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6D beam-beam interaction step-by-step

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Summary

This document describes in detail the numerical method used in different codes for the simulation of beam-beam interactions using the "Synchro Beam Mapping" approach, in order to correctly model the coupling introduced by beam-beam between the longitudinal and the transverse plane. The goal is to provide in a compact, complete and self-consistent manner the set of equations needed for the implementation in a numerical code. The effect of a "crossing angle" in an arbitrary "crossing plane" with respect to the assigned reference frame is taken into account with a suitable coordinate transformation. The employed description of the strong beam allows correctly accounting for the hour-glass effect as well as for linear coupling at the interaction point.

1 Introduction

This document describes in detail the numerical method used in different codes, and in particular in SixTrack [1], for the simulation of beam-beam interactions in the weak-strong framework using the "Synchro Beam Mapping" approach [2]. This allows correctly modeling the coupling introduced by beam-beam between the longitudinal and transverse planes. The goal of this document is in particular to provide in a compact, complete and self-consistent manner, the set of equations that are needed for the implementation in a numerical code. Complementary information can be found in [3], including graphical representations of the procedure presented in this note and several validation tests.

The effect of a "crossing angle" in an arbitrary "crossing plane" with respect to the assigned reference frame is taken into account with a suitable coordinate transformation following the approach described in [4, 5]. The employed description of the strong beam allows the correct inclusion of the hour-glass effect as well as the linear coupling at the interaction point, following the treatment presented in [5].

If not differently stated in an explicit way in the following, all coordinates are given in the reference system defined by the closed orbit of the weak beam, which is traveling with positive speed along the s direction. The Interaction Point (IP) is located at s=0 and the crossing plane is defined by as the angle that the strong beam forms with the s-axis. In the presence of an offset between the beams (separation), the orientation of the reference system is defined by the closed orbit of the weak beam and the system is centered at the IP location as defined for the strong beam. Therefore the strong beam passes always through the origin of the reference frame.

2 Direct Lorentz boost (for the weak beam)

We want to transform the coordinates by moving to a Lorentz boosted frame in which the collision is head-on (i.e. $p_x = p_y = 0$ for the strong beam and for the reference particle of the weak beam). We call ϕ the half crossing angle and α the angle that the crossing plane makes with respect to the $x - z$ plane. For this purpose, we perform a transformation which actually includes four operations (more details can be found in Appendix A1 and in [3, 5]):

- Transform the accelerator positions and momenta into Cartesian coordinates (which can then be Lorentz boosted);
- Rotate particle coordinates to the "barycentric" reference frame;
- Perform the Lorentz boost;

- Drift all the particles back to $s = 0$ (as not all particles with $s = 0$ are fixed points of the transformation, and we are tracking with respect to s and not with respect to time).

We name the original accelerator coordinates (as defined in the SixTrack Physics Manual [6]):

$$(x, p_x, y, p_y, \sigma, \delta) \tag{1}$$

and the transformed coordinates:

$$\left(x^*, p_x^*, y^*, p_y^*, \sigma^*, \delta^*\right) \tag{2}$$

We start by computing the drift Hamiltonian in the original coordinates (we are doing a Lorentz transformation, therefore constants matter as we are assuming that h is the total energy of the particle):

$$h = \delta + 1 - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \tag{3}$$

We transform the momenta:

$$p_x^* = \frac{p_x}{\cos\phi} - h\cos\alpha\frac{\tan\phi}{\cos\phi} \tag{4}$$

$$p_y^* = \frac{p_y}{\cos\phi} - h\sin\alpha\frac{\tan\phi}{\cos\phi} \tag{5}$$

$$\delta^* = \delta - p_x\cos\alpha\tan\phi - p_y\sin\alpha\tan\phi + h\tan^2\phi \tag{6}$$

In order to calculate the angles in the transformed frame, we evaluate:

$$p_z^* = \sqrt{(1+\delta^*)^2 - p_x^{*2} - p_y^{*2}} \tag{7}$$

We can now evaluate the following derivatives of the transformed Hamiltonian (from Hamilton's equations it can be easily seen that these are the angles in the boosted frame):

$$h_x^* = \frac{\partial h^*}{\partial p_x^*} = \frac{p_x^*}{p_z^*} \tag{8}$$

$$h_y^* = \frac{\partial h^*}{\partial p_y^*} = \frac{p_y^*}{p_z^*} \tag{9}$$

$$h_\sigma^* = \frac{\partial h^*}{\partial \delta} = 1 - \frac{\delta^* + 1}{p_z^*} \tag{10}$$

These can be used to build the following matrix:

$$L = \begin{pmatrix} (1 + h_x^* \cos\alpha \sin\phi) & h_x^* \sin\alpha \sin\phi & \cos\alpha \tan\phi \\ h_y^* \cos\alpha \sin\phi & \left(1 + h_y^* \sin\alpha \sin\phi\right) & \sin\alpha \tan\phi \\ h_\sigma^* \cos\alpha \sin\phi & h_\sigma^* \sin\alpha \sin\phi & \frac{1}{\cos\phi} \end{pmatrix} \tag{11}$$

which can then be used to transform the test-particle positions:

$$\begin{pmatrix} x^* \\ y^* \\ \sigma^* \end{pmatrix} = L \begin{pmatrix} x \\ y \\ \sigma \end{pmatrix} \tag{12}$$

3 Syncrho-beam mapping

Following the approach introduced in [2], the strong beam is sliced along z. A common approach is to use constant-charge slices (see Appendix A2). For each particle in the weak beam and for each slice in the strong beam we perform the following.

