# Statistical Description of Particle Beams

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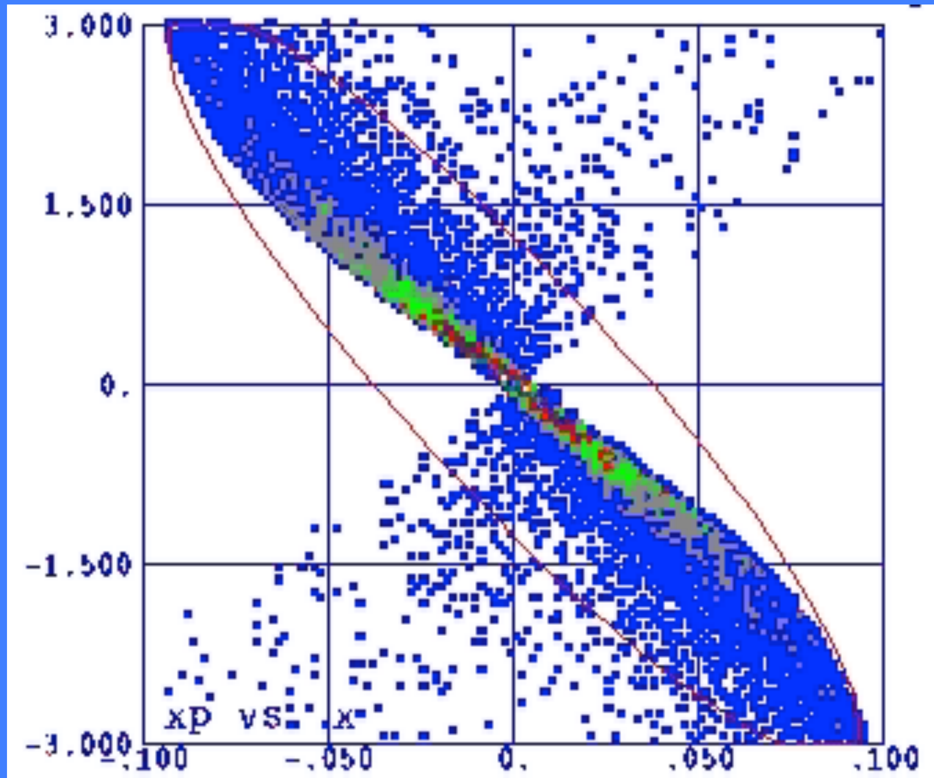
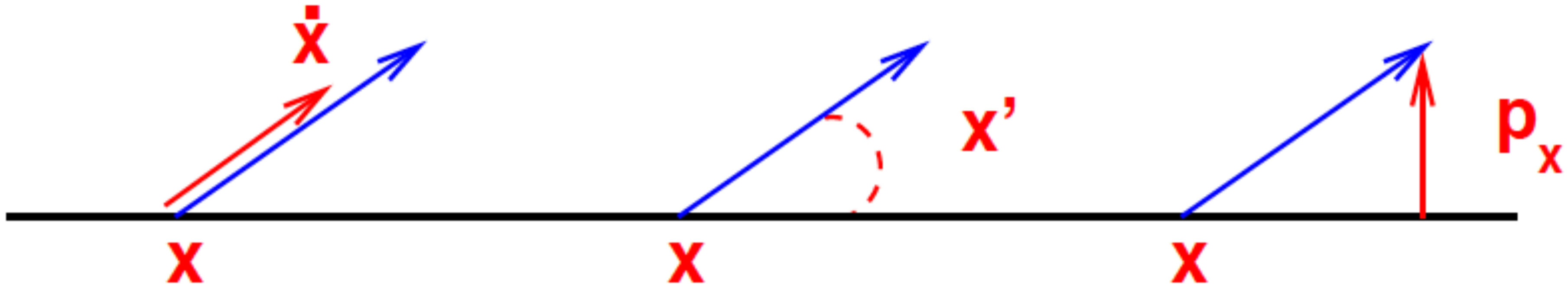



The CERN Accelerator School

Introduction to Accelerator Physics - Constanta – 17 September 2018

# Typical coordinates to describe the particle motion
(6 per particle)

Configuration Space | Trace Space | Phase Space




position velocity | position angle | position momentum

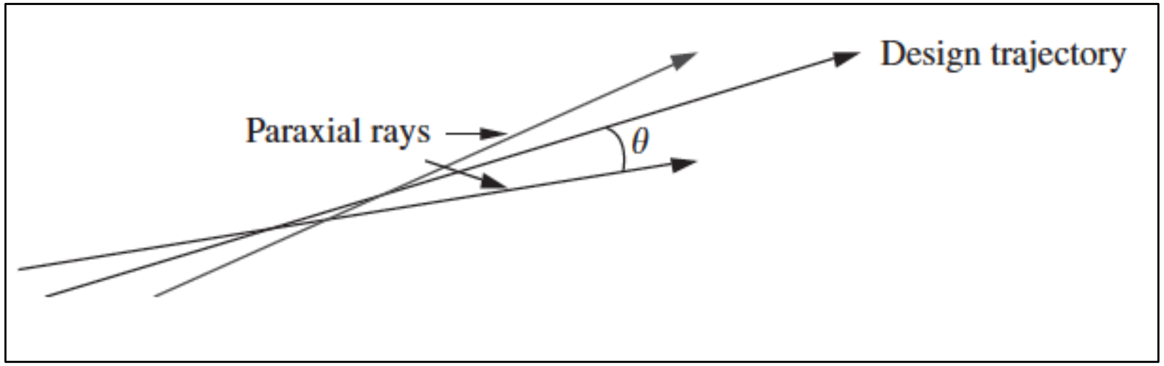
$$\boxed{\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \qquad p_x << p_z \end{cases}}$$

$$\begin{aligned} p_z &= \gamma m_o v_z \\ &= \beta_z \gamma m_o c \end{aligned}$$

# Paraxial Ray Approximation

**paraxial rays** => vector representations of the local trajectory which, by definition, have an angle with respect to a design trajectory that is much smaller than unity.

Trajectories of interest in beam physics are often paraxial: one must confine the beam inside of small, near-axis regions.




In a locally Cartesian coordinate system, we take the distance along the design trajectory to be z. The horizontal offset is designated by x and the horizontal angle is $\theta_x$ .

# Paraxial Ray Approximation

The angle is given in terms of the momenta as:

$$\tan\theta_x = \frac{p_x}{p_z} = \frac{v_x}{v_z} \qquad\qquad p_{x,y} \ll p_z$$

In order to use z as the independent variable, we must be able to write equations of motion in terms of z:

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{v_z}\frac{\mathrm{d}}{\mathrm{d}t}$$

The derivative of a horizontal offset with respect to z is given by:

$$x' = \frac{\mathrm{d}x}{\mathrm{d}z} = \tan(\theta_x) \qquad\qquad \boxed{x' = \tan(\theta_x) \cong \theta_x \cong \sin(\theta_x) \ll 1}$$

# Example

Focusing systems often resemble simple harmonic oscillators, where the transverse force is **linear** in offset:

$$\boxed{F_x = -Kx}$$

And the equation of motion is:

$$\ddot{x} + \omega^2 x = 0 \qquad \omega^2 = K/\gamma m_0$$

in terms of a ray description:

$$\boxed{x'' + k^2 x = 0} \qquad \boxed{k \equiv \omega/v_z}$$

The solutions are of the form:

