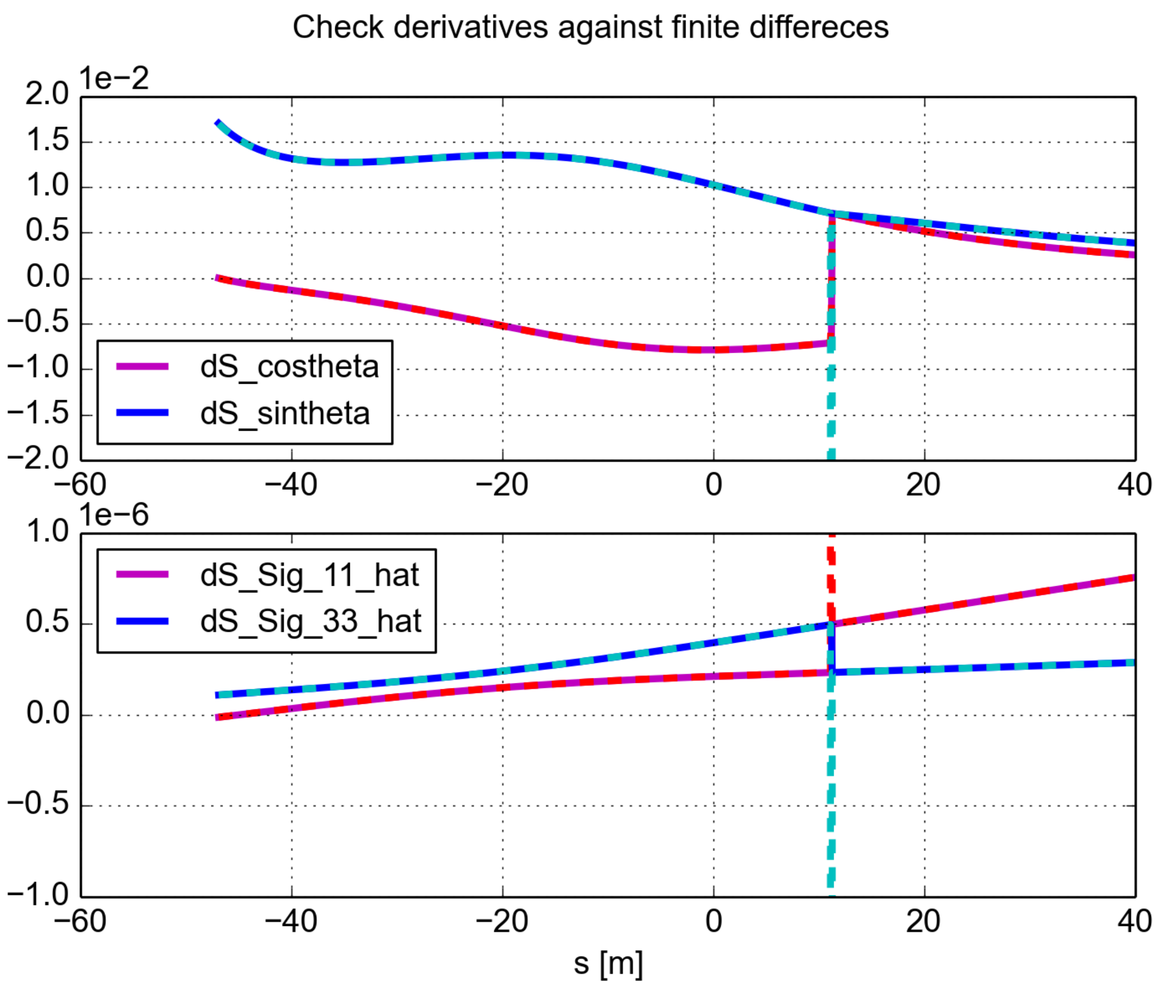
Σ-matrix transformation to un-coupled frame

Library tested against **numerical diagonalization** of the Σ-matrix



Σ-matrix transformation to un-coupled frame

Library tested against **numerical diagonalization** of the Σ-matrix



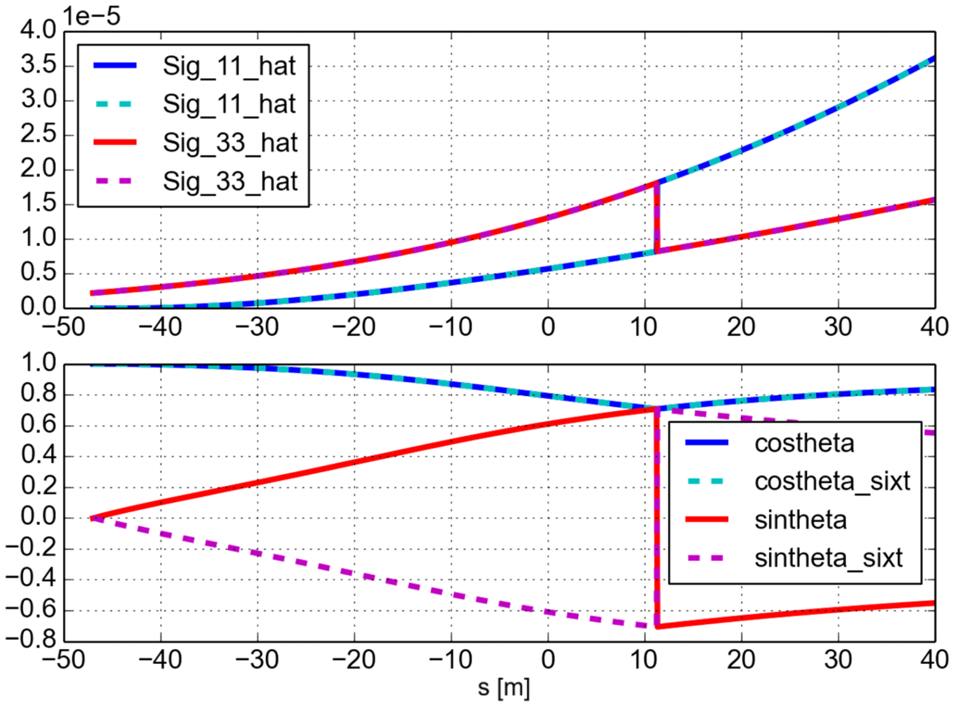
Σ-matrix transformation to un-coupled frame

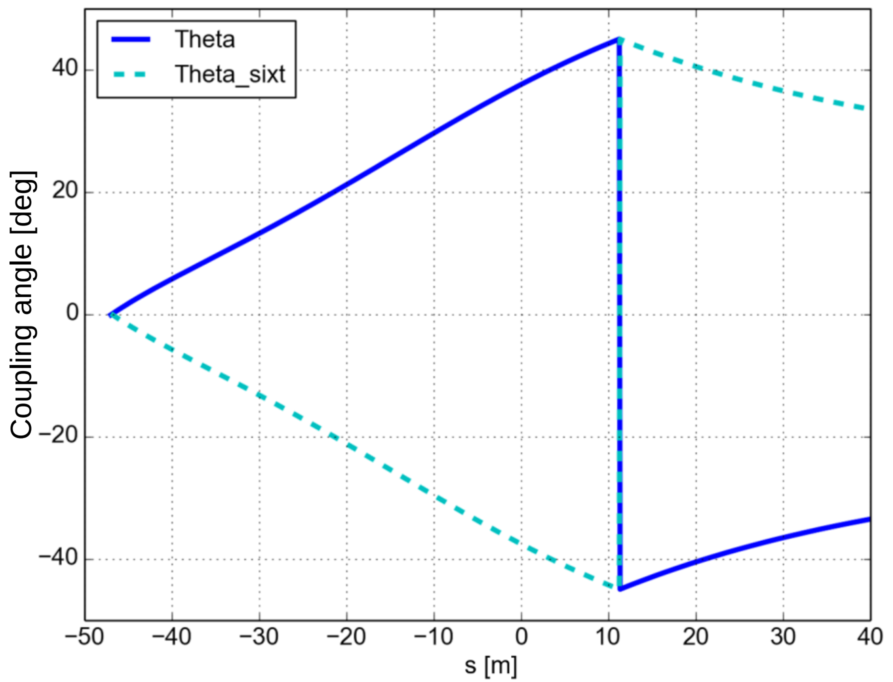
SixTrack tested against library: test failed!