We identify the position of the Collision Point (CP):

$$S = \frac{\sigma^* - \sigma^*_{\mathrm{sl}}}{2} \tag{13}$$

Here σ^* is defined in the reference system of the weak beam ($\sigma^* > 0$ for particles at the head of the weak bunch) while σ^*_{sl} is defined in the reference system of the strong beam ($\sigma^*_{\mathrm{sl}} > 0$ for particles at the head of the strong bunch). S is the coordinate of the collision point in the reference system of the weak beam (from Eq. 13, we can see that particles at the head of the weak bunch, collide with particles at the tail of the strong bunch at $S > 0$).

N.B. Here we are making an approximation since we are assuming that particles are moving at the speed of light along z independently on their angles. This means that the presented approach works only for small particle angles. It is for this reason that we need to Lorentz boost to get rid of the crossing angle and we cannot just move to the reference of the strong beam using a rotation (in this case the weak beam would have large angles).

We now evaluate the transverse position of the particle at the CP, with respect to the centroid of the slice, taking into account the particle angles :

$$\overline{x}^* = x^* + p_x^* S - x^*_{\mathrm{sl}} \tag{14}$$
$$\overline{y}^* = y^* + p_y^* S - y^*_{\mathrm{sl}} \tag{15}$$

Here x^*_{sl} and y^*_{sl} are defined in the coordinate system of the weak beam.

3.1 Propagation of the strong beam to the collision point

The distribution of the strong beam in the transverse phase-space can be written using the Σ-matrix [7]:

$$f(\eta) = f_0 e^{-\eta^{\mathrm{T}} \Sigma^{-1} \eta} \tag{16}$$

where:

$$\eta = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{17}$$

Points having same phase space density lie on hyper-elliptic manifolds defined by the equation:

$$\eta^{\mathrm{T}} \Sigma^{-1} \eta = \text{const.} \tag{18}$$

Further considerations on the Σ-matrix can be found in Appendix A3.
We transform the Σ-matrix at the Interaction Point to take into account the Lorentz Boost:

$$\begin{align}
\Sigma^{*0}_{11} &= \Sigma^{0}_{11} \tag{19}\\
\Sigma^{*0}_{12} &= \Sigma^{0}_{12} / \cos\phi \tag{20}\\
\Sigma^{*0}_{13} &= \Sigma^{0}_{13} \tag{21}\\
\Sigma^{*0}_{14} &= \Sigma^{0}_{14} / \cos\phi \tag{22}\\
\Sigma^{*0}_{22} &= \Sigma^{0}_{22} / \cos^2\phi \tag{23}\\
\Sigma^{*0}_{23} &= \Sigma^{0}_{23} / \cos\phi \tag{24}\\
\Sigma^{*0}_{24} &= \Sigma^{0}_{24} / \cos^2\phi \tag{25}\\
\Sigma^{*0}_{33} &= \Sigma^{0}_{33} \tag{26}\\
\Sigma^{*0}_{34} &= \Sigma^{0}_{34} / \cos\phi \tag{27}\\
\Sigma^{*0}_{44} &= \Sigma^{0}_{44} / \cos^2\phi \tag{28}\\
& \tag{29}
\end{align}$$

We transport the position part of the boosted Σ-matrix to the CP (here we are taking into account hourglass effect, assuming that we are in a drift space):

$$\Sigma^*_{11} = \Sigma^{*0}_{11} + 2\Sigma^{*0}_{12}S + \Sigma^{*0}_{22}S^2 \tag{30}$$
$$\Sigma^*_{33} = \Sigma^{*0}_{33} + 2\Sigma^{*0}_{34}S + \Sigma^{*0}_{44}S^2 \tag{31}$$
$$\Sigma^*_{13} = \Sigma^{*0}_{13} + \left(\Sigma^{*0}_{14} + \Sigma^{*0}_{23}\right) S + \Sigma^{*0}_{24}S^2 \tag{32}$$

The Σ-matrix is given in the reference system of the weak beam.
For singular cases we will also need to transport the other terms:

$$\Sigma^*_{12} = \Sigma^{*0}_{12} + \Sigma^{*0}_{22}S \tag{33}$$
$$\Sigma^*_{14} = \Sigma^{*0}_{14} + \Sigma^{*0}_{24}S \tag{34}$$
$$\Sigma^*_{22} = \Sigma^{*0}_{22} \tag{35}$$
$$\Sigma^*_{23} = \Sigma^{*0}_{23} + \Sigma^{*0}_{24}S \tag{36}$$
$$\Sigma^*_{24} = \Sigma^{*0}_{24} \tag{37}$$
$$\Sigma^*_{34} = \Sigma^{*0}_{34} + \Sigma^{*0}_{44}S \tag{38}$$
$$\Sigma^*_{44} = \Sigma^{*0}_{44} \tag{39}$$

We introduce the following three auxiliary quantities:

$$R\left(S\right) = \Sigma^*_{11} - \Sigma^*_{33} \tag{40}$$
$$W\left(S\right) = \Sigma^*_{11} + \Sigma^*_{33} \tag{41}$$
$$T\left(S\right) = R^2 + 4{\Sigma^*_{13}}^2 \tag{42}$$