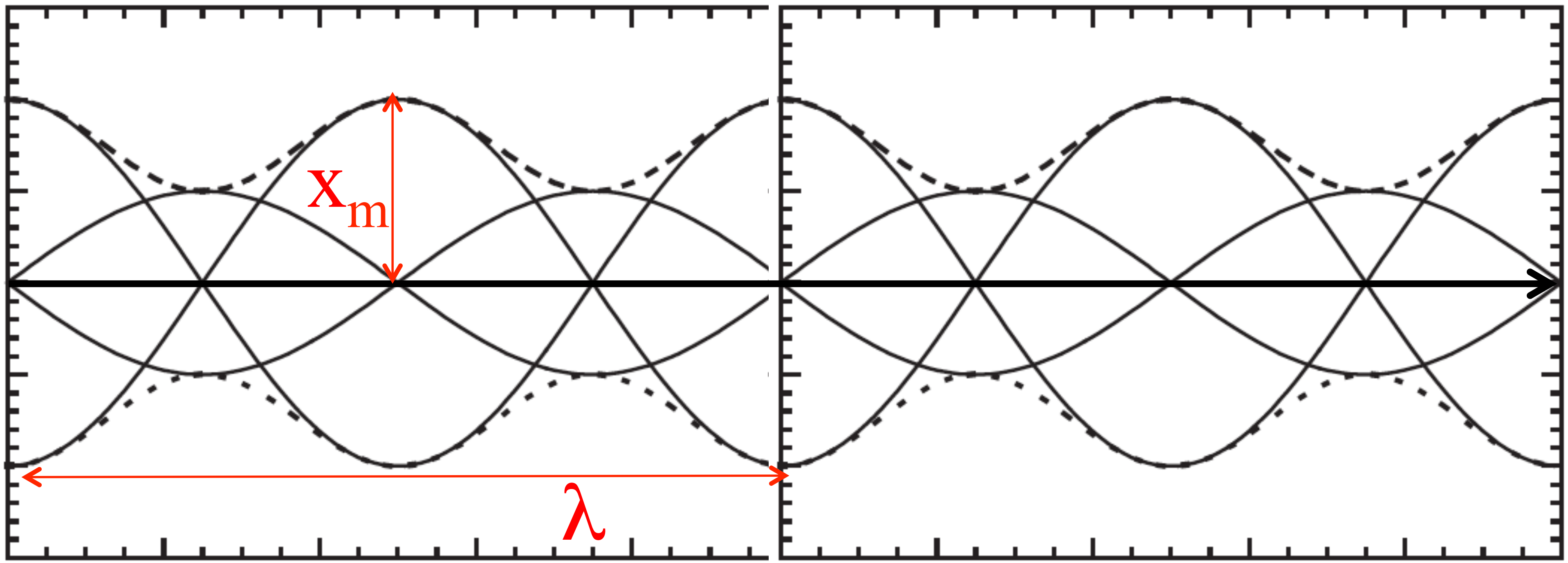
$$x = x_{\mathrm{m}} \cos(kz + \phi)$$

with an angle:

$$\boxed{x' = \left|-kx_m \sin(kz + \phi)\right| << 1 \quad \Rightarrow \quad x_m << \frac{1}{k}}$$

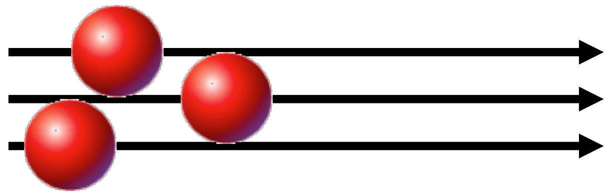
the paraxial approximation is valid for offsets smaller than the characteristic oscillation wavenumber.

$$x = x_{\mathrm{m}} \cos(kz + \phi)$$

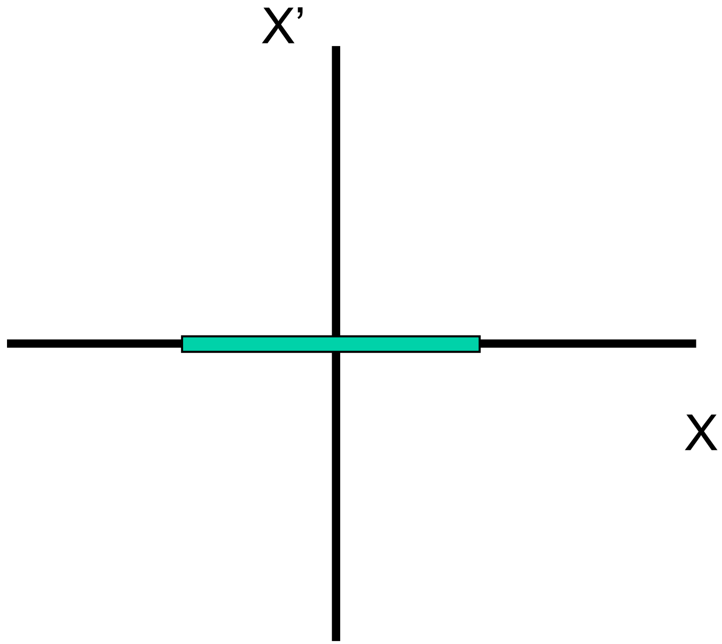



$$\boxed{x_m << \frac{1}{k} = \frac{\lambda}{2\pi}}$$

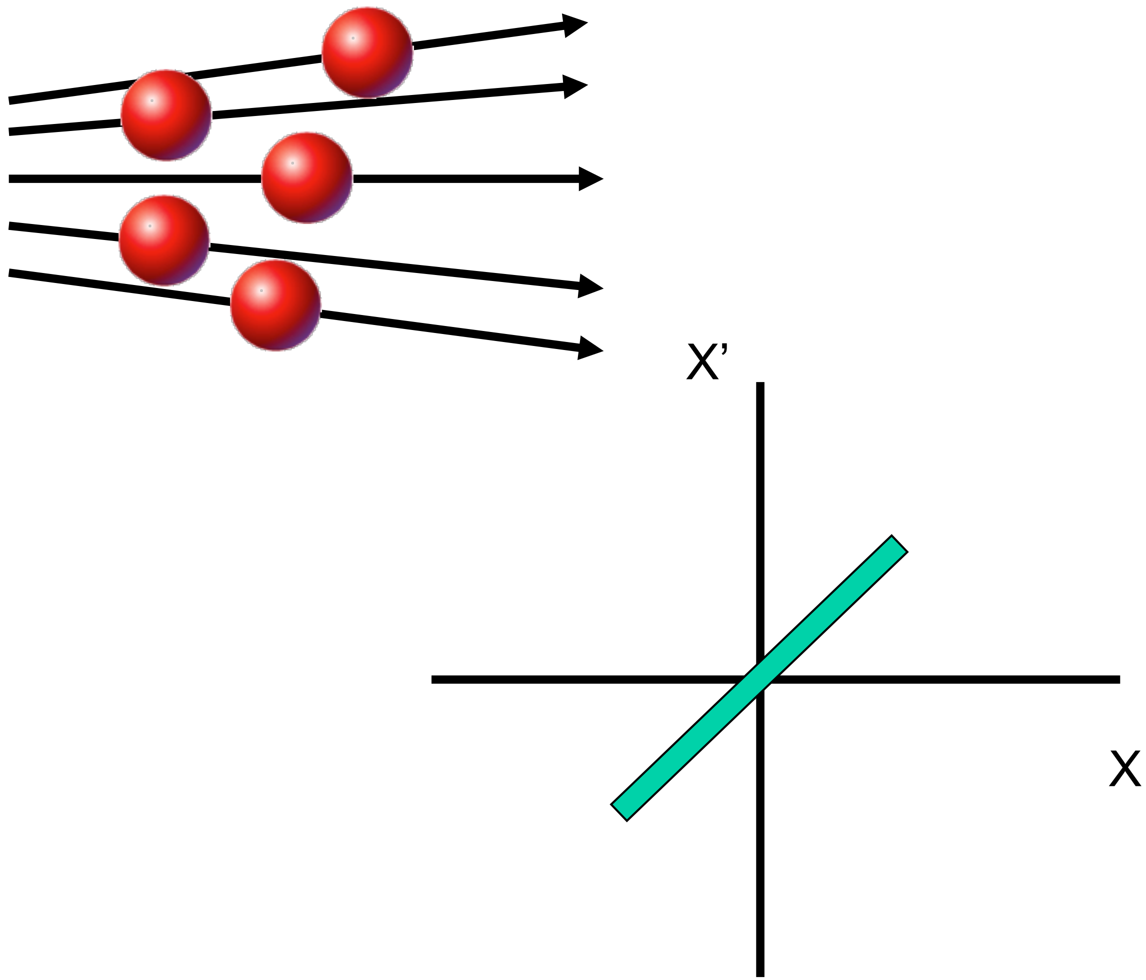
# Trace space of an ideal beam



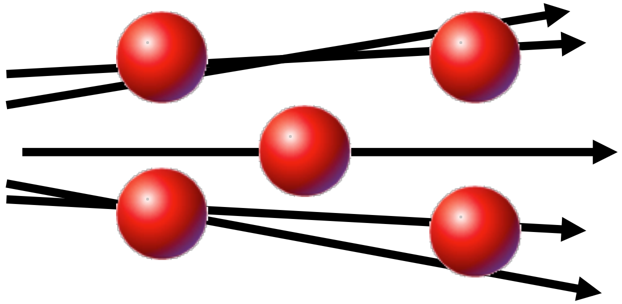
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

