CAS – Introduction to Accelerator Physics

Collective effects

Part II: Space charge – wake fields and impedances

In the last lecture, we have learned about the **dynamics and the representation of multiparticle systems**. We have seen how we differentiate between **incoherent and coherent motion**. Linked to this, we looked at the phenomenon of **filamentation with decoherence and emittance blow-up**.

We also discussed a first collective effect – **direct space charge**. We saw that direct space charge leads to an incoherent tune shift and the **characteristic tune footprint**.

We will now look into some of the **mitigation methods** for direct space charge and then discuss some of the effects of **indirect space charge**. We will then move to studying the **concept of wake fields and impedances**.

- Part 2: Direct- and indirect space charge, wake fields and impedances
  - Direct space charge – mitigation techniques
  - Indirect space charge
  - From indirect space charge to (resistive) wall wakes
  - Concept of wake fields

Direct space charge effects have the undesired property of **generating incoherent tune spreads** which can be a problem for dynamic aperture and emittance preservation.

We will look at a few of the different techniques used for **the mitigation of direct space charge** effects.

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Direct space charge tune shift

After some reshuffling we notice some of the **fundamental properties of the direct space charge tune shift**:

$$\left.\begin{aligned} \Delta Q_{x,y} &= -\frac{r_0 R\lambda}{e\beta^2\gamma^3}\left\langle \frac{\beta_{x,y}(s)}{a^2(s)} \right\rangle \\ a(s) &= \sqrt{\frac{\beta_{x,y}(s)\hat{\varepsilon}^n_{x,y}}{\beta\gamma}} \end{aligned}\right\} \Longrightarrow \boxed{\Delta Q_{x,y} = -\frac{r_0 R\lambda}{e\beta\gamma^2\hat{\varepsilon}^n_{x,y}}}$$

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} = \begin{cases} 1.54 \cdot 10^{-18}\ \mathrm{m\ (proton)} \\ 2.82 \cdot 10^{-15}\ \mathrm{m\ (electron)} \end{cases}$$

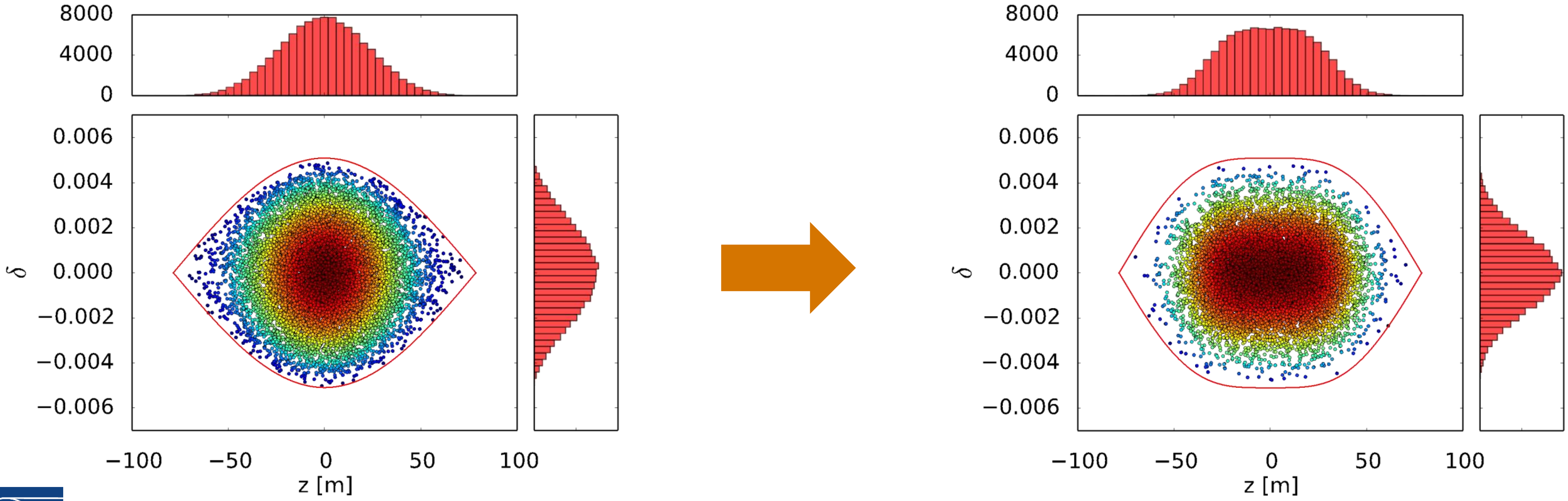
- is negative, because space charge transversely always defocuses
- is proportional **to the line density** and thus to the number of particles in the beam
- decreases **with energy like $1/(\beta\gamma^2)$** (when expressed in terms of normalized emittance) and therefore vanishes in the ultra-relativistic limit
- does not depend on the local beta functions or beam sizes but is inversely proportional to the normalized emittance (here the emittance includes all particles!)