Sign error in the computation of the coupling angle

Original source code:

```
if(abs(sinth).gt.pieni) then
   sinth=(-1d0*sfac)*sqrt(sinth)
else
   sinth=zero
endif
```





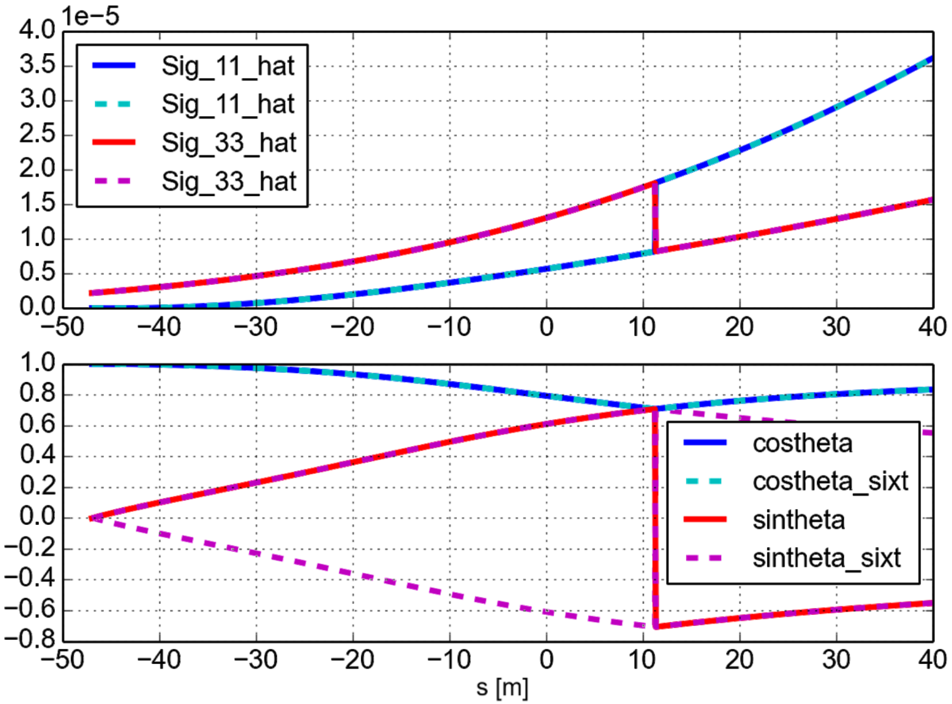
Σ-matrix transformation to un-coupled frame

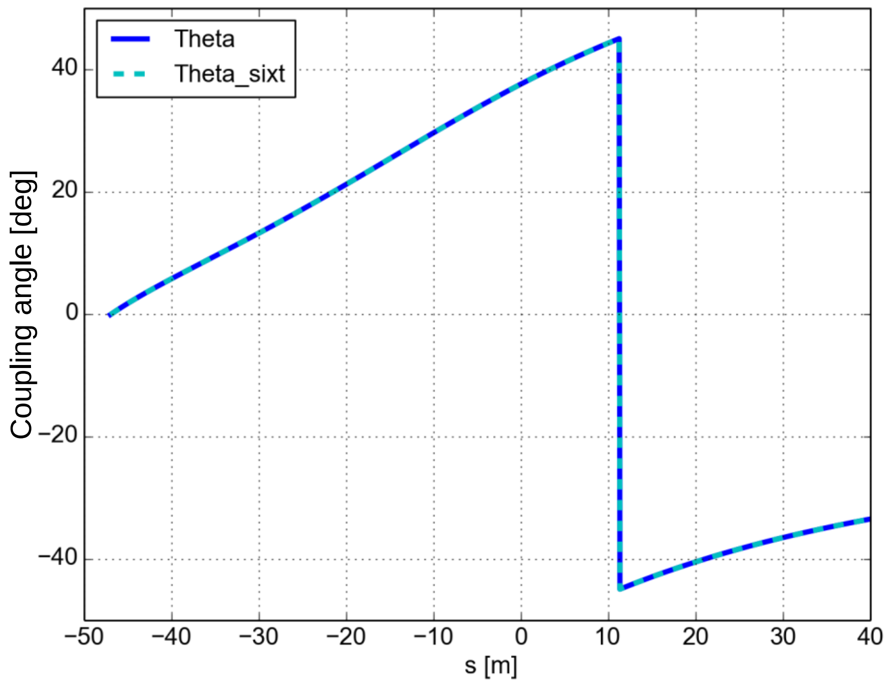
SixTrack tested against library: test failed!

Sign error in the computation of the coupling angle

Corrected source code:

```
if(abs(sinth).gt.pieni) then
   sinth=(sfac)*sqrt(sinth)
else
   sinth=zero
endif
```





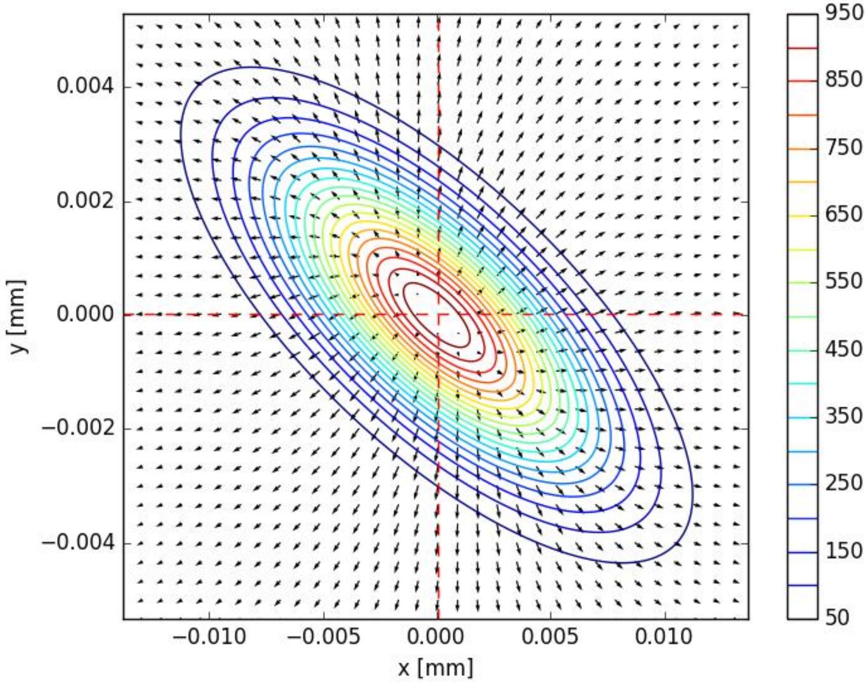
Σ-matrix transformation to un-coupled frame

Input sigma matrix:

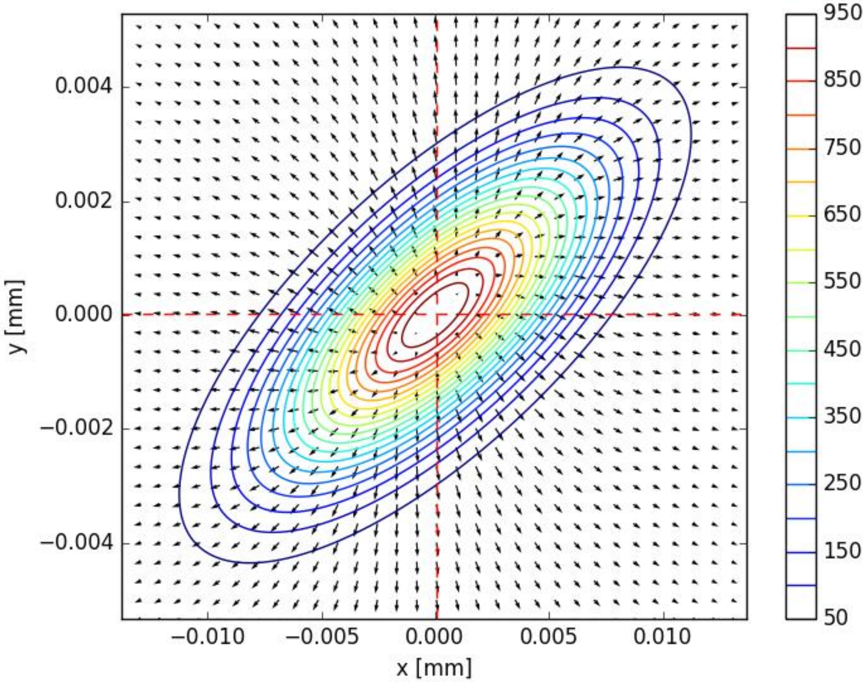
{'Sig_11_0': 2.1046670129999999e-05,
'Sig_12_0': 2.7725426699999999e-07,
'Sig_13_0': 5.9207071659999999e-06,
'Sig_14_0': 1.2224001670000001e-07,
'Sig_22_0': 3.6622825020000002e-09,
'Sig_23_0': 7.4141336339999994e-08,
'Sig_24_0': 1.495491124e-09,
'Sig_33_0': 3.165637487e-06,
'Sig_34_0': 7.9058234540000002e-08,
'Sig_44_0': 2.040387648e-09}

Checked by Kyrre using full SixTrack simulations (numerical divergence of the computed kicks)