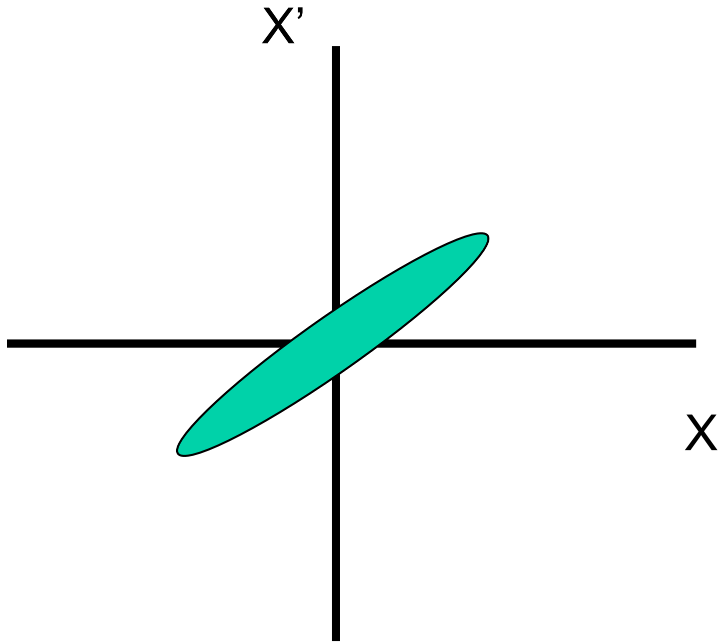


# Trace space of a laminar beam



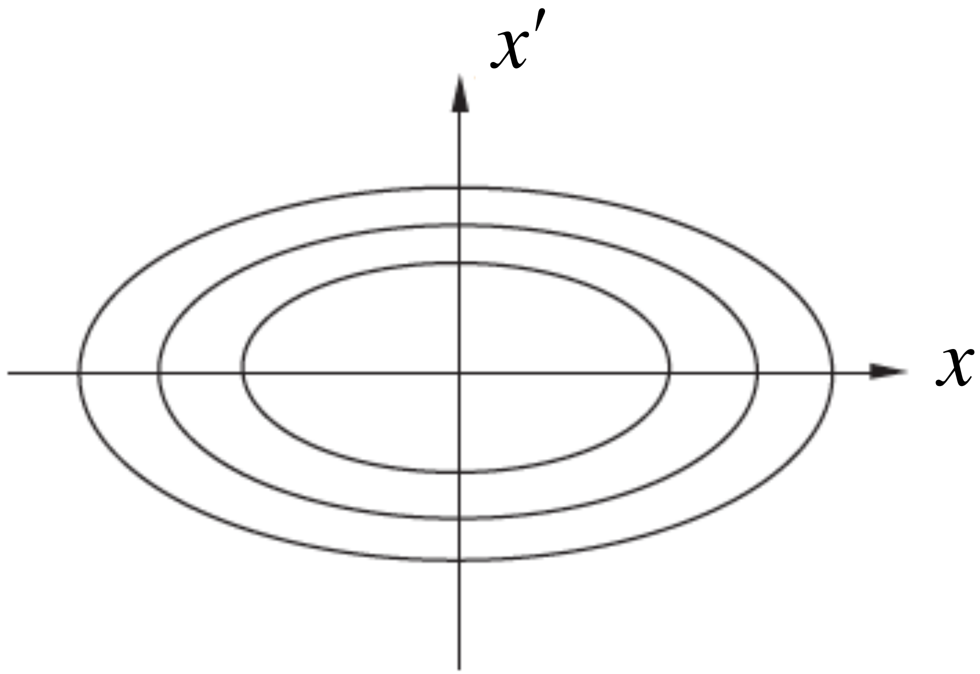
# Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x' trace space.

$$\boxed{\ddot{x} + k^2 x = 0}$$



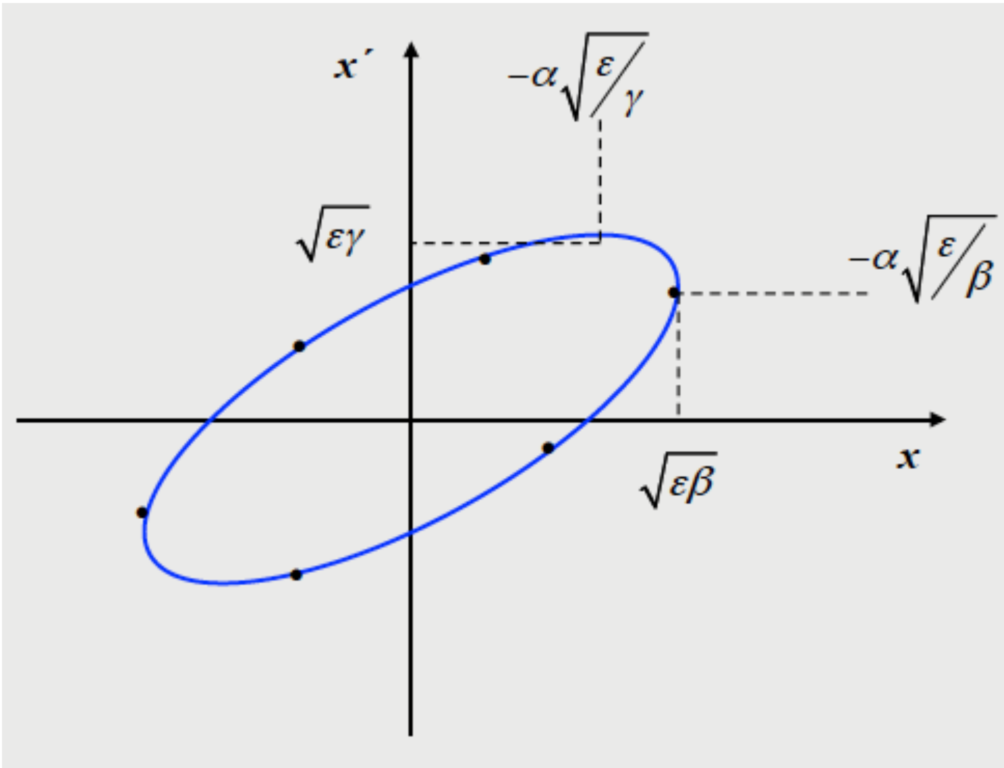

➔ Emittance Concept

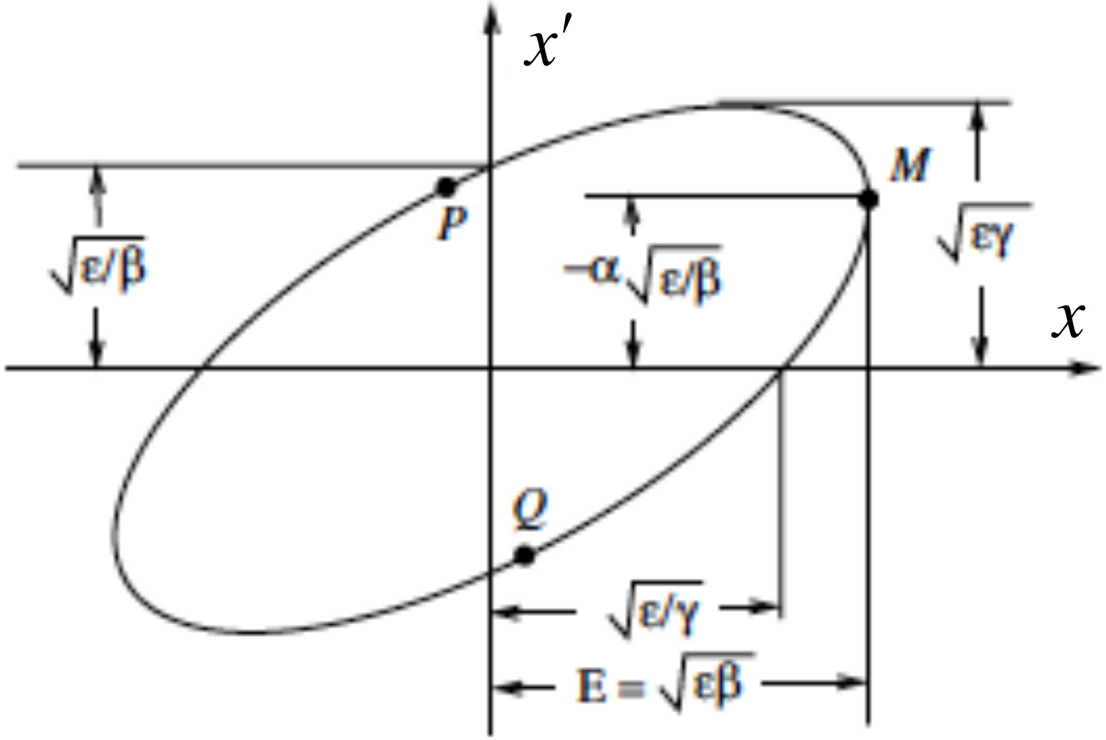
Geometric emittance: $\boxed{\varepsilon_g}$