The following derivatives will be needed in the following:

$$\frac{\partial R}{\partial S} = 2\left(\Sigma^0_{12} - \Sigma^0_{34}\right) + 2S\left(\Sigma^0_{22} - \Sigma^0_{44}\right) \tag{43}$$

$$\frac{\partial W}{\partial S} = 2\left(\Sigma^0_{12} + \Sigma^0_{34}\right) + 2S\left(\Sigma^0_{22} + \Sigma^0_{44}\right) \tag{44}$$

$$\frac{\partial \Sigma^*_{13}}{\partial S} = \Sigma^0_{14} + \Sigma^0_{23} + 2\Sigma^0_{24} S \tag{45}$$

$$\frac{\partial T}{\partial S} = 2R\frac{\partial R}{\partial S} + 8\Sigma^*_{13}\frac{\partial \Sigma^*_{13}}{\partial S} \tag{46}$$

We will now compute, at the location of the CP, the coupling angle θ, defining a reference frame in which the beam is decoupled. We will call $\hat{x}$ and $\hat{y}$ the coordinates in the decoupled frame and $\hat{\Sigma}^*_{11}$, $\hat{\Sigma}^*_{33}$ the corresponding squared beam sizes. The angle θ is defined as the angle between the $\hat{x}$-axis and the x-axis.

These quantities can be found by diagonalizing the $x - y$ block of the Σ-matrix. We will make determination choices (Eqs. 49, 52 and 56) so that the set $(\theta, \hat{\Sigma}^*_{11}, \hat{\Sigma}^*_{33})$ is uniquely defined and the coupling angle θ lies in the interval:

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4} \tag{47}$$

Different cases need to be treated separately:

Case T>0, $\left|\Sigma^*_{13}\right|$>0

We evaluate the coupling angle at the position of the CP in the boosted frame:

$$\cos 2\theta = \mathrm{sgn}\left(\Sigma^*_{11} - \Sigma^*_{33}\right)\frac{\Sigma^*_{11} - \Sigma^*_{33}}{\sqrt{\left(\Sigma^*_{11} - \Sigma^*_{33}\right)^2 + 4{\Sigma^*_{13}}^2}} \tag{48}$$

Or more synthetically:

$$\cos 2\theta = \mathrm{sgn}(R)\frac{R}{\sqrt{T}} \tag{49}$$

In the following we will need also the derivative of this quantity:

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(R)\left(\frac{\partial R}{\partial S}\frac{1}{\sqrt{T}} - \frac{R}{2\left(\sqrt{T}\right)^3}\frac{\partial T}{\partial S}\right) \tag{50}$$

It can be proved that [5]:

$$\cos\theta = \sqrt{\frac{1}{2}\left(1+\cos 2\theta\right)} \tag{51}$$

$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\sqrt{\frac{1}{2}\left(1-\cos 2\theta\right)} \tag{52}$$

The corresponding derivatives are given by (see Eq. 2.64 in [5]):

$$\frac{\partial}{\partial S}\cos\theta = \frac{1}{4\cos\theta}\frac{\partial}{\partial S}\cos 2\theta \tag{53}$$

$$\frac{\partial}{\partial S}\sin\theta = -\frac{1}{4\sin\theta}\frac{\partial}{\partial S}\cos 2\theta \tag{54}$$

The squared beam sizes in the rotated (un-coupled) boosted frame are given by:

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left[\left(\Sigma^*_{11}+\Sigma^*_{33}\right) + \mathrm{sgn}\left(\Sigma^*_{11}-\Sigma^*_{33}\right)\sqrt{\left(\Sigma^*_{11}-\Sigma^*_{33}\right)^2 + 4{\Sigma^*_{13}}^2}\right] \tag{55}$$

$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left[\left(\Sigma^*_{11}+\Sigma^*_{33}\right) - \mathrm{sgn}\left(\Sigma^*_{11}-\Sigma^*_{33}\right)\sqrt{\left(\Sigma^*_{11}-\Sigma^*_{33}\right)^2 + 4{\Sigma^*_{13}}^2}\right] \tag{56}$$

Equation 56 can be written in a compact form as:

$$\hat{\Sigma}^*_{11} = \frac{1}{2}\left(W + \mathrm{sgn}(R)\sqrt{T}\right) \tag{57}$$

$$\hat{\Sigma}^*_{33} = \frac{1}{2}\left(W - \mathrm{sgn}(R)\sqrt{T}\right) \tag{58}$$

The corresponding derivatives, which will be needed in the following, are given by:

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} + \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right) \tag{59}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\left(\frac{\partial W}{\partial S} - \mathrm{sgn}(R)\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial S}\right) \tag{60}$$

Case T>0, $|\Sigma^*_{13}|$=0:

The treatment of the previous case is still applicable with the exception of Eq. 54 in which the denominator becomes zero. This happens when Σ^*_{13}=0, which implies $\sqrt{T} = |R|$ and therefore $\cos 2\theta = 1$. The case $T = 0$ will be treated separately later, therefore here we can assume $|R| > 0$. We can expand with respect to Σ^*_{13}/R obtaining:

$$\cos 2\theta = \frac{|R|}{\sqrt{R^2 + 4{\Sigma^*_{13}}^2}} = \frac{1}{\sqrt{1 + 4\frac{{\Sigma^*_{13}}^2}{R^2}}} \simeq \frac{1}{1 + 2\frac{{\Sigma^*_{13}}^2}{R^2}} \simeq 1 - 2\frac{{\Sigma^*_{13}}^2}{R^2} \tag{61}$$