Original



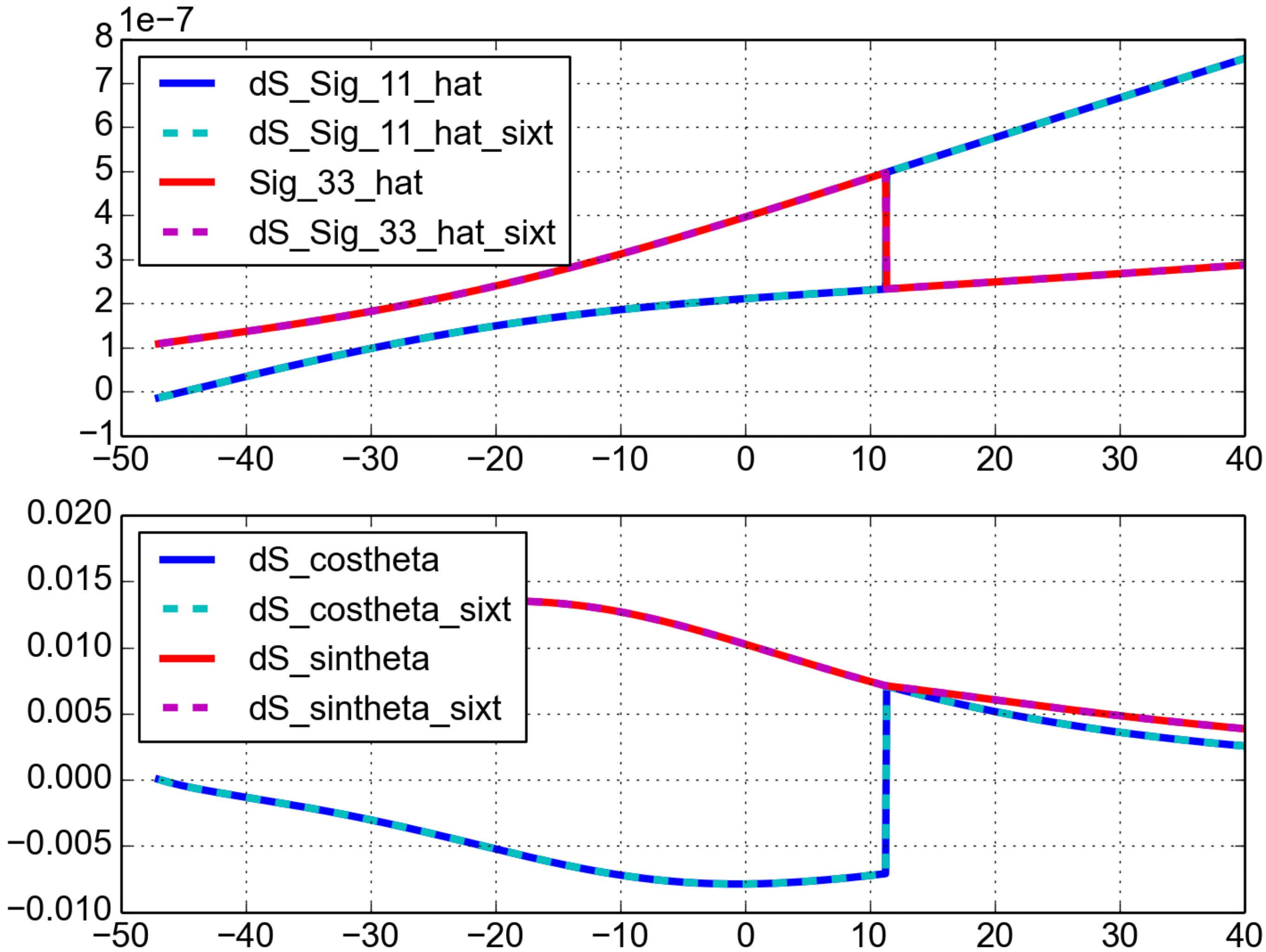
Corrected



More info at: https://github.com/SixTrack/SixTrack/issues/267#issuecomment-307333656

Σ-matrix transformation to un-coupled frame

After bug correction **derivatives were also found to be ok**



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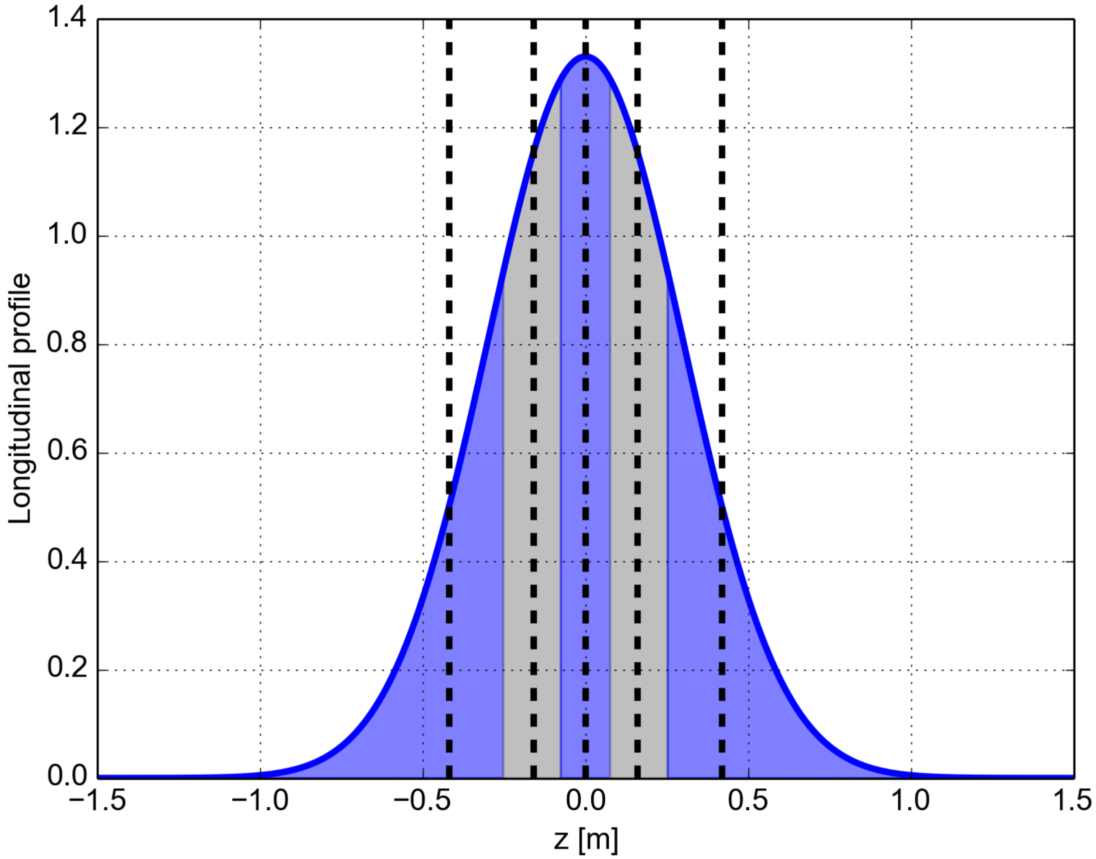
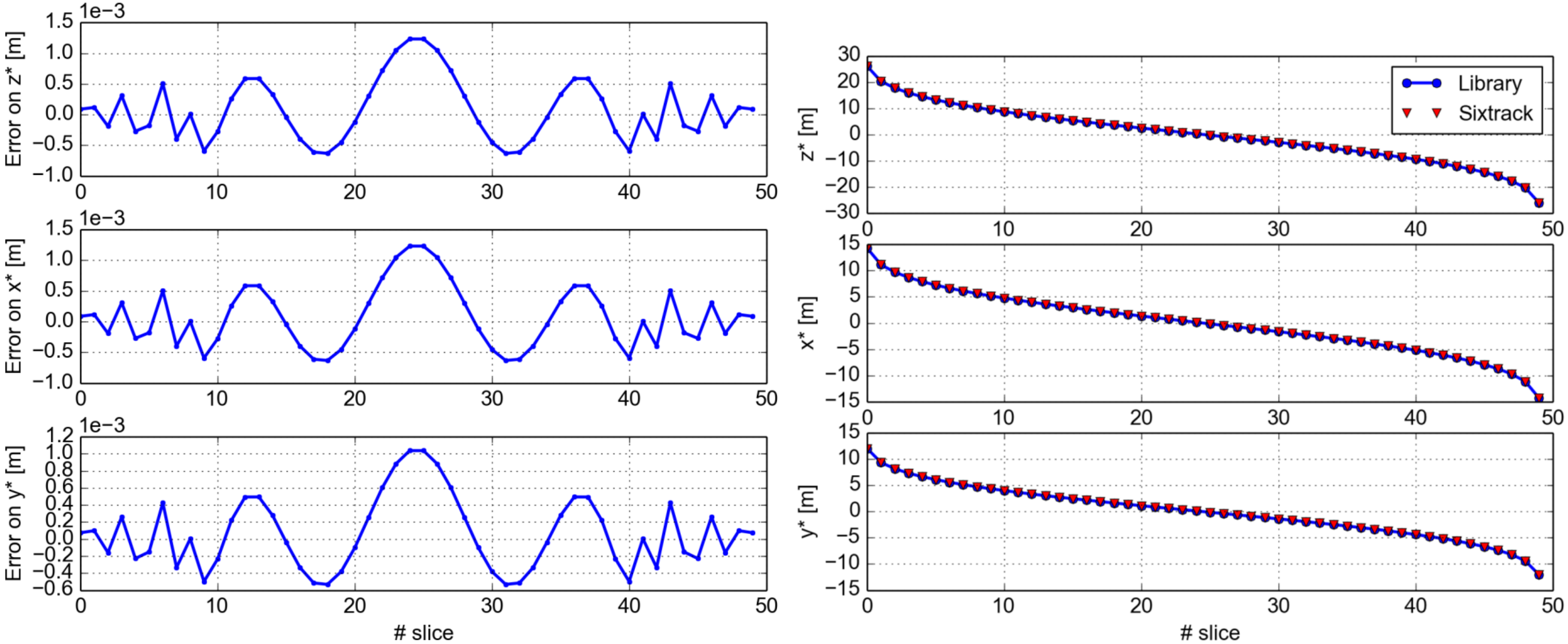
  - Complete multi-slice interaction
- **Handling the denominators**

Constant charge slicing

Library: slicing could be easily re-implemented using python inverse error function



Sixtrack: implementation is correct but not very accurate



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  - **Complete multi-slice interaction**
- Handling the denominators

Complete multi-slice interaction

Sixtrack (corrected) vs library: agreement to the 6th digit!

```
Compare kicks against sixtrack:
D_x -2.32123980148e-07 -2.32123980355e-07 err=2.08e-16
D_px 4.62575633839e-08 4.62575633839e-08 err=0.00e+00
D_y -1.95977011284e-07 -1.9597701092e-07 err=-3.64e-16
D_py 3.88258677153e-08 3.88258677153e-08 err=0.00e+00
D_sigma -5.29477794942e-10 -5.29477350852e-10 err=-4.44e-16
D_delta 6.18915584942e-08 6.18915584951e-08 err=-8.67e-19
```

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- **Handling the denominators**

Case T>0, $\left|\Sigma^*_{13}\right|$>0

We use the expression that we have derived before:

$$R\,(S) = \Sigma^*_{11} - \Sigma^*_{33}$$
$$W\,(S) = \Sigma^*_{11} + \Sigma^*_{33}$$
$$T\,(S) = R^2 + 4{\Sigma^*_{13}}^2$$