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$






The slope of the beam envelope at point M: $E' = \frac{\beta'}{2}\sqrt{\frac{\varepsilon}{\beta}}$

is the same as the slope of the particular ray defined by point M: $x'(M) = -\alpha\sqrt{\frac{\varepsilon}{\beta}}$

By equating it follows that: $\beta' = -2\alpha$

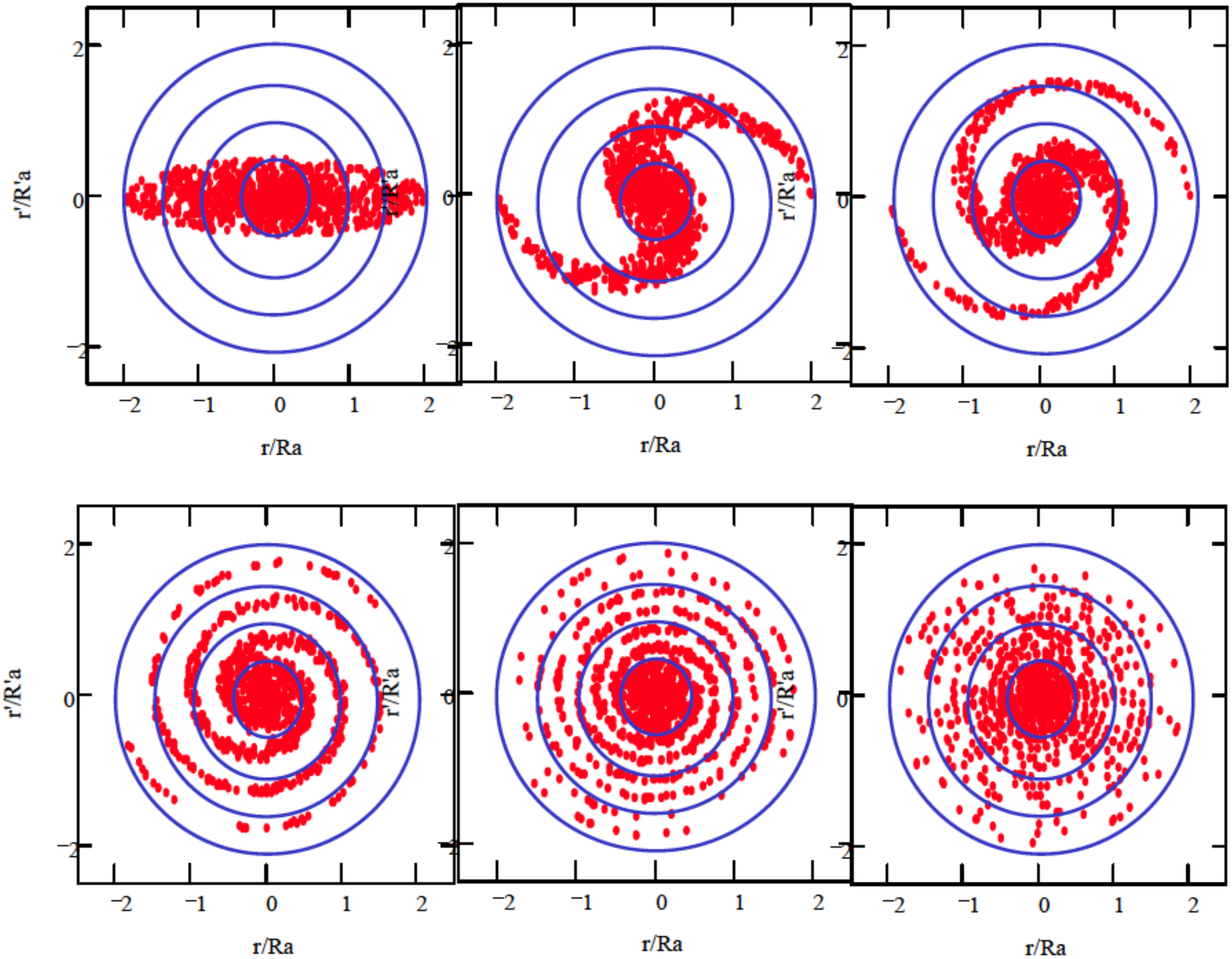
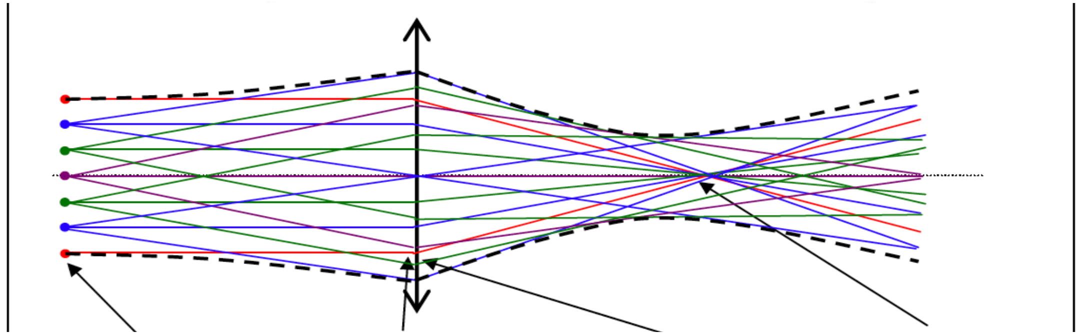
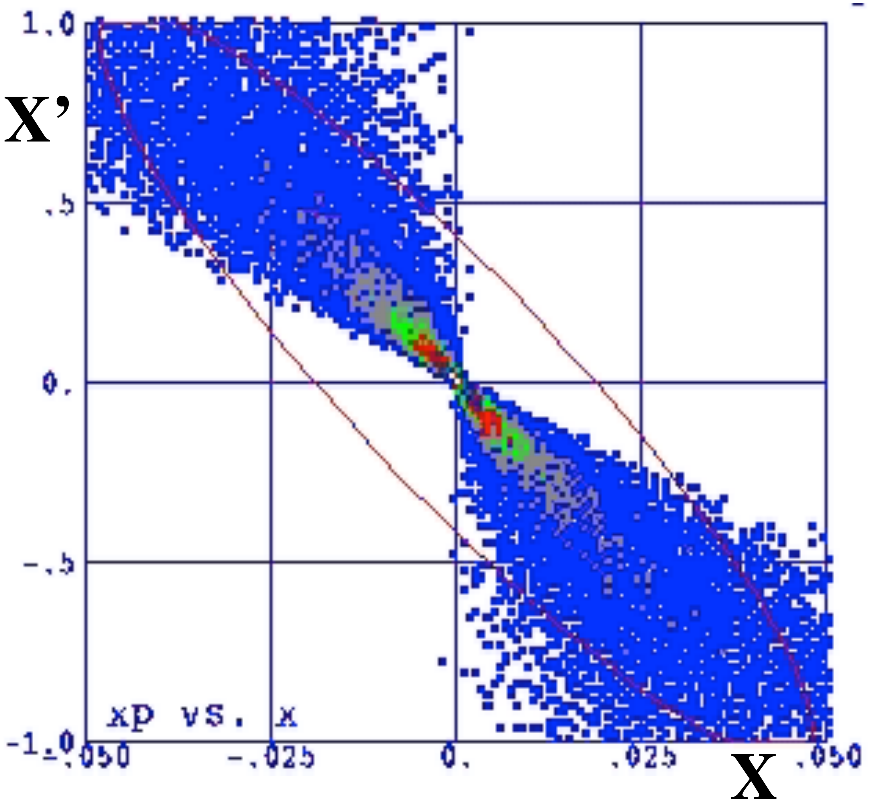