Replacing these result in Eq. 52 we obtain:

$$\sin\theta = \mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13})\frac{|\Sigma^*_{13}|}{|R|} = \frac{\Sigma^*_{13}}{R} \tag{62}$$

We call S_0 the location at which $\Sigma^*_{13} = 0$. At this location we define the auxiliary quantities:

$$c = \Sigma^*_{14} + \Sigma^*_{23} \tag{63}$$

$$d = \Sigma^*_{24} \tag{64}$$

We introduce $\Delta S = S - S_0$ and we can write using Eqs. 30–32:

$$\Sigma^*_{13} = c\Delta S + d\Delta S^2 \tag{65}$$

By taking the derivative of Eq. 62 and using Eq. 65 we obtain:

$$\frac{\partial}{\partial S} \sin\theta = \frac{1}{R^2}\left[(c + 2d\Delta S)\, R \;-\; \frac{\partial R}{\partial S}\left(c\Delta S + d\Delta S^2\right)\right] \tag{66}$$

In the implementation we need only the value for ΔS=0, which is simply given by:

$$\frac{\partial}{\partial S} \sin\theta = \frac{c}{R} \tag{67}$$

Case T=0, $|c|$>0

Special care has to be taken at sections S_0 at which $\Sigma^*_{11} = \Sigma^*_{33}$ and $\Sigma^*_{13} = 0$ as Eqs. 49 and 60 cannot be evaluated directly. Also in this case we define:

$$\Delta S = S - S_0 \tag{68}$$

At the location of the apparent singularity (ΔS=0) we define the auxiliary quantities:

$$\begin{align} a =& \Sigma^*_{12} - \Sigma^*_{34} \tag{69}\\ b =& \Sigma^*_{22} - \Sigma^*_{44} \tag{70}\\ c =& \Sigma^*_{14} + \Sigma^*_{23} \tag{71}\\ d =& \Sigma^*_{24} \tag{72}\end{align}$$

and therefore, using Eqs. 30–32, we can write:

$$R = 2a\Delta S + b\Delta S^2 \tag{73}$$

$$\Sigma^*_{13} = c\Delta S + d\Delta S^2 \tag{74}$$

With these definitions the function T (defined by Eq. 42) can be expanded around $\Delta S = 0$ (using the Eqs. 30, 31, 32):

$$T = \Delta S^2 \left[(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2 \right] \tag{75}$$

Replacing Eq. 75 in Eq. 49 allows removing the apparent singularity:

$$\cos 2\theta = \frac{|2a + b\Delta S|}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}} \tag{76}$$

This can be derived obtaining:

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(2a + b\Delta S)\left[\frac{b}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}} - \frac{(2a + b\Delta S)(2ab + b^2\Delta s + 4cd + 4d^2\Delta S)}{\left(\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}\right)^3} \right] \tag{77}$$

Similarly, replacing Eq. 75 in Eq. 58 we obtain:

$$\hat{\Sigma}^*_{11} = \frac{W}{2} + \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)|\Delta S|\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} \tag{78}$$

$$\hat{\Sigma}^*_{33} = \frac{W}{2} - \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)|\Delta S|\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} \tag{79}$$

This can be derived obtaining:

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\frac{\partial W}{\partial S} + \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)\mathrm{sgn}(\Delta S)\left[\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} + \frac{\Delta S\left(2ab + b^2\Delta s + 4cd + 4d^2\Delta S\right)}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}} \right] \tag{80}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\frac{\partial W}{\partial S} - \frac{1}{2}\mathrm{sgn}\left(2a\Delta S + b\Delta S^2\right)\mathrm{sgn}(\Delta S)\left[\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2} + \frac{\Delta S\left(2ab + b^2\Delta s + 4cd + 4d^2\Delta S\right)}{\sqrt{(2a + b\Delta S)^2 + 4\,(c + d\Delta S)^2}} \right] \tag{81}$$

In the implementation only the values at ΔS=0 are needed. For this case the obtained results above can be simplified as:

$$\cos 2\theta = \frac{|2a|}{2\sqrt{a^2+c^2}} \tag{82}$$

$$\frac{\partial}{\partial S}\left[\cos 2\theta\right] = \mathrm{sgn}(2a)\left[\frac{b}{2\sqrt{a^2+c^2}} - \frac{a(ab+2cd)}{2\left(\sqrt{a^2+c^2}\right)^3}\right] \tag{83}$$

$$\hat{\Sigma}^*_{11} = \frac{W}{2} \tag{84}$$

$$\hat{\Sigma}^*_{33} = \frac{W}{2} \tag{85}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\frac{\partial W}{\partial S} + \mathrm{sgn}(2a)\sqrt{a^2+c^2} \tag{86}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\frac{\partial W}{\partial S} - \mathrm{sgn}(2a)\sqrt{a^2+c^2} \tag{87}$$

Eqs. 52 and 54 can still be used to evaluate $\sin\theta$ and $\cos\theta$ and the corresponding derivatives, once we assume that $\mathrm{sgn}(0) = 1$ and noticing from Eqs. 73 and 74 that for small ΔS:

$$\mathrm{sgn}(R)\mathrm{sgn}(\Sigma^*_{13}) = \mathrm{sgn}(a)\mathrm{sgn}(c) \tag{88}$$

Case T=0, c=0, $|a|$>0

The treatment of the previous case is still applicable with the exception of Eq. 54 in which the denominator becomes zero.