Mitigation techniques

Maximum tune shift (circular Gaussian)

$$\Delta\hat{Q}_{x,y} = -\frac{r_0 C\hat{\lambda}}{2\pi e\beta\gamma^2}\frac{1}{2\,\varepsilon^n_{x,y}}$$

- Decrease the peak line density by
  - maximizing the bunch length
  - flattening the bunch profile with a specially configured (double harmonic) RF system



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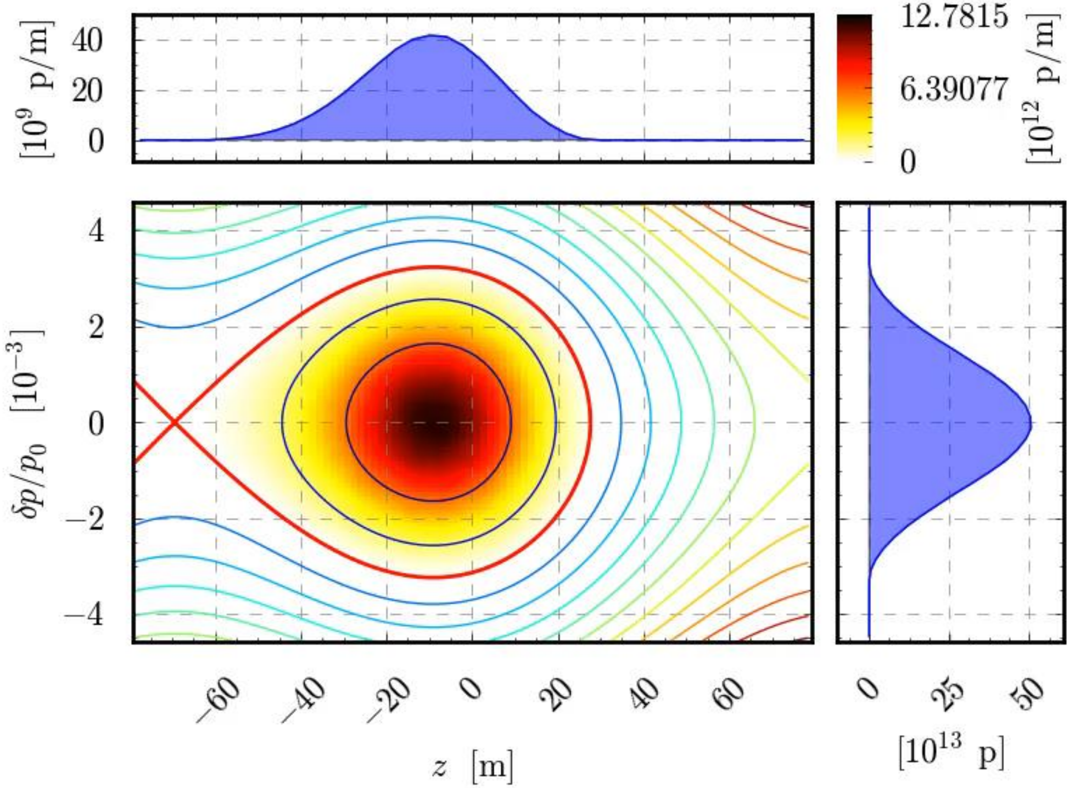
  - using **bunch distributions with small peak density** (e.g. parabolic instead of Gaussian)
  - reducing the central density of the particle distribution (e.g. "hollow bunches")



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- Increase the beam energy by
  - accelerating the beam as quickly as possible
  - increasing the injection energy (usually requires an upgrade of the pre-injector)