**Fig. 17:** Filamentation of mismatched beam in non-linear force

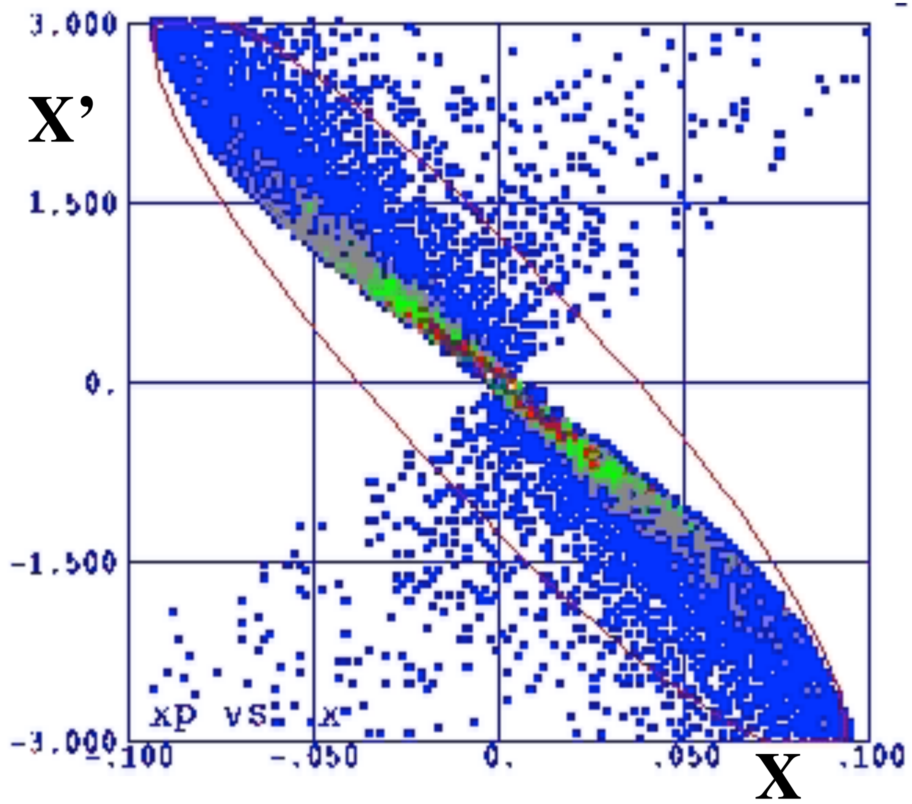
# Trace space evolution



No space charge => cross over

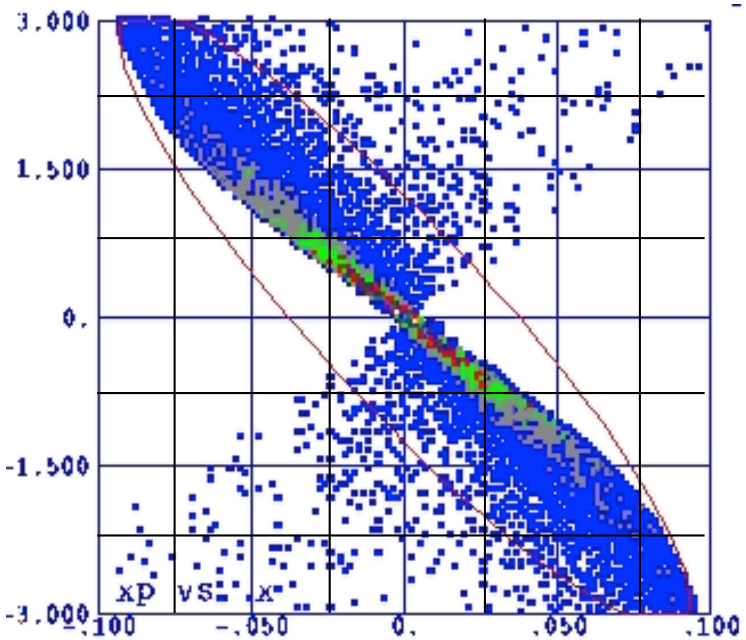



With space charge => no cross over



In charged particle beam dynamics, we are commonly not interested in the phase-space location of individual particles, a statistical mechanics approach is appropriate using function f (x , p , t)

The distribution function f is viewed as a smooth probability function in a 2M – dimensional space f (x , p , t)




The number of particles found in a differential volume dV = $d^3x$ $d^3p$ in the neighborhood of a location x , p at a time t is simply given by f (x , p , t) dV .

The total time derivative of the phase space distribution function is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}p_i}{\mathrm{d}t}\frac{\partial f}{\partial p_i} \right)$$

If the forces are derivable from a Hamiltonian, then

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial f}{\partial p_i} \right)$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_i} \frac{\mathrm{d}f}{\mathrm{d}H} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_i} \frac{\mathrm{d}f}{\mathrm{d}H} \right) = \frac{\partial f}{\partial t}.$$

If neither creation nor destruction of the particles are allowed, we also have:

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = 0.}$$

This result is termed Liouville' s theorem, and it states that the phase space density encountered as one travels with a particle in a Hamiltonian system is conserved => The phase space Volume is also conserved

The full distribution function contains all of the information needed to describe the state of a non-interacting ensemble of beam particles.

One may not need all of this information, though, and so it is often the case that moments of the distribution are used to arrive at a simpler description of the distribution's evolution.

These moments are formally written in the Trace Space as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n (x')^m f_x(x,x')\,\mathrm{d}x\,\mathrm{d}x'.$$

where m and n are equal to zero or a positive integer, and the quantity m + nnis referred to as the order of the moment.

The zeroth-order moment is simply the normalization condition on the distribution,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x,x')\,\mathrm{d}x\,\mathrm{d}x' = 1.$$

The first-order moments are the centroids of the distribution

$$\langle x\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_x(x,x')\,\mathrm{d}x\,\mathrm{d}x',$$

$$\langle x'\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x' f_x(x,x')\,\mathrm{d}x\,\mathrm{d}x',$$

which vanish when a beam is simmetric to its design axis

The second moments are written in standard notation as the distribution variances,

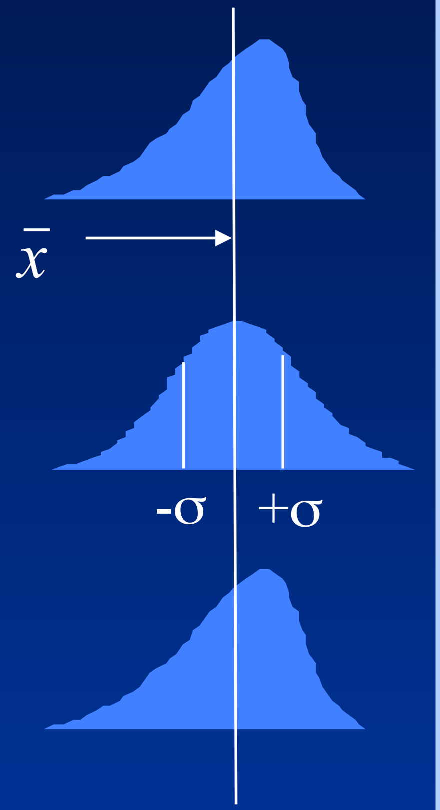
$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

$$\sigma_x'^2 = \langle x'^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x'^2 f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

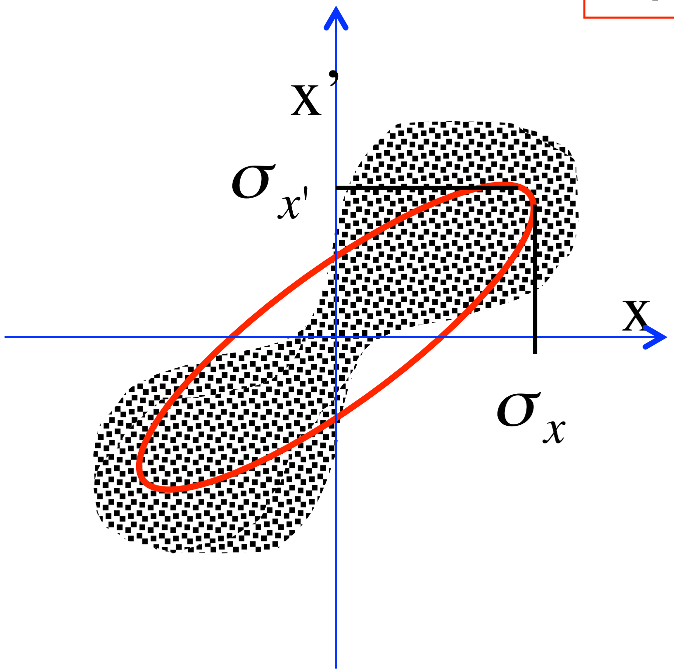
$$\sigma_{xx'} = \langle xx' \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