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

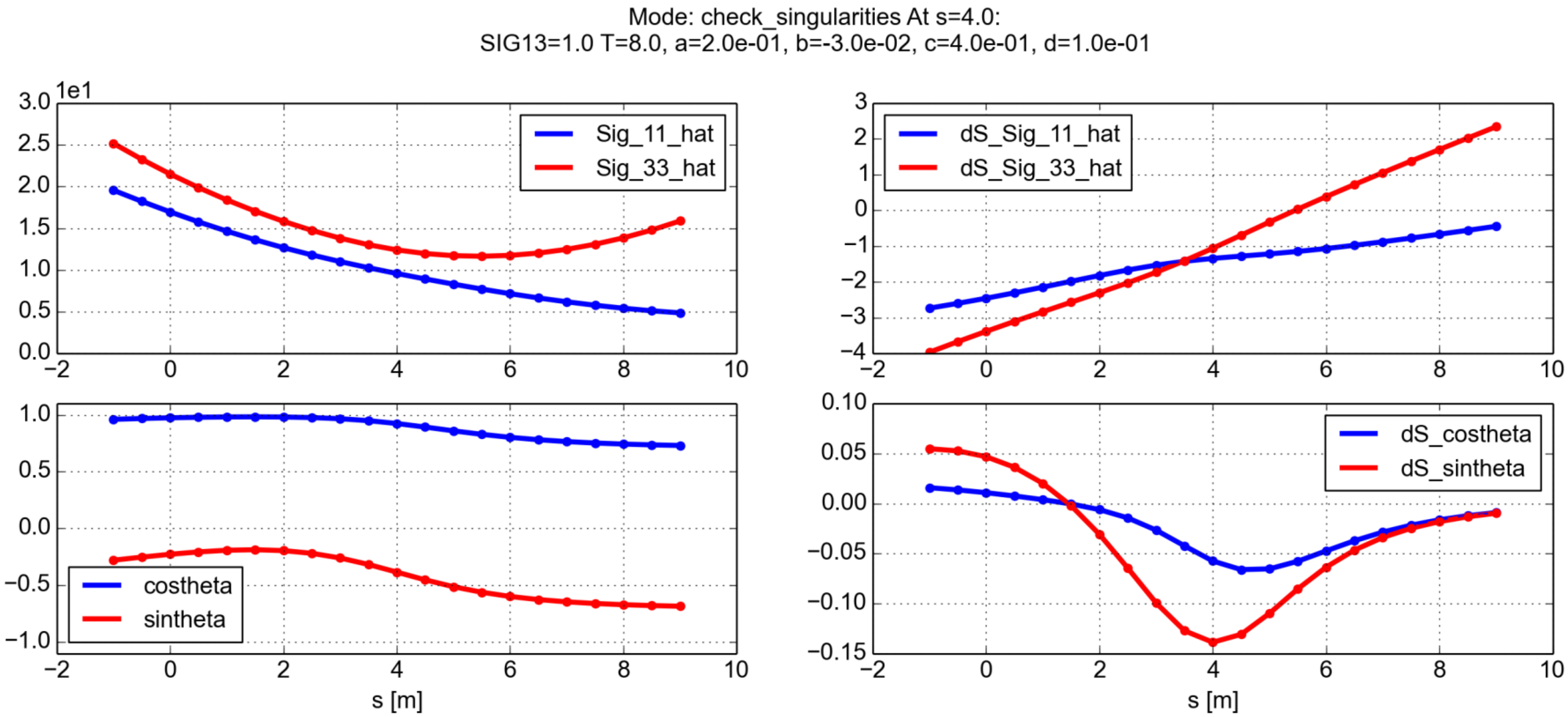
$$\cos\theta = \sqrt{\frac{1}{2}\left(1 + \cos 2\theta\right)}$$
$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1 - \cos 2\theta\right)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Handling the denominators: case #0

Case T>0, $|\Sigma^*_{13}|>0$

Tests:

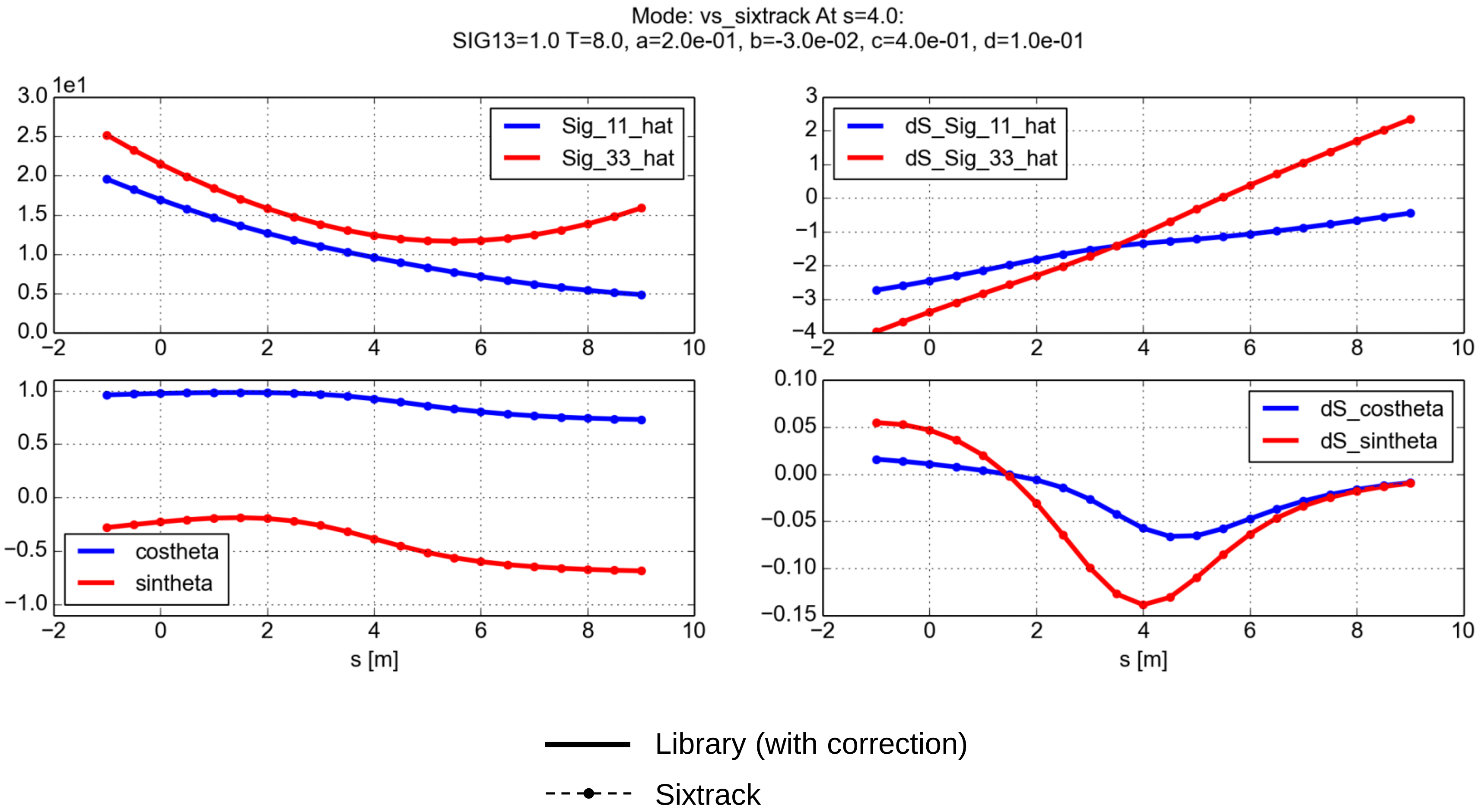


Expression with denominator (apparently singular)

Expression with correction

Case T>0, $|\Sigma_{13}^*|>0$

Tests against Sixtrack:



Case T>0, $\left|\Sigma^*_{13}\right|$=0:

The highlighted formulas break and **alternative expressions** need to be found:

$$R(S) = \Sigma^*_{11} - \Sigma^*_{33}$$
$$W(S) = \Sigma^*_{11} + \Sigma^*_{33}$$
$$T(S) = R^2 + 4{\Sigma^*_{13}}^2$$

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

$$\cos\theta = \sqrt{\frac{1}{2}\left(1 + \cos 2\theta\right)}$$
$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1 - \cos 2\theta\right)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Case T>0, $\left|\Sigma^*_{13}\right|$=0:

$$\cos 2\theta = \mathrm{sgn}\left(\Sigma^*_{11} - \Sigma^*_{33}\right) \frac{\Sigma^*_{11} - \Sigma^*_{33}}{\sqrt{\left(\Sigma^*_{11} - \Sigma^*_{33}\right)^2 + 4{\Sigma^*_{13}}^2}} \Rightarrow \sin\theta = 0$$

$$\Rightarrow \frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$

~~$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$~~

Case T>0, $\left|\Sigma^*_{13}\right|$=0:

Around the singular point we can write:

$$\Sigma^*_{13} = c\Delta S + d\Delta S^2$$

with

$$\begin{aligned} a &= \Sigma^*_{12} - \Sigma^*_{34} \\ b &= \Sigma^*_{22} - \Sigma^*_{44} \\ c &= \Sigma^*_{14} + \Sigma^*_{23} \\ d &= \Sigma^*_{24} \end{aligned}$$

$$\cos 2\theta = \frac{|R|}{\sqrt{R^2 + 4{\Sigma^*_{13}}^2}} = \frac{1}{\sqrt{1 + 4\frac{{\Sigma^*_{13}}^2}{R^2}}} \simeq \frac{1}{1 + 2\frac{{\Sigma^*_{13}}^2}{R^2}} \simeq 1 - 2\frac{{\Sigma^*_{13}}^2}{R^2}$$