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- Increase the beam energy by
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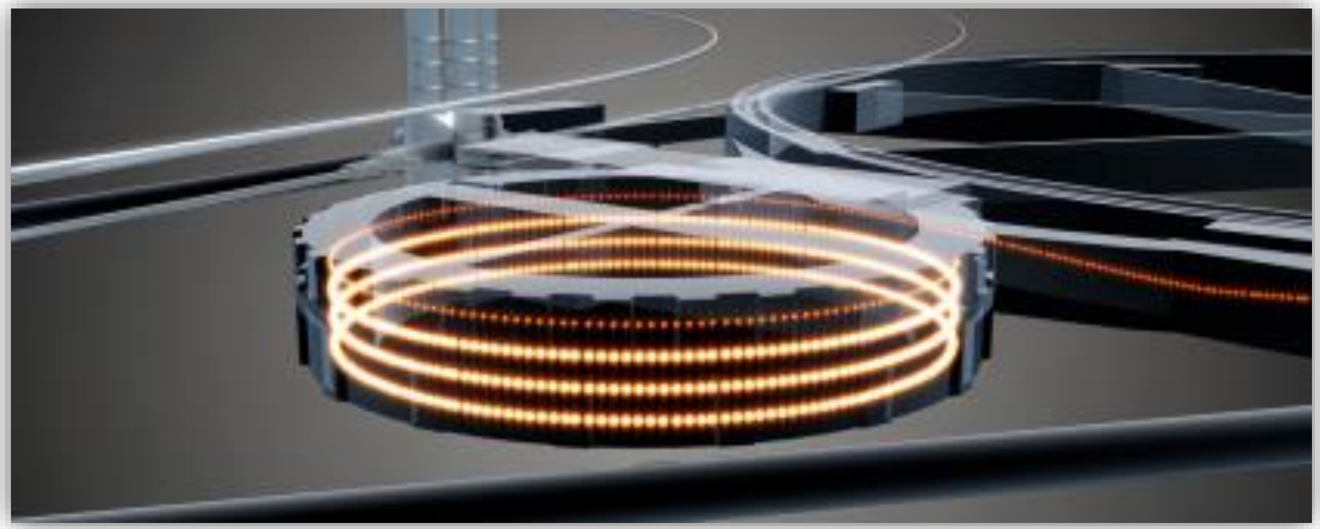
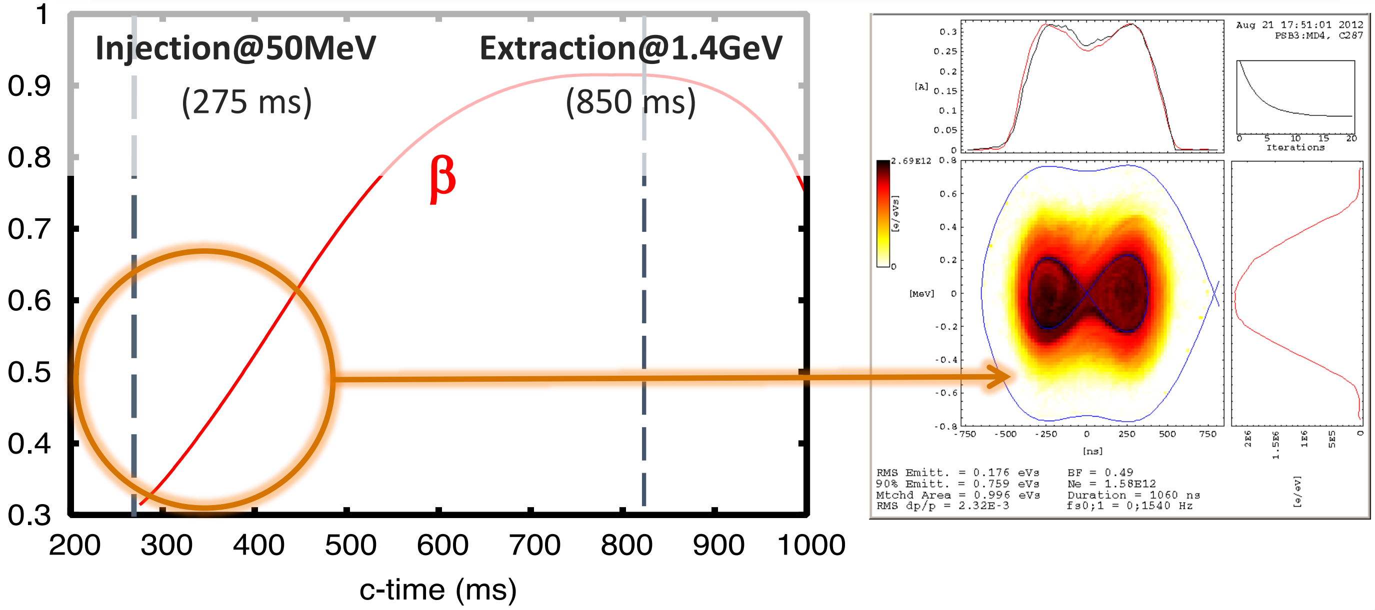
- Minimize the machine circumference
  - when designing/building a new accelerator since the **space charge detuning is an integrated effect**

Direct space charge in the CERN PSB

An accelerator that is particularly subject to space charge effects and makes use of **several mitigation schemes at the same time** is the CERN PS Booster:

- The PSB accelerates bright beams from 50 MeV to 1.4 GeV over 530 ms
- Instead of one large ring is it made up **of 4 smaller rings** to reduce the integrated effect
- Space charge is important, especially in first part of the cycle – **bunch is flattened** through a second harmonic RF system
- A future upgrade (LIU) foresees to **increase the injection energy** from 50MeV to 160MeV





```
RMS Emitt. = 0.176 eVs      BF = 0.49
90% Emitt. = 0.759 eVs      Ne = 1.58E12
Mtchd Area = 0.996 eVs      Duration = 1060 ns
RMS dp/p = 2.32E-3          fs0;1 = 0;1540 Hz
```

Direct space charge leads to a **purely incoherent tune shift** and a tune spread leading to the characteristic tune footprint with its **typical necktie shape** for bunched beams. Direct space charge does not lead to any coherent tune shifts (on the centroid motion).