For this case we can write (from Eq. 76) around the point where this condition is verified:

$$\cos 2\theta = \frac{1}{\sqrt{1+\frac{4d^2\Delta S^2}{(2a+b\Delta S)^2}}} \simeq 1 - \frac{2d^2\Delta S^2}{(2a+b\Delta S)^2} \tag{89}$$

We notice from Eqs. 73 and 74 that for small ΔS:

$$\mathrm{sgn}(R)\mathrm{sgn}(\Sigma_{13}^*) = \mathrm{sgn}(a)\mathrm{sgn}(d)\mathrm{sgn}(\Delta S) \tag{90}$$

Replacing Eq. 89 and 90 into in Eq. 52 we obtain:

$$\sin\theta = \frac{d\Delta S}{2a}\left|1 - \frac{b\Delta S}{2a}\right| \tag{91}$$

which can be derived in $\Delta S = 0$ obtaining:

$$\frac{\partial}{\partial S}\sin\theta = \frac{d}{2a} \tag{92}$$

The case in which also $d = 0$ is (or is equivalent to) the uncoupled case as Σ_{13}^* is zero for all S.

Case T=0, c=0, a=0

Around the apparently singular point we can write:

$$R = b\Delta S^2 \tag{93}$$

$$\Sigma_{13}^* = d\Delta S^2 \tag{94}$$

Therefore:

$$T = S^4\left(b^2 + 4d^2\right) \tag{95}$$

and:

$$\cos 2\theta = \frac{|b|}{\sqrt{b^2+4d^2}} \tag{96}$$

which is a constant. Eqs. 52 and 54 can still be used to evaluate $\sin\theta$ and $\cos\theta$ while the corresponding derivatives vanish: This can be derived obtaining:

$$\frac{\partial}{\partial S}\cos\theta = 0 \tag{97}$$

$$\frac{\partial}{\partial S}\sin\theta = 0 \tag{98}$$

Replacing $a = c = 0$ into Eq 79 we obtain:

$$\hat{\Sigma}^*_{11} = \frac{W}{2} + \frac{1}{2}\mathrm{sgn}(b)\Delta S^2\sqrt{b^2 + 4d^2} \tag{99}$$

$$\hat{\Sigma}^*_{33} = \frac{W}{2} - \frac{1}{2}\mathrm{sgn}(b)\Delta S^2\sqrt{b^2 + 4d^2} \tag{100}$$

and:

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{11}\right] = \frac{1}{2}\frac{\partial W}{\partial S} \tag{101}$$

$$\frac{\partial}{\partial S}\left[\hat{\Sigma}^*_{33}\right] = \frac{1}{2}\frac{\partial W}{\partial S} \tag{102}$$

The case in which also $d = 0$ is (or is equivalent to the uncoupled case) as Σ^*_{13} is zero for all S.

3.2 Forces and kicks on weak beam particles

The positions of the weak beam particle in the un-coupled boosted frame are given by:

$$\hat{\overline{x}}^* = \overline{x}^*\cos\theta + \overline{y}^*\sin\theta \tag{103}$$

$$\hat{\overline{y}}^* = -\overline{x}^*\sin\theta + \overline{y}^*\cos\theta \tag{104}$$

In the following we will also need to evaluate:

$$\frac{\partial}{\partial S}\left[\hat{\overline{x}}^*\left(\theta(S)\right)\right] = \overline{x}^* \frac{\partial}{\partial S}\left[\cos\theta\right] + \overline{y}^* \frac{\partial}{\partial S}\left[\sin\theta\right] \tag{105}$$

$$\frac{\partial}{\partial S}\left[\hat{\overline{y}}^*\left(\theta(S)\right)\right] = -\overline{x}^* \frac{\partial}{\partial S}\left[\sin\theta\right] + \overline{y}^* \frac{\partial}{\partial S}\left[\cos\theta\right] \tag{106}$$

In this boosted, rotated and re-centered frame, closed formulas exist to evaluate the following quantities:

$$\hat{F}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\overline{x}}^*}\left(\hat{\overline{x}}^*, \hat{\overline{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \tag{107}$$

$$\hat{F}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\overline{y}}^*}\left(\hat{\overline{x}}^*, \hat{\overline{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \tag{108}$$

$$\hat{G}_x^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*}\left(\hat{\overline{x}}^*, \hat{\overline{y}}^*, \hat{\Sigma}_{11}^*, \hat{\Sigma}_{33}^*\right) \tag{109}$$

$$\hat{G}_y^* = -K_{sl}\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*}\left(\hat{\overline{x}}^*, \hat{\overline{y}}^*, \hat{\Sigma}_{33}^*, \hat{\Sigma}_{33}^*\right) \tag{110}$$

where $\hat{U}^*$ is the electric potential associated to the normalized transverse distribution and:

$$K_{sl} = \frac{N_{sl} q_{sl} q_0}{P_0 c} \tag{111}$$

where N_{sl} is the number of particles in the strong-beam slice, q_{sl} and q_0 are the particle charges for the strong and weak beam respectively, P_0 is the reference momentum of the weak beam.
The minus sign in the Eqs. 107-110 comes from the definition of electric potential, i.e. $E = -\nabla U$.