$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\frac{|\Sigma^*_{13}|}{|R|} = \frac{\Sigma^*_{13}}{R}$$

$$\frac{\partial}{\partial S}\sin\theta = \frac{1}{R^2}\left[(c + 2d\Delta S)\,R - \frac{\partial R}{\partial S}\left(c\Delta S + d\Delta S^2\right)\right]$$

At the singular point

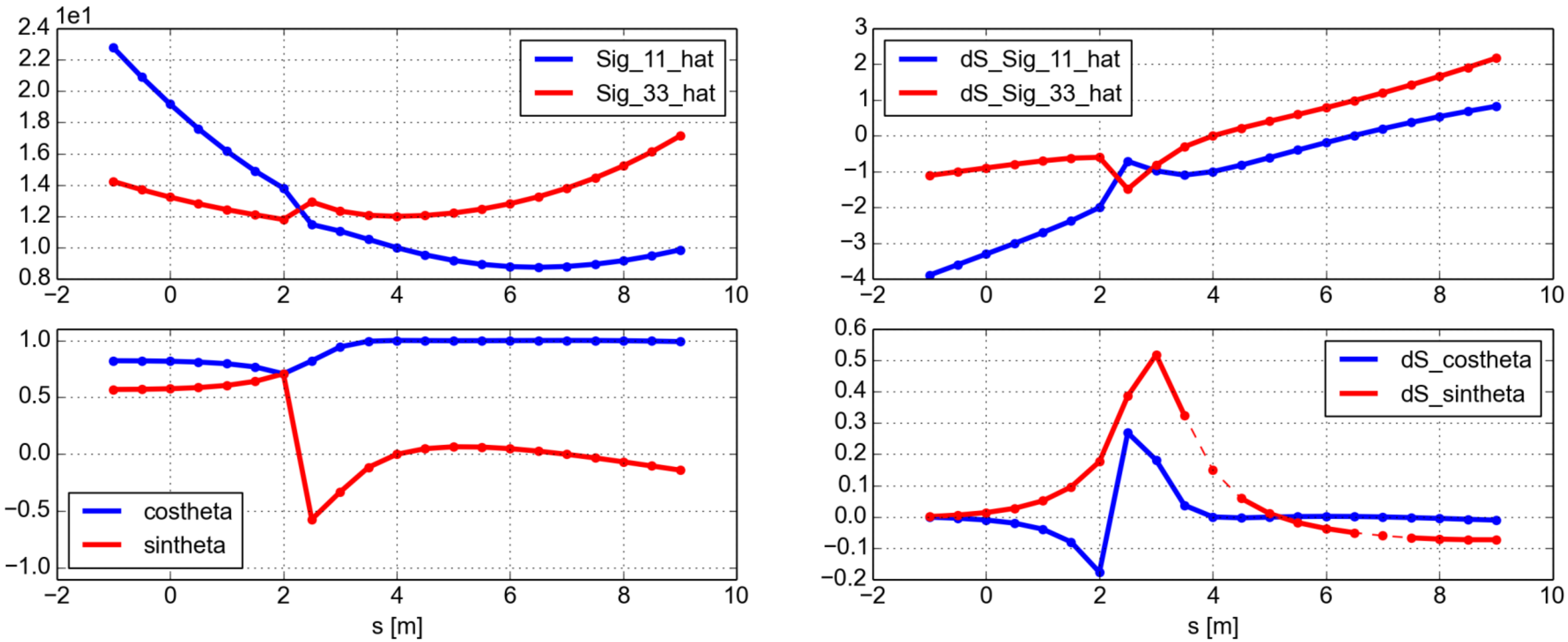
$$\frac{\partial}{\partial S}\sin\theta = \frac{c}{R}$$

Which is always regular once we assume T>0 and therefore R^2>0

Case T>0, $\left|\Sigma^*_{13}\right|$=0:

Tests:

Mode: check_singularities At s=4.0:
SIG13=0.0 T=4.0, a=-5.0e-01, b=0.0, c=-3.0e-01, d=1.0e-01



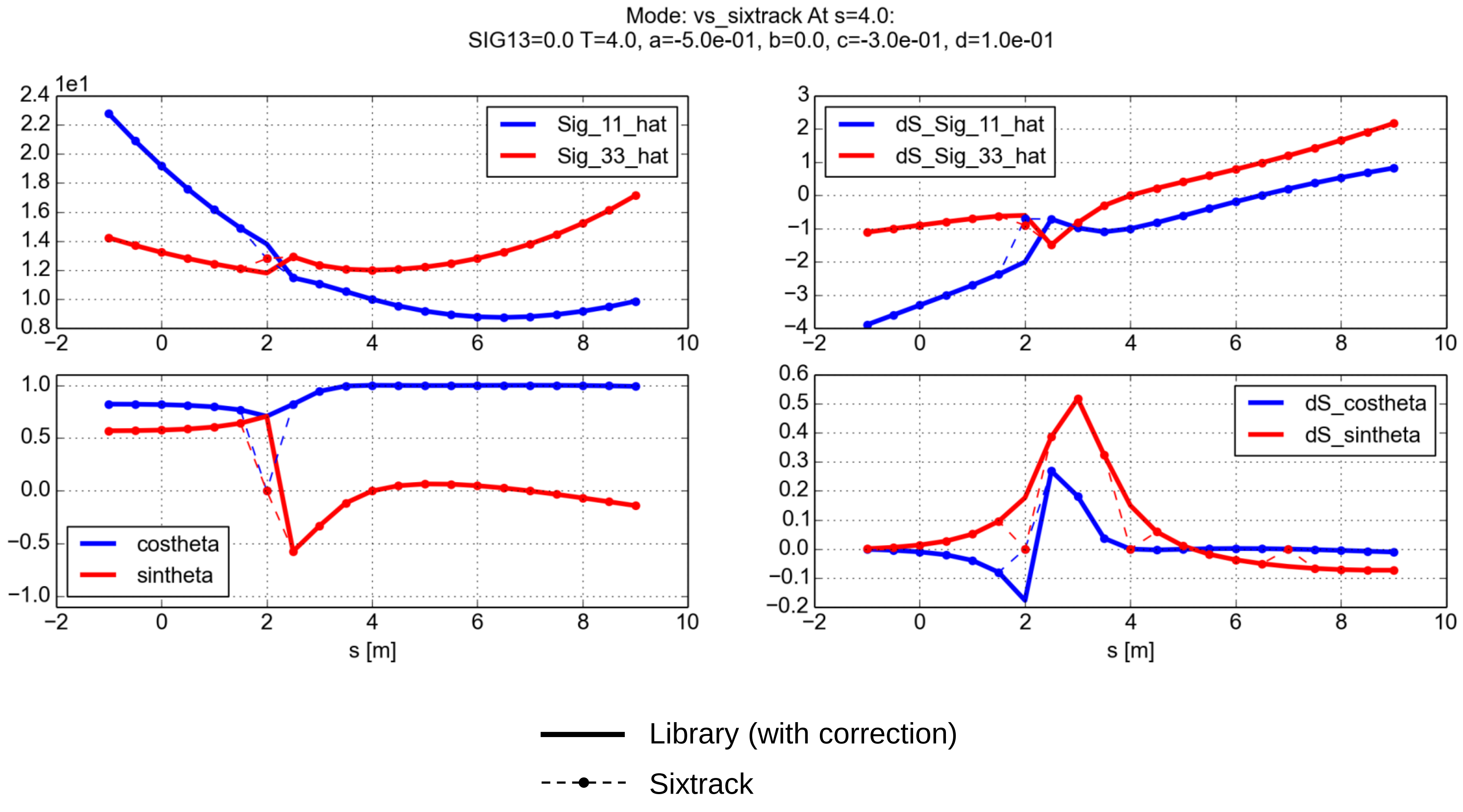
Expression with denominator (apparently singular)

Expression with correction

Handling the denominators: case #1

Case T>0, $\left|\Sigma_{13}^{*}\right|$=0:

Tests against Sixtrack:



Case T=0, $|c|>0$

The highlighted formulas break and **alternative expressions** need to be found:

$$R(S) = \Sigma_{11}^* - \Sigma_{33}^*$$
$$W(S) = \Sigma_{11}^* + \Sigma_{33}^*$$
$$T(S) = R^2 + 4{\Sigma_{13}^*}^2$$

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}_{11}^* = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}_{33}^* = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}[\hat{\Sigma}_{11}^*] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}[\hat{\Sigma}_{33}^*] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

$$\frac{\partial}{\partial S}[\cos 2\theta] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

$$\cos\theta = \sqrt{\frac{1}{2}(1+\cos 2\theta)}$$
$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma_{13}^*)\sqrt{\frac{1}{2}(1-\cos 2\theta)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Case T=0, $|c|$>0

Around the singular point we can write:

$$
\begin{aligned}
a &= \Sigma^*_{12} - \Sigma^*_{34} \\
b &= \Sigma^*_{22} - \Sigma^*_{44} \\
c &= \Sigma^*_{14} + \Sigma^*_{23} \\
d &= \Sigma^*_{24}
\end{aligned}
$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2\left[(2a + b\Delta S)^2 + 4(c + d\Delta S)^2\right]$$

$$\cos 2\theta = \frac{|2a + b\Delta S|}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}}$$

$$\frac{\partial}{\partial S}[\cos 2\theta] = \mathrm{sgn}(2a + b\Delta S)\left[\frac{b}{\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}} - \frac{(2a + b\Delta S)(2ab + b^2\Delta s + 4cd + 4d^2\Delta S)}{\left(\sqrt{(2a + b\Delta S)^2 + 4(c + d\Delta S)^2}\right)^3}\right]$$