# Moments of a Distribution

- First Moment:
  - ***mean*** **- measure of location**
- Second Moment:
  - ***standard deviation*** **- measure of spread**
- Third Moment:
  - ***skewness*** **- measure of symmetry**



# rms emittance $\boxed{\varepsilon_{rms}}$




$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1 \qquad \boxed{f'(x,x')=0}$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\frac{\beta'}{2}$

it follows: $\alpha = -\frac{1}{2\varepsilon_{rms}} \frac{d}{dz}\langle x^2 \rangle = -\frac{\langle x x' \rangle}{\varepsilon_{rms}} = -\frac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}} \\ \sigma_{x'} &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}} \\ \sigma_{xx'} &= \langle x x' \rangle = -\alpha \varepsilon_{rms}\end{aligned}}$$

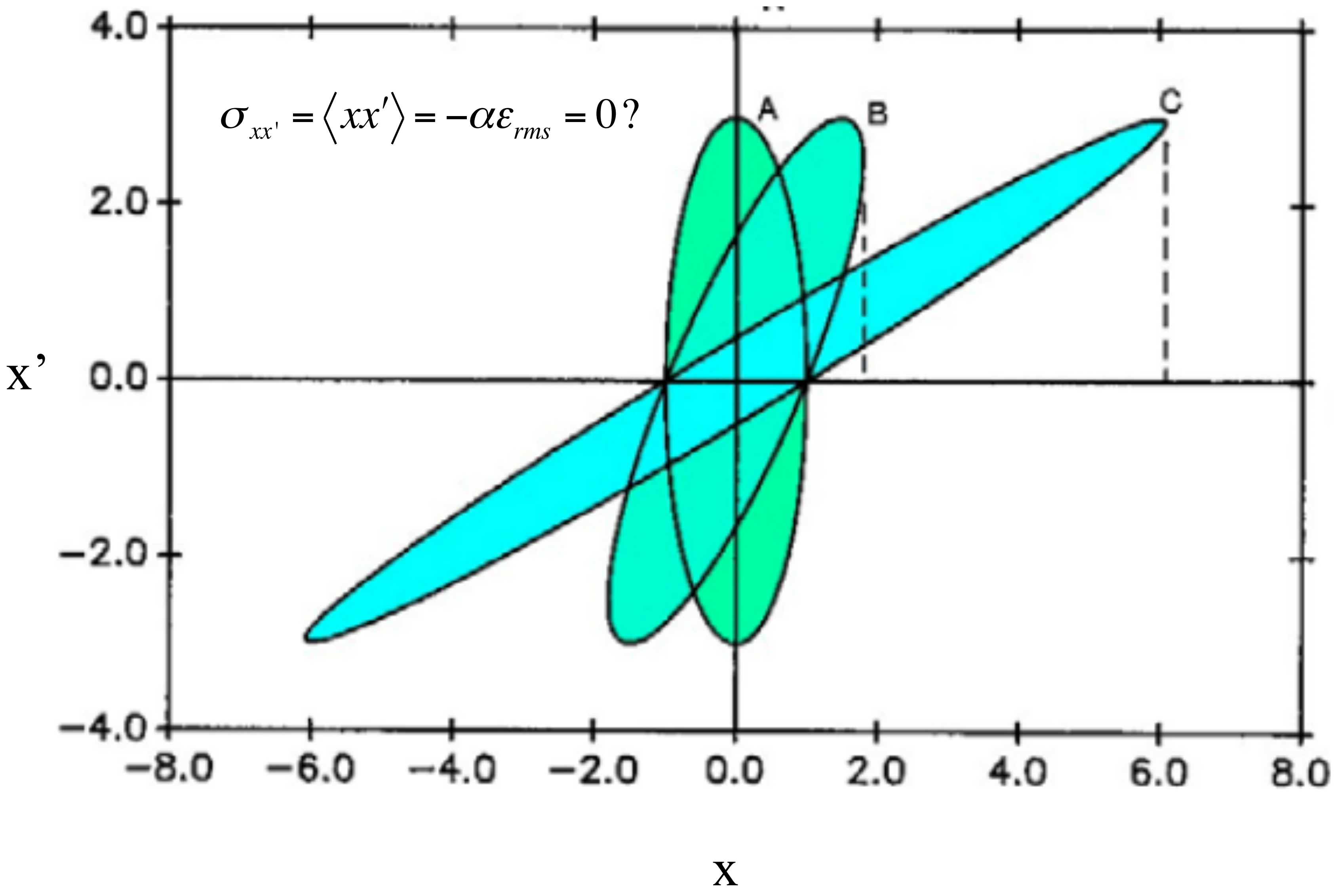
It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting $\alpha, \beta, \gamma$ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

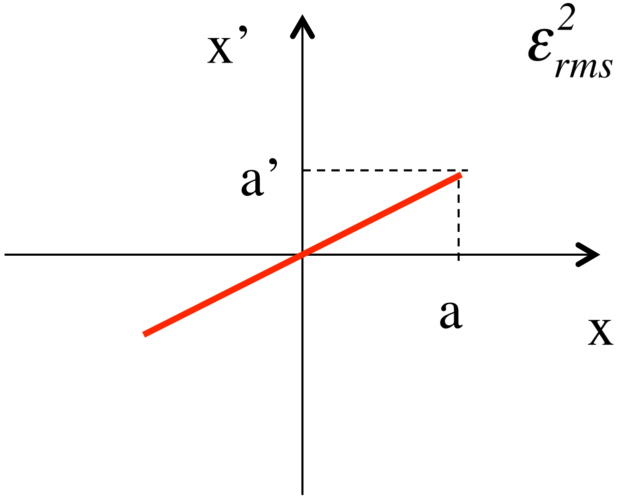
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2\right)}} \qquad x' = \frac{p_x}{p_z}$$

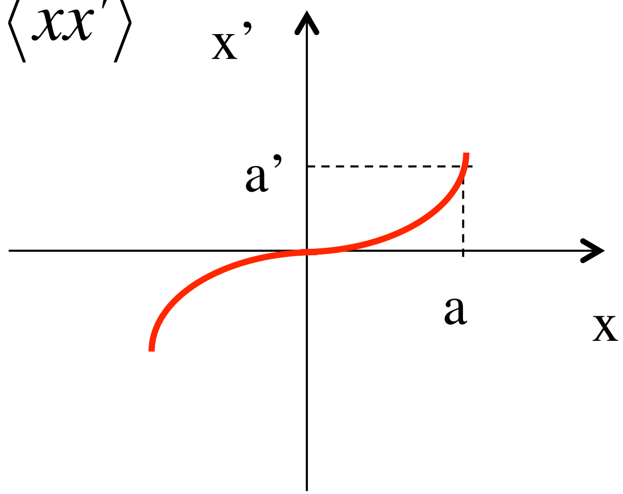
# Which distribution has no correlations?



What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?




$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$




Assuming a generic $x, x'$ correlation of the type: $x' = C x^n$

$$\varepsilon_{rms}^2 = C^2 \left( \langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

# Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 0$$

And without acceleration: $\boxed{x' = \frac{p_x}{p_z}}$

# Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $\qquad p_z \approx p$

$$\boxed{\varepsilon_{n,rms} = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)}}$$

Liouville theorem: the density of particles $n$, or the volume V occupied by a given number of particles in phase space $(x,p_x,y,p_y,z,p_z)$ remains invariant.

$$\boxed{\frac{dn}{dt} = 0}$$

It hold also in the projected phase spaces $(x,p_x),(y,p_y)(,z,p_z)$ provided that there are no couplings

# Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables ($x,p_x$). According to Heisenberg:

$$\boxed{\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of $\hbar^3$ in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \dfrac{1}{2}\dfrac{\hbar}{m_o c} = \dfrac{\lambdabar_c}{2} = 1.9 \times 10^{-13}\, m & \text{quantum limit} \end{cases}$$

Where $\boxed{\lambdabar_c}$ is the reduced Compton wavelength.