For a bi-Gaussian beam (elliptic) [2]:

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Im}\left[w\left(\frac{\hat{\bar{x}}^* + i\hat{\bar{y}}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{\bar{x}}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{\bar{y}}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right] \tag{112}$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\epsilon_0\sqrt{2\pi\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\mathrm{Re}\left[w\left(\frac{\hat{\bar{x}}^* + i\hat{\bar{y}}^*}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right) - \exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) w\left(\frac{\hat{\bar{x}}^*\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}} + i\hat{\bar{y}}^*\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}}{\sqrt{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}}\right)\right] \tag{113}$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = -\frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{33}^*}{\hat{\Sigma}_{11}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\} \tag{114}$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left(\hat{\Sigma}_{11}^* - \hat{\Sigma}_{33}^*\right)}\left\{\hat{\bar{x}}^*\hat{E}_x^* + \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\left[\sqrt{\frac{\hat{\Sigma}_{11}^*}{\hat{\Sigma}_{33}^*}}\exp\left(-\frac{(\hat{\bar{x}}^*)^2}{2\hat{\Sigma}_{11}^*} - \frac{(\hat{\bar{y}}^*)^2}{2\hat{\Sigma}_{33}^*}\right) - 1\right]\right\} \tag{115}$$

where w is the Faddeeva function.

For a round beam, i.e. $\hat{\Sigma}_{11}^* = \hat{\Sigma}_{33}^* = \hat{\Sigma}^*$:

$$\hat{f}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\pi\epsilon_0}\left[1 - \exp\left(-\frac{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2}{2\hat{\Sigma}^*}\right)\right]\frac{x}{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2} \tag{116}$$

$$\hat{f}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\bar{x}}^*} = \frac{1}{2\pi\epsilon_0}\left[1 - \exp\left(-\frac{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2}{2\hat{\Sigma}^*}\right)\right]\frac{y}{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2} \tag{117}$$

$$\hat{g}_x^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{11}^*} = \frac{1}{2\left[(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2\right]}\left[\hat{\bar{y}}^*\hat{E}_y^* - \hat{\bar{x}}^*\hat{E}_x^* + \frac{1}{2\pi\epsilon_0}\frac{\left(\hat{\bar{x}}^*\right)^2}{\hat{\Sigma}^*}\exp\left(-\frac{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2}{2\hat{\Sigma}^*}\right)\right] \tag{118}$$

$$\hat{g}_y^* = -\frac{\partial \hat{U}^*}{\partial \hat{\Sigma}_{33}^*} = \frac{1}{2\left[(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2\right]}\left[\hat{\bar{x}}^*\hat{E}_x^* - \hat{\bar{y}}^*\hat{E}_y^* + \frac{1}{2\pi\epsilon_0}\frac{\left(\hat{\bar{y}}^*\right)^2}{\hat{\Sigma}^*}\exp\left(-\frac{(\hat{\bar{x}}^*)^2 + (\hat{\bar{y}}^*)^2}{2\hat{\Sigma}^*}\right)\right] \tag{119}$$

We have used lower-case symbols to indicate that the factor given by Eq. 111 is not yet applied.

The transverse kicks in the coupled (but still boosted) reference frame are given by:

$$F_x^* = \hat{F}_x^* \cos\theta - \hat{F}_y^* \sin\theta \tag{120}$$

$$F_y^* = \hat{F}_x^* \sin\theta + \hat{F}_y^* \cos\theta \tag{121}$$

To compute the longitudinal kick we notice from Eq. 13 that:

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial S} \tag{122}$$

Therefore:

$$F_z^* = \frac{1}{2}\frac{\partial}{\partial S}\left[\hat{U}^*\left(\hat{\bar{x}}^*\left(\theta(S)\right), \hat{\bar{y}}^*\left(\theta(S)\right), \hat{\Sigma}_{11}^*(S), \hat{\Sigma}_{33}^*(S)\right)\right] \tag{123}$$

This can be rewritten as:

$$F_z^* = \frac{1}{2}\left(\hat{F}_x^* \frac{\partial}{\partial S}\left[\hat{\bar{x}}^*\left(\theta(S)\right)\right] + \hat{F}_y^* \frac{\partial}{\partial S}\left[\hat{\bar{y}}^*\left(\theta(S)\right)\right] + \hat{G}_x^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{11}^*(S)\right] + \hat{G}_y^* \frac{\partial}{\partial S}\left[\hat{\Sigma}_{33}^*(S)\right]\right) \tag{124}$$

where all the terms have been evaluated before.
The quantities evaluated so far can be used to compute the effect of the beam-beam interaction on the particles coordinates and momenta [2]:

$$x_{new}^* = x^* - SF_x^* \tag{125}$$
$$p_{x,new}^* = p_x^* + F_x^* \tag{126}$$
$$y_{new}^* = y^* - SF_y^* \tag{127}$$
$$p_{y,new}^* = p_y^* + F_y^* \tag{128}$$
$$z_{new}^* = z^* \tag{129}$$
$$\delta_{new}^* = \delta^* + F_z^* + \frac{1}{2}\left[F_x^*\left(p_x^* + \frac{1}{2}F_x^*\right) + F_y^*\left(p_y^* + \frac{1}{2}F_y^*\right)\right] \tag{130}$$

The physical meaning of the different terms in these equations is illustrated in [3].