$\Delta S = 0$

$$\frac{\partial}{\partial S}[\cos 2\theta] = \mathrm{sgn}(2a)\left[\frac{b}{2\sqrt{a^2 + c^2}} - \frac{a(ab + 2cd)}{2\left(\sqrt{a^2 + c^2}\right)^3}\right]$$

Case T=0, $|c|>0$

$$a = \Sigma^*_{12} - \Sigma^*_{34}$$
$$b = \Sigma^*_{22} - \Sigma^*_{44}$$
$$c = \Sigma^*_{14} + \Sigma^*_{23}$$
$$d = \Sigma^*_{24}$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2 \right]$$

$$\hat{\Sigma}^*_{11} = \frac{W}{2} + \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right) |\Delta S| \sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}$$

$$\hat{\Sigma}^*_{33} = \frac{W}{2} - \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right) |\Delta S| \sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\frac{\partial W}{\partial S} + \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)\mathrm{sgn}(\Delta S)\left[\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} + \frac{\Delta S\,(2ab + b^2\Delta s + 4cd + 4d^2\Delta S)}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}}\right]$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\frac{\partial W}{\partial S} - \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)\mathrm{sgn}(\Delta S)\left[\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} + \frac{\Delta S\,(2ab + b^2\Delta s + 4cd + 4d^2\Delta S)}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}}\right]$$

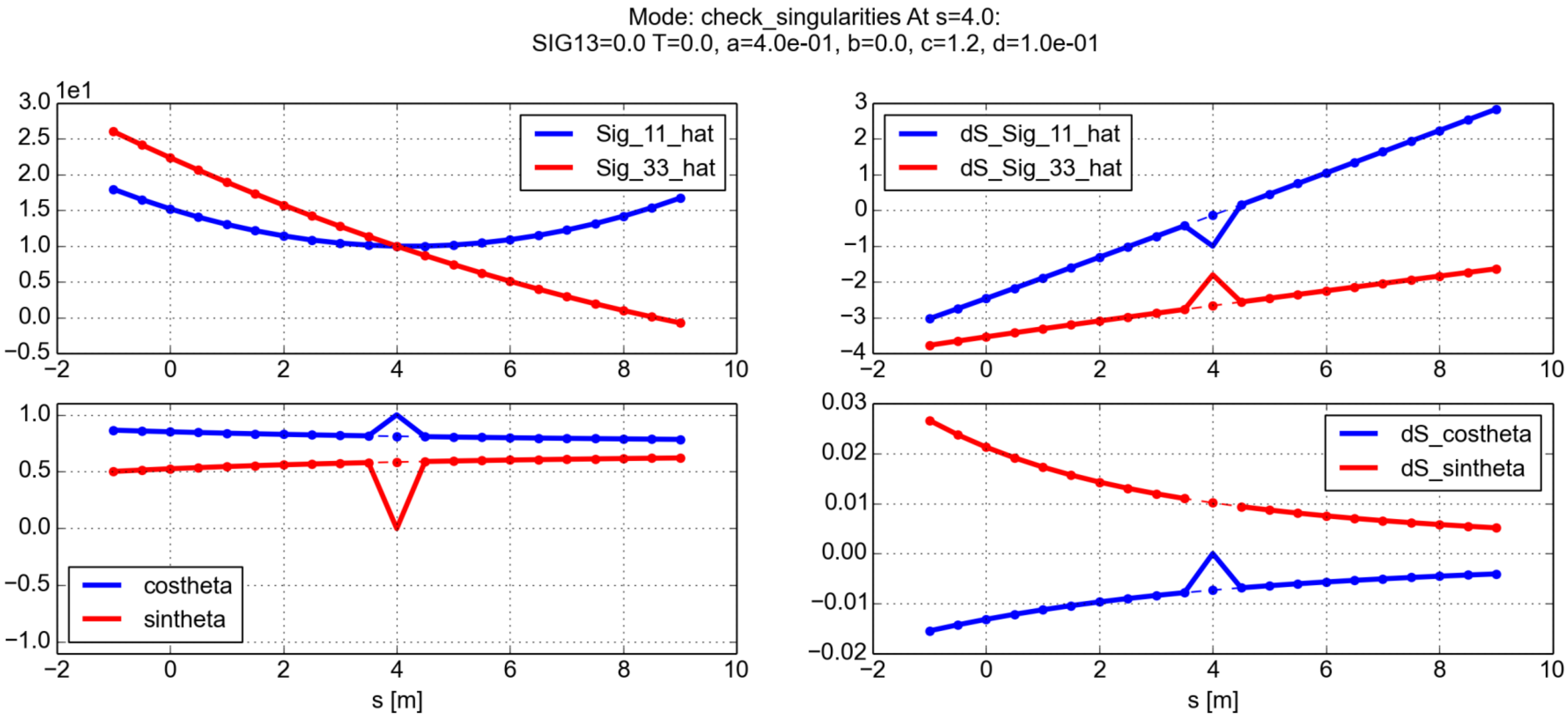
$\Delta S = 0$ ⇒

$$\hat{\Sigma}^*_{11} = \frac{W}{2} \qquad \frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\frac{\partial W}{\partial S} + \mathrm{sgn}(2a)\sqrt{a^2 + c^2}$$

$$\hat{\Sigma}^*_{33} = \frac{W}{2} \qquad \frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\frac{\partial W}{\partial S} - \mathrm{sgn}(2a)\sqrt{a^2 + c^2}$$

Case T=0, $|c|>0$

Tests:

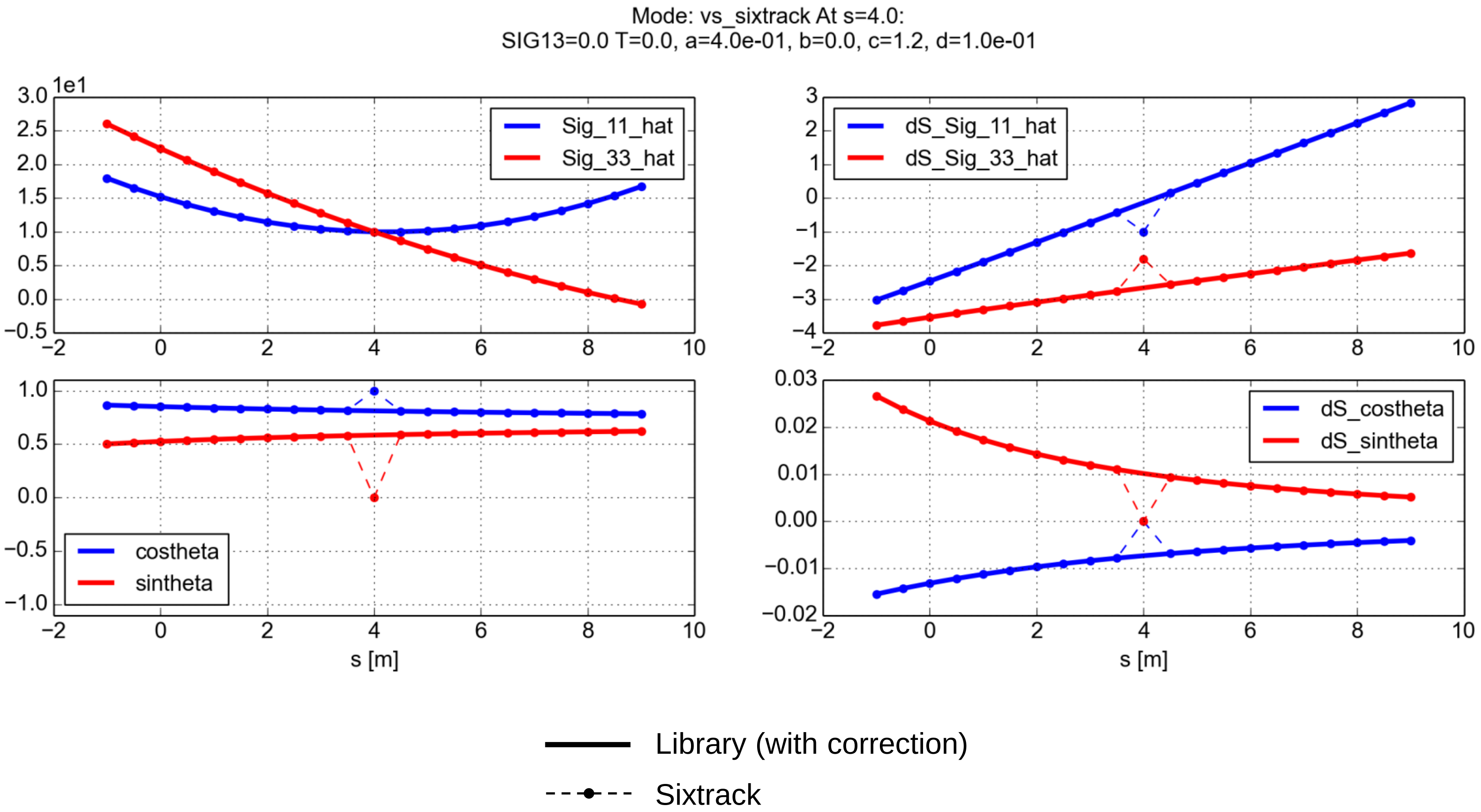


Expression with denominator (apparently singular)

Expression with correction

Case T=0, $|c|>0$

Tests against Sixtrack:



Case T=0, c=0, $|a|$>0

The highlighted formulas break and **alternative expressions** need to be found:

$$R(S) = \Sigma^*_{11} - \Sigma^*_{33}$$
$$W(S) = \Sigma^*_{11} + \Sigma^*_{33}$$
$$T(S) = R^2 + 4\Sigma^{*\,2}_{13}$$

$$\cos 2\theta = \text{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left(W + \text{sgn}(R)\sqrt{T}\right)$$
$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left(W - \text{sgn}(R)\sqrt{T}\right)$$

$$\frac{\partial}{\partial S}[\hat{\Sigma}^*_{11}] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \text{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$
$$\frac{\partial}{\partial S}[\hat{\Sigma}^*_{33}] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \text{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right)$$

$$\frac{\partial}{\partial S}[\cos 2\theta] = \text{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right)$$

$$\cos\theta = \sqrt{\frac{1}{2}(1+\cos 2\theta)}$$
$$\sin\theta = \text{sgn}(R)\text{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}(1-\cos 2\theta)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$
$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Handling the denominators: case #3

Case T=0, c=0, $|a|$>0

$$\begin{aligned} a &= \Sigma_{12}^* - \Sigma_{34}^* \\ b &= \Sigma_{22}^* - \Sigma_{44}^* \\ c &= \Sigma_{14}^* + \Sigma_{23}^* \\ d &= \Sigma_{24}^* \end{aligned}$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2 \right]$$

We proceed as before:

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}} \Rightarrow \cos 2\theta = \frac{|2a + b\Delta S|}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}} \Rightarrow \cos 2\theta = \frac{|2a|}{2\sqrt{a^2 + c^2}}$$

$$\cos\theta = \sqrt{\frac{1}{2}\,(1 + \cos 2\theta)}$$

$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma_{13}^*)\sqrt{\frac{1}{2}\,(1 - \cos 2\theta)}$$

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta$$

$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta$$

Same as before but this denominator becomes zero

Case T=0, c=0, $|a|>0$

$$a = \Sigma^*_{12} - \Sigma^*_{34}$$
$$b = \Sigma^*_{22} - \Sigma^*_{44}$$
$$c = \Sigma^*_{14} + \Sigma^*_{23}$$
$$d = \Sigma^*_{24}$$

$$R = 2a\Delta S + b\Delta S^2$$

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2 \right]$$

We need to **expand to higher order:**

$$\cos 2\theta = \frac{1}{\sqrt{1 + \frac{4d^2\Delta S^2}{(2a+b\Delta S)^2}}} \simeq 1 - \frac{2d^2\Delta S^2}{(2a + b\Delta S)^2}$$

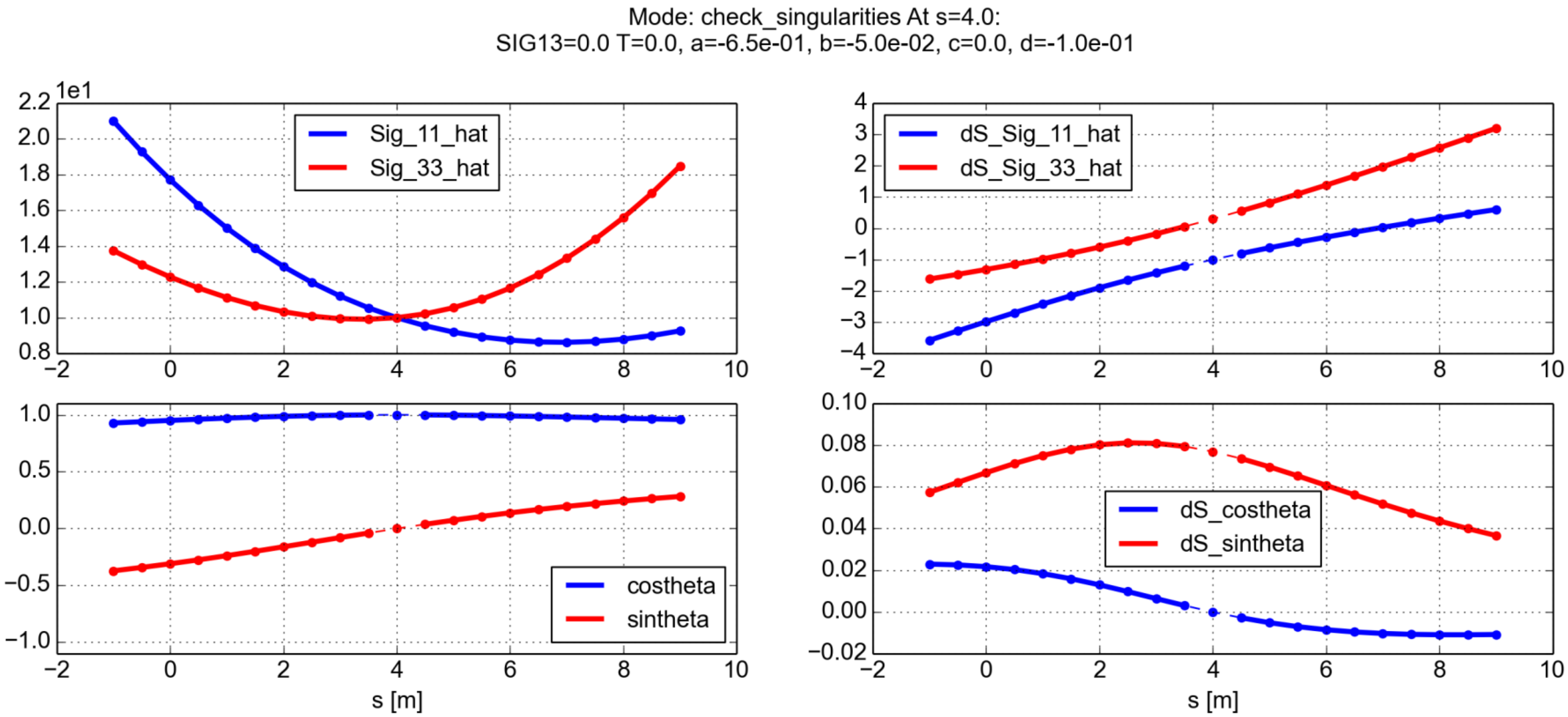
$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1 - \cos 2\theta\right)}$$

$$\sin\theta = \frac{d\Delta S}{2a}\left|1 - \frac{b\Delta S}{2a}\right|$$

$$\frac{\partial}{\partial S}\sin\theta = \frac{d}{2a}$$

Case T=0, c=0, $|a|>0$

Tests:

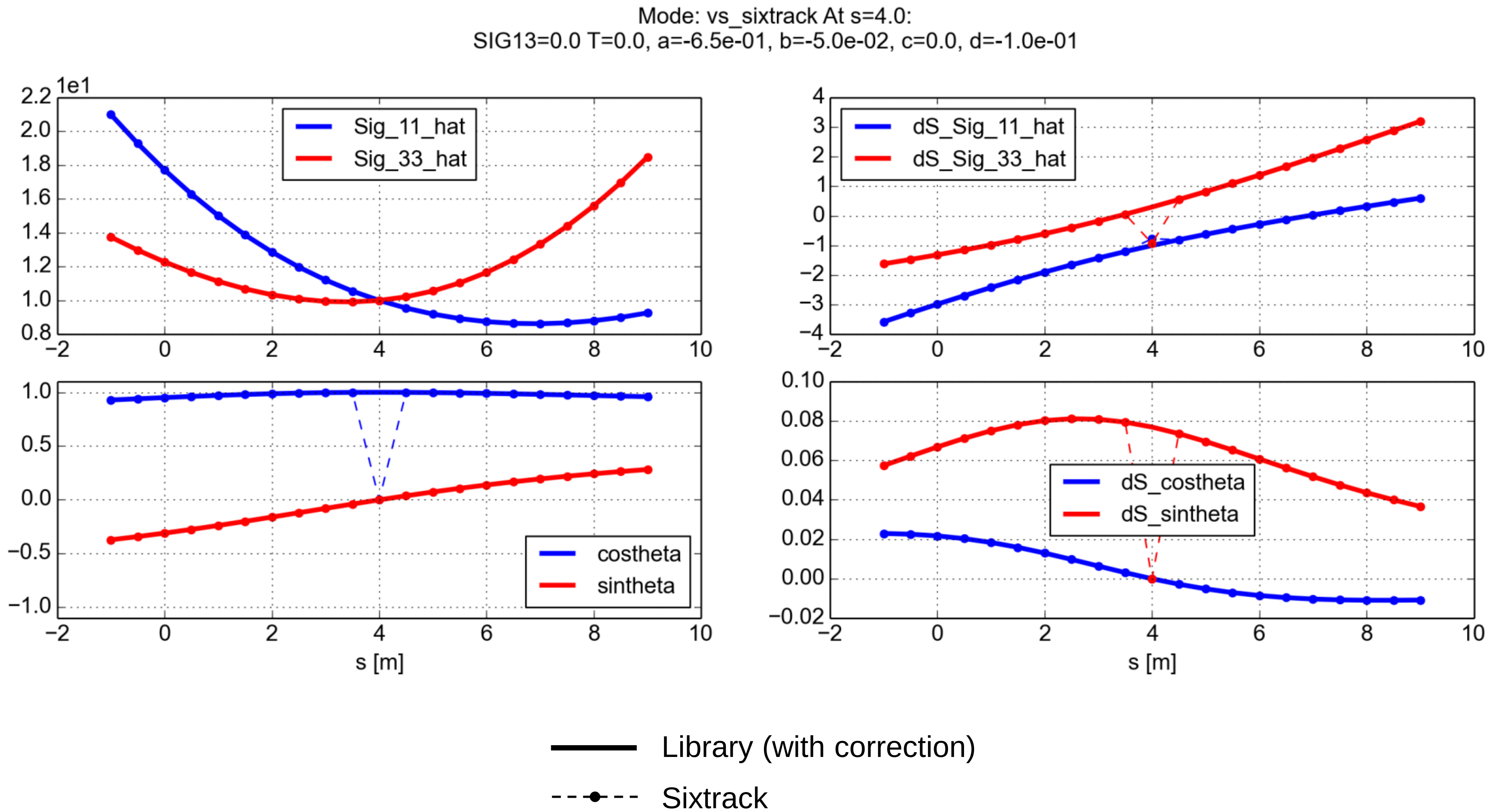


Expression with denominator (apparently singular)