## Normalized and un-normalized emittances

$$p_x = p_z x' = m_o c \beta\gamma x'$$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2\right)} = \sqrt{\left(\langle x^2\rangle\langle(\beta\gamma x')^2\rangle - \langle x\beta\gamma x'\rangle^2\right)} = \langle\beta\gamma\rangle\varepsilon_{rms}$$

Assuming **small energy** spread within the beam, the normalized and un-normalized emittances can be related by the above approximated relation.

This approximation that is often used in conventional accelerators may be strongly misleading when adopted to describe beams with significant energy spread, as the one at present produced by plasma accelerators.

When the correlations between the energy and transverse positions are negligible (as in a drift without collective effects) we can write:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \langle x^2 \rangle \langle x'^2 \rangle - \langle \beta\gamma \rangle^2 \langle xx' \rangle^2$$

Considering now the definition of relative energy spread:

$$\sigma_\gamma^2 = \frac{\langle \beta^2\gamma^2 \rangle - \langle \beta\gamma \rangle^2}{\langle \beta\gamma \rangle^2}$$

which can be inserted in the emittance definition to give:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \sigma_\gamma^2 \langle x^2 \rangle \langle x'^2 \rangle + \langle \beta\gamma \rangle^2 \left( \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)$$

Assuming relativistic particles (β=1) we get:

$$\boxed{\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left( \sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)}$$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle} = \sqrt{\beta\varepsilon_{rms}}\\ \sigma'_x &= \sqrt{\langle x'^2\rangle} = \sqrt{\gamma\varepsilon_{rms}}\\ \sigma_{xx'} &= \langle xx'\rangle = -\alpha\varepsilon_{rms}\end{aligned}}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting $\alpha, \beta, \gamma$ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}}\frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)}}$$

$$x' = \frac{p_x}{p_z}$$

# Envelope Equation without Acceleration

Now take the derivatives:

$$\boxed{\frac{d\sigma_x}{dz}} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle = \frac{1}{2\sigma_x}2\left\langle xx' \right\rangle = \boxed{\frac{\sigma_{xx'}}{\sigma_x}}$$

$$\boxed{\frac{d^2\sigma_x}{dz^2}} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\left\langle x'^2 \right\rangle + \left\langle xx' \right\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \boxed{\frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}}$$

And simplify: $\boxed{\sigma_x''} = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \boxed{\frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\left\langle xx'' \right\rangle}{\sigma_x}}$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\boxed{\sigma_x'' - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}} \qquad \boxed{\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P}$$

# Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m\left\langle v_x^2 \right\rangle \qquad T = \frac{1}{3}\left(T_x + T_y + T_z\right) \qquad E_k = \frac{1}{2}m\left\langle v^2 \right\rangle = \frac{3}{2}k_B T$$

Definition of beam temperature in analogy:

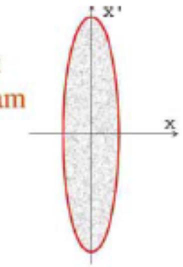
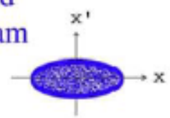
$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle \qquad \left\langle v_x^2 \right\rangle = \beta^2 c^2 \left\langle x'^2 \right\rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

We get:

$$\boxed{k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}}$$

$$\boxed{P_{beam,x} = n k_B T_{beam,x} = n\gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_L \sigma_x^2}}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

| Property | Hot beam | Cold beam |
|---|---|---|
| ion mass ($m_o$) | heavy ion | light ion |
| ion energy ($\beta\gamma$) | high energy | low energy |
| beam emittance ($\varepsilon$) | large emittance | small emittance |
| lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$) | strong focus (low $\beta$ ) | high $\beta$ |
| phase space portrait | hot beam  | cold beam  |

**Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.**

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## EMITTANCE, ENTROPY AND INFORMATION

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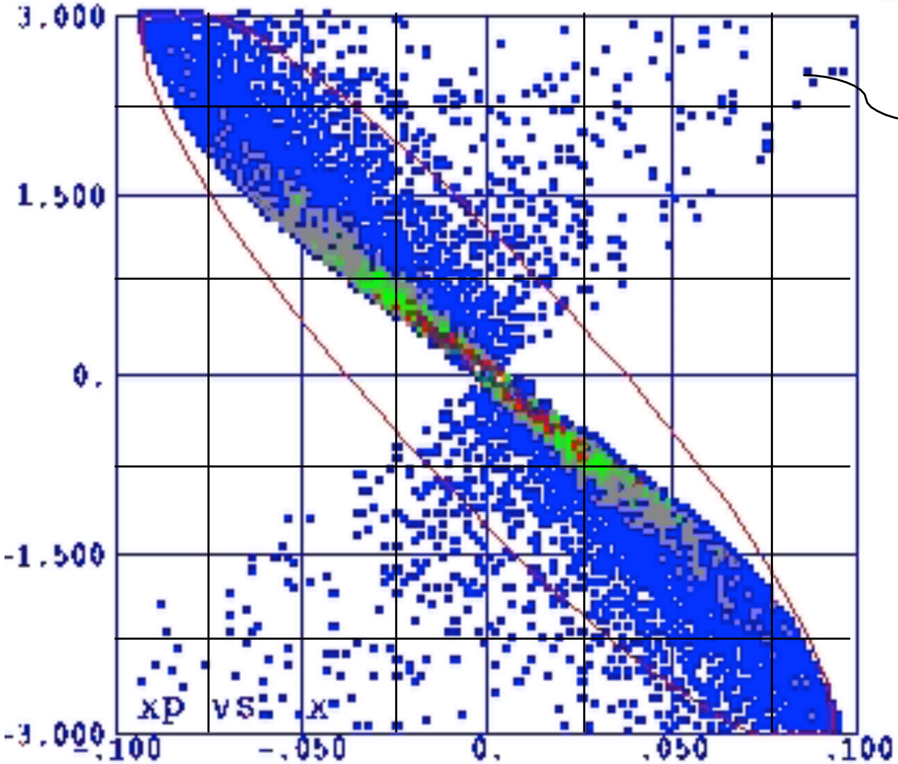
$$S = kN \log(\pi\varepsilon)$$

# Emittance and Entropy




$$A = \delta x \delta x'$$

The entropy of the distribution is by definition:

$$\boxed{S = k \log W}$$

$$W = \frac{N!}{n_1! n_2! \ldots n_M!}$$

is the number of ways in which the points can be assigned to the cells to produce the given distribution

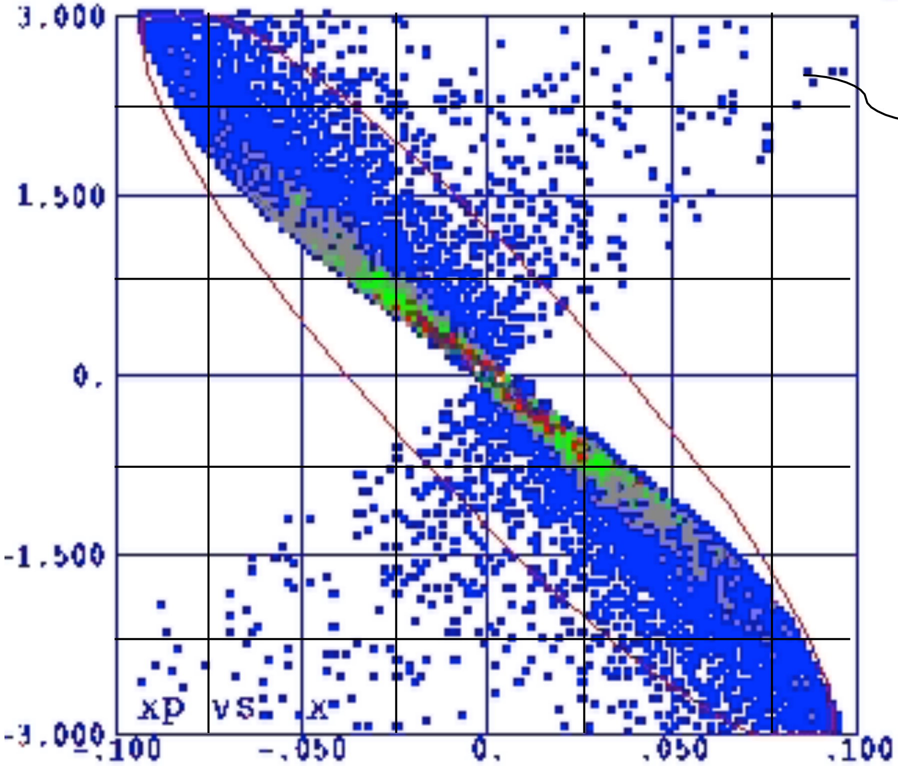
For large N, n, Stirling's formula gives: $\log W = N \log N - \sum_{i=1}^{M} n_i \log n_i.$