4 Inverse Lorentz boost (for the weak beam)

Now we need to go back to the accelerator coordinates by undoing the transformation described in Sec. 2.
As before we evaluate:

$$p_z^* = \sqrt{(1+\delta^*)^2 - p_x^{*2} - p_y^{*2}} \tag{131}$$

and then:

$$h_x^* = \frac{p_x^*}{p_z^*} \tag{132}$$

$$h_y^* = \frac{p_y^*}{p_z^*} \tag{133}$$

$$h_\sigma^* = 1 - \frac{\delta^* + 1}{p_z^*} \tag{134}$$

We invert the matrix (11) using Cramer's rule:

$$\mathrm{Det}\,(L) = \frac{1}{\cos\phi} + \left(h_x^*\cos\alpha + h_y^*\sin\alpha - h_\sigma^*\sin\phi\right)\tan\phi \tag{135}$$

$$L^{\mathrm{inv}} = \frac{1}{\mathrm{Det}\,(L)} \times$$
$$\begin{pmatrix} \left(\frac{1}{\cos\phi} + \sin\alpha\tan\phi\left(h_y^* - h_\sigma^*\sin\alpha\sin\phi\right)\right) & \sin\alpha\tan\phi\,(h_\sigma^*\cos\alpha\sin\phi - h_x^*) & -\tan\phi\left(\cos\alpha - h_x^*\sin^2\alpha\sin\phi + h_y^*\cos\alpha\sin\alpha\sin\phi\right) \\ \cos\alpha\tan\phi\left(-h_y^* + h_\sigma^*\sin\alpha\sin\phi\right) & \left(\frac{1}{\cos\phi} + \cos\alpha\tan\phi\,(h_x^* - h_\sigma^*\cos\alpha\sin\phi)\right) & -\tan\phi\left(\sin\alpha - h_y^*\cos^2\alpha\sin\phi + h_x^*\cos\alpha\sin\alpha\sin\phi\right) \\ -h_\sigma^*\cos\alpha\sin\phi & -h_\sigma^*\sin\alpha\sin\phi & \left(1 + h_x^*\cos\alpha\sin\phi + h_y^*\sin\alpha\sin\phi\right) \end{pmatrix} \tag{136}$$

This can be used to transform the positions:

$$\begin{pmatrix} x \\ y \\ \sigma \end{pmatrix} = L^{\mathrm{inv}} \begin{pmatrix} x^* \\ y^* \\ \sigma^* \end{pmatrix} \tag{137}$$

The Hamiltonian can be transformed with a re-scaling:

$$h = h^* \cos^2\phi = \left(\delta^* + 1 - \sqrt{(1+\delta^*)^2 - p_x^{*2} - p_y^{*2}}\right)\cos^2\phi \tag{138}$$

This can be used to transform the transverse momenta (inverting Eqs. 4 and following):

$$p_x = p_x^* \cos\phi + h\cos\alpha\tan\phi \tag{139}$$
$$p_y = p_y^* \cos\phi + h\sin\alpha\tan\phi \tag{140}$$

The longitudinal momentum can be calculated using directly Eq. 6:

$$\delta = \delta^* + p_x\cos\alpha\tan\phi + p_y\sin\alpha\tan\phi - h\tan^2\phi \tag{141}$$

Appendices

A1 Detailed explanation of "the boost" transformation

The reference frame transformation used in Sec. 2 can be written as [4, 5]:

$$\begin{pmatrix} \sigma^* \\ x^* \\ s^* \\ y^* \end{pmatrix} = A^{-1} {R_{\mathrm{CP}}}^{-1} L_{\mathrm{boost}} R_{\mathrm{CA}} R_{\mathrm{CP}} A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix} \tag{142}$$

Here A is the matrix transforming the accelerator coordinates (Courant-Snyder) to Cartesian coordinates:

$$\begin{pmatrix} ct \\ X \\ Z \\ Y \end{pmatrix} = A \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ x \\ s \\ y \end{pmatrix} \tag{143}$$

R_{CP} is a rotation matrix bringing the crossing plane to the $X - Z$ plane:

$$R_{\mathrm{CP}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \tag{144}$$

R_{CA} is a rotation matrix moving to the barycentric reference frame (in which the two beams are symmetric with respect to the s-axis):

$$R_{\mathrm{CA}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{145}$$

L_{boost} is the matrix defining a Lorentz boost in the direction of the rotated X-axis:

$$L_{\mathrm{boost}} = \begin{pmatrix} 1/\cos\phi & -\tan\phi & 0 & 0 \\ -\tan\phi & 1/\cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{146}$$

The momenta are transformed similarly [5]:

$$\begin{pmatrix} \delta^* \\ p_x^* \\ h^* \\ p_y^* \end{pmatrix} = B^{-1} R_{\mathrm{CP}}{}^{-1} L_{\mathrm{boost}} R_{\mathrm{CA}} R_{\mathrm{CP}} B \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix} \tag{147}$$

where the transformation from accelerator to Cartesian coordinates given by:

$$\begin{pmatrix} E/c - p_0 \\ P_x \\ P_z - p_0 \\ P_y \end{pmatrix} = p_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \\ p_x \\ h \\ p_y \end{pmatrix} \tag{148}$$

As explaned in Sec.2 not all particles with $s = 0$ are fixed points of the transformation, therefore a drift back to s=0 needs to be performed as we are tracking w.r.t. s and not w.r.t. time. The net effect of the transformation is to move from the reference frame of the weak beam to the boosted barycentric frame.