Expression with correction

Case T=0, c=0, $|a|>0$

Tests against Sixtrack:



Case T=0, c=0, a=0

$$a = \Sigma^*_{12} - \Sigma^*_{34}$$
$$b = \Sigma^*_{22} - \Sigma^*_{44}$$
$$c = \Sigma^*_{14} + \Sigma^*_{23}$$
$$d = \Sigma^*_{24}$$

$$R = b\Delta S^2$$
$$\Sigma^*_{13} = d\Delta S^2$$
$$T(S) = R^2 + 4{\Sigma^*_{13}}^2$$

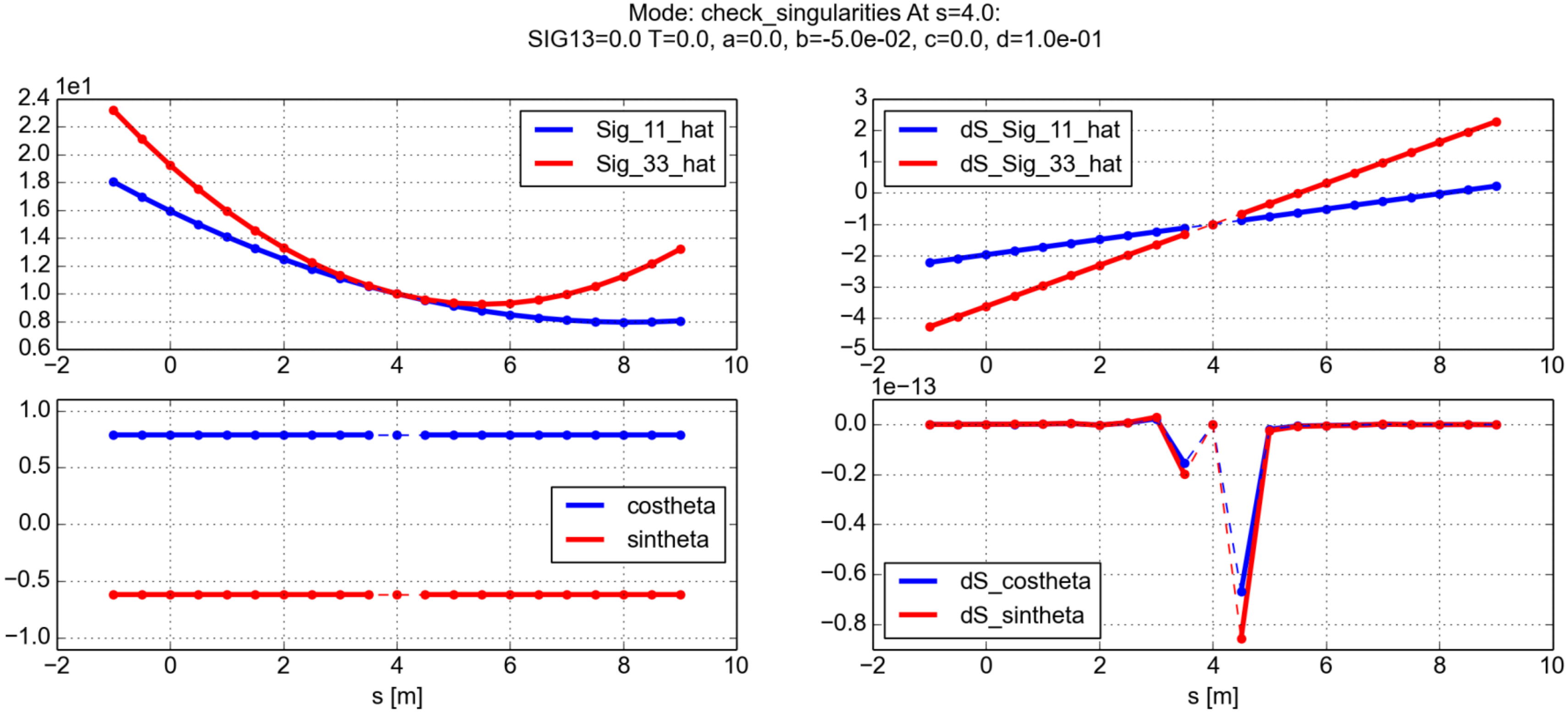
$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}}$$

$$\cos 2\theta = \frac{|b|}{\sqrt{b^2 + 4d^2}}$$

which is a constant…

Case T=0, c=0, a=0

Tests:

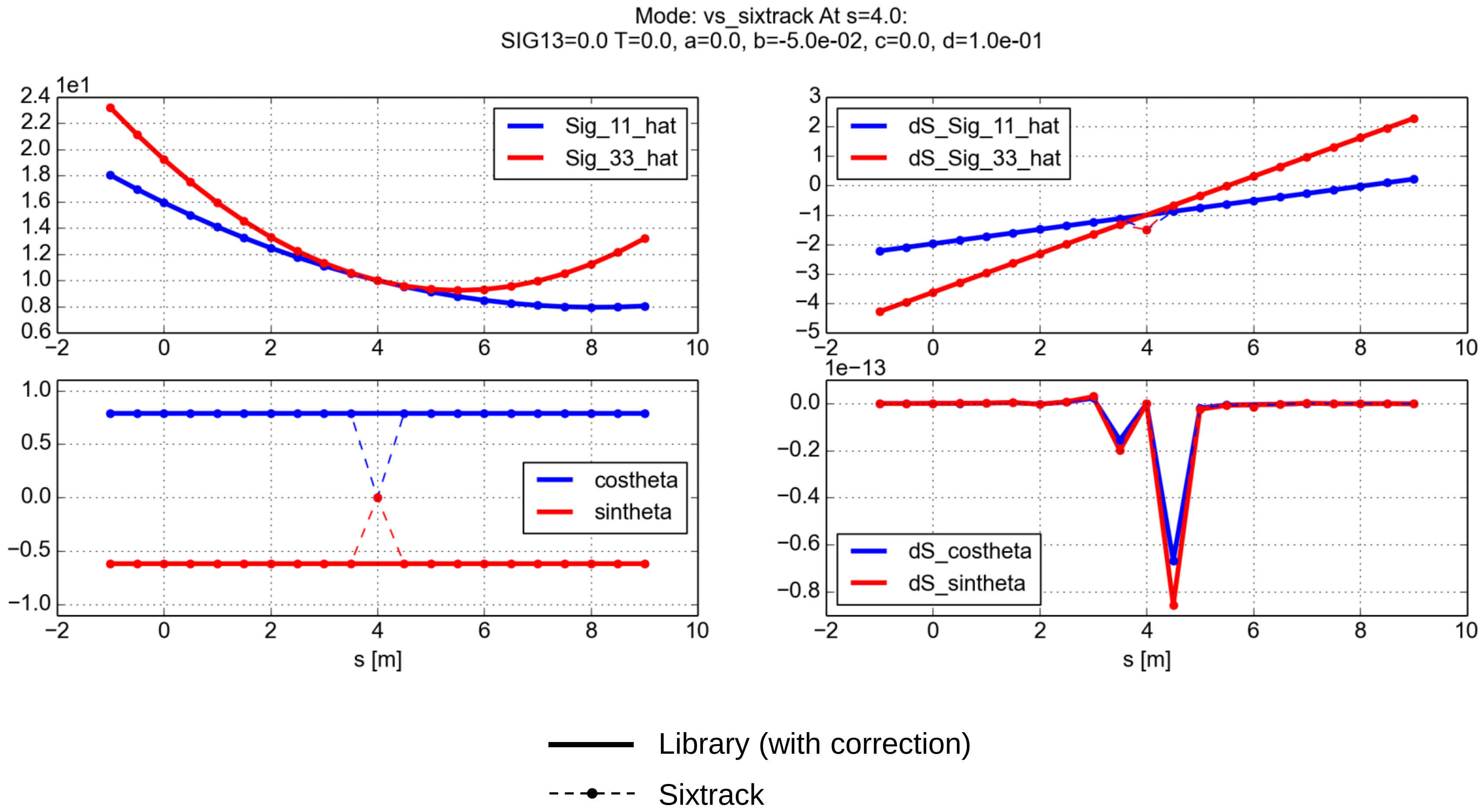


Expression with denominator (apparently singular)

Expression with correction

Case T=0, c=0, a=0

Tests against Sixtrack:



Summary

- Complete **mathematical derivation** needed for implementation available in the prepared note (present version here)
- Implemented in a **Python/C library** for usage in other simulation codes (SixtrackLib, PyHEADTAIL) and compatible with **GPU**
  - **"Stress tests"** performed on the different functional blocks of the library → **Passed**
- **Source code** including all tests available on github
- **SixTrack implementation tested** against library. Outcome:
  - **Uncoupled case:**
    - **Bug identified** in "inverse boost" → **corrected** (now in the production version)
    - **Other tests passed**
  - **Coupled case:**
    - Suffering from a **serious bug** (wrong sign) → **corrected** (now in the production version)
    - **Apparently singular cases** (denominators) not correctly handled → **strategy to be defined** (requires serious re-structuring, should we just replace everything with the library code?)
- **Next steps:**
  - Tests on GPU
  - Performance profiling and, if needed, optimization
  - Real life usage (fancy GPUs in Bologna should be coming soon☺)