If A is sufficiently small, the summation may be replaced by an integral to give:

$$S/kN = S_0 = \log N - \frac{1}{N}\int \rho \log(A\,\rho) \mathrm{d}x\, \mathrm{d}x' \qquad \rho = n/A$$

$$\int \rho\, dx\, dx' = N$$

# Emittance and Entropy




$$A = \delta x \delta x'$$

The entropy of the distribution is by definition:

$$\boxed{S = k \log W}$$

Consider a distribution in which the density is uniform and bounded by an ellipse of area $\pi\varepsilon$ and:

$$\rho = \frac{n}{A} = \frac{N}{\pi\varepsilon} \qquad \int \rho \, dx \, dx' = N$$

$$S/kN = S_0 = \log N - \frac{1}{N}\int \rho \log A \, \rho \, \mathrm{d}x \, \mathrm{d}x'$$

$$\boxed{S_o = \log(N) - \log\left(\frac{AN}{\pi\varepsilon}\right) = \log(\pi\varepsilon) - \log(A)}$$

# Envelope Equation with Linear Focusing

$$\boxed{\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

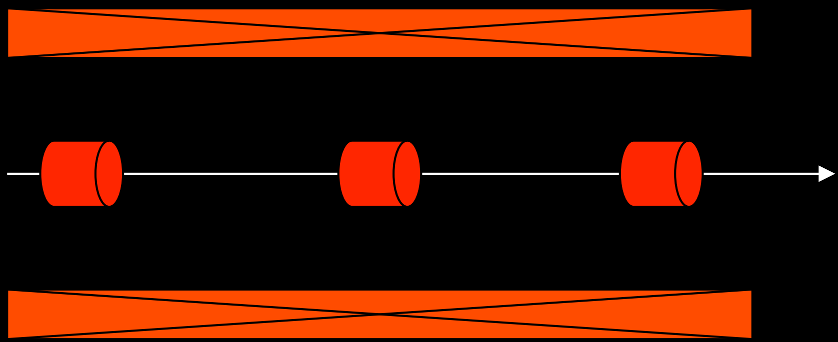
Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\boxed{\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

# Matching Condition:



$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$\sigma_r'' + k_r^2\sigma_r = \frac{\varepsilon_{rms}^2}{\sigma_r^3}$$

Equilibrium solution: $$\sigma_r = \sqrt{\frac{\varepsilon_{rms}}{k_x}}$$ Matched beam

Lets now consider for example the simple case with $\langle xx'' \rangle = 0$ describing a beam drifting in the free space.

The envelope equation reduces to:

$$\boxed{\sigma_x^3 \sigma_x'' = \varepsilon_{rms}^2}$$

With initial conditions $\sigma_o, \sigma_o'$ at $z_o$, depending on the upstream transport channel, the equation has a hyperbolic solution:

$$\sigma(z) = \sqrt{\left(\sigma_o + \sigma_o'(z - z_o)\right)^2 + \frac{\varepsilon_{rms}^2}{\sigma_o^2}(z - z_o)^2}$$

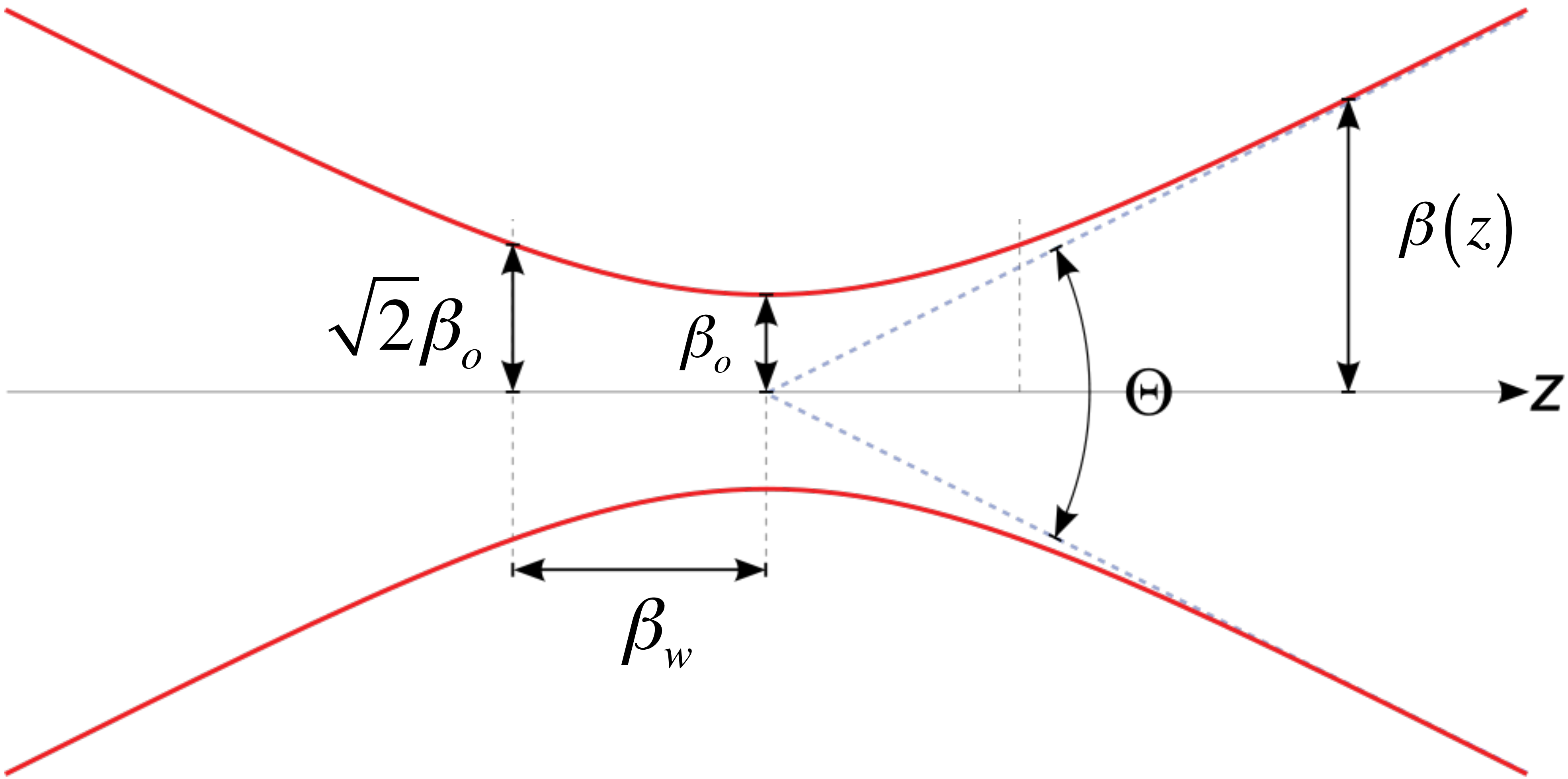
Considering the case $\sigma'_o = 0$ (beam at waist)

and using the definition $\sigma_x = \sqrt{\beta \varepsilon_{rms}}$

the solution is often written in terms of the $\beta$ function as:

$$\sigma(z) = \sigma_o \sqrt{1 + \left(\frac{z - z_o}{\beta_w}\right)^2}$$

This relation indicates that without any external focusing element the beam envelope increases from the beam waist by a factor $\sqrt{2}$ with a characteristic length $\beta_w = \dfrac{\sigma_o^2}{\varepsilon_{rms}}$




For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line.

# Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(px'\right) = 0$$

$$\boxed{p = \beta\gamma m_o c}$$

$$\boxed{x'' = -\frac{(\beta\gamma)'}{\beta\gamma}x'}$$

$$x'' + \frac{p'}{p}x' = 0$$

$$\boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x}}$$

$$\langle xx''\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\langle xx'\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_{xx'} = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_x\sigma_x'$$

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_{rms}}$$

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