A2 Constant charge slicing

We consider a Gaussian longitudinal bunch distribution:

$$\lambda(z) = \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_z^2}} \tag{149}$$

We introduce the cumulative distribution function:

$$Q(z) = \int_{-\infty}^{z} \lambda(z')dz' = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{z}{\sqrt{2}\sigma_z}\right) \tag{150}$$

We define longitudinal cuts z_n^{cut} such that the bunch is sliced in N sections having the same charge:

$$Q(z_n^{\text{cut}}) = \frac{n}{N} \tag{151}$$

Replacing 151 in 150 we obtain:

$$z_n^{\text{cut}} = \sqrt{2}\sigma_z \text{erf}^{-1}\left(\frac{2n}{N} - 1\right) \tag{152}$$

For each slice we need to find the longitudinal centroid position. For generic slice having edges z_1 and z_2 the centroid position can be written as:

$$z^{centroid} = \frac{1}{Q(z_2) - Q(z_1)} \int_{z_1}^{z_2} z\lambda(z)dz = \frac{\sigma_z}{\sqrt{2\pi}\,(Q(z_2) - Q(z_1))} \left(e^{-\frac{z_1^2}{2\sigma_z^2}} - e^{-\frac{z_2^2}{2\sigma_z^2}}\right) \tag{153}$$

A3 Considerations on the Σ-matrix description

Given the reduced Σ-matrix of the beam (including only position terms, no momenta):

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{13} & \Sigma_{33} \end{pmatrix} \tag{154}$$

the distribution for a Gaussian beam can be written as:

$$\rho(\mathbf{x}) = \rho_0 e^{-\mathbf{x}^{\mathrm{T}}\Sigma^{-1}\mathbf{x}} \tag{155}$$

Points having same density lie con ellipses defined by the equation:

$$\mathbf{x}^{\mathrm{T}}\Sigma^{-1}\mathbf{x} = \text{const.} \tag{156}$$

As Σ is symmetric, it can be diagonalized:

$$\Sigma = \mathbf{V}\mathbf{W}\mathbf{V}^T \tag{157}$$

where the matrix $\mathbf{V}$ has in its columns the eigenvectors of Σ and $\mathbf{W}$ is a diagonal matrix with the corresponding eigenvalues:

$$W = \begin{pmatrix} \hat{\Sigma}_{11} & 0 \\ 0 & \hat{\Sigma}_{33} \end{pmatrix} \tag{158}$$

$\mathbf{V}$ is a unitary matrix (eigenvectors are ortho-normal):

$$\mathbf{V}\mathbf{V}^T = \mathbf{I} \Rightarrow \mathbf{V}^{-1} = \mathbf{V}^T \tag{159}$$

$\mathbf{V}$ can be used to transform coordinates from the initial frame to the de-coupled frame:
where $\hat{\mathbf{x}}$ are the coordinates in the decoupled frame, i.e. the projections of of $\mathbf{x}$ on the eigenvectors:

$$\hat{\mathbf{x}} = \mathbf{V}^T\mathbf{x} \tag{160}$$

Combining Eqs. 157 and 159 we can write:

$$\Sigma^{-1} = \mathbf{V}\mathbf{W}^{-1}\mathbf{V}^{T} \tag{161}$$

This can be replaced in Eq. 156, re-writing the equation of the ellipse as:

$$\mathbf{x}^{\mathrm{T}}\mathbf{V}\mathbf{W}^{-1}\mathbf{V}^{T}\mathbf{x} = \text{const.} \tag{162}$$

Using Eq. 160 we obtain the equation of the ellipse in the reference system of the eigenvectors:

$$\hat{\mathbf{x}}^{\mathrm{T}}\mathbf{W}^{-1}\hat{\mathbf{x}} = \text{const.} \tag{163}$$

which can be rewritten in the familiar form:

$$\frac{\hat{x}^2}{\hat{\Sigma}_{11}} + \frac{\hat{y}^2}{\hat{\Sigma}_{33}} = \text{const.} \tag{164}$$

Once the Σ-matrix is assigned, the one-sigma ellipse can be drawn by the following procedure:

- We diagonalize Σ and we generate an auxiliary matrix defined as:

$$\mathbf{A} = \mathbf{V}\sqrt{\mathbf{W}}\mathbf{V}^{T} \tag{165}$$

- We generate a set of points in the unitary circle

$$\mathbf{z} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \tag{166}$$

- We apply $\mathbf{A}$ to t to generate points on the one-sigma ellipse:

$$\mathbf{x}_{1\sigma} = \mathbf{A}\mathbf{z} \tag{167}$$

This can be verified as follows:

$$\mathbf{x}_{1\sigma}{}^{\mathrm{T}}\Sigma^{-1}\mathbf{x}_{1\sigma} = \mathbf{z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\Sigma^{-1}\mathbf{A}\mathbf{z} = \mathbf{z}^{\mathrm{T}}(\mathbf{V}\sqrt{\mathbf{W}}\mathbf{V})(\mathbf{V}^{T}\mathbf{W}^{-1}\mathbf{V}^{T})(\mathbf{V}\sqrt{\mathbf{W}}\mathbf{V}^{T})\mathbf{z} = \mathbf{z}^{\mathrm{T}}\mathbf{V}\sqrt{\mathbf{W}}\mathbf{W}^{-1}\sqrt{\mathbf{W}}\mathbf{V}^{T}\mathbf{z} = \mathbf{z}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{T}\mathbf{z} = \mathbf{z}^{\mathrm{T}}\mathbf{z} = 1 \tag{168}$$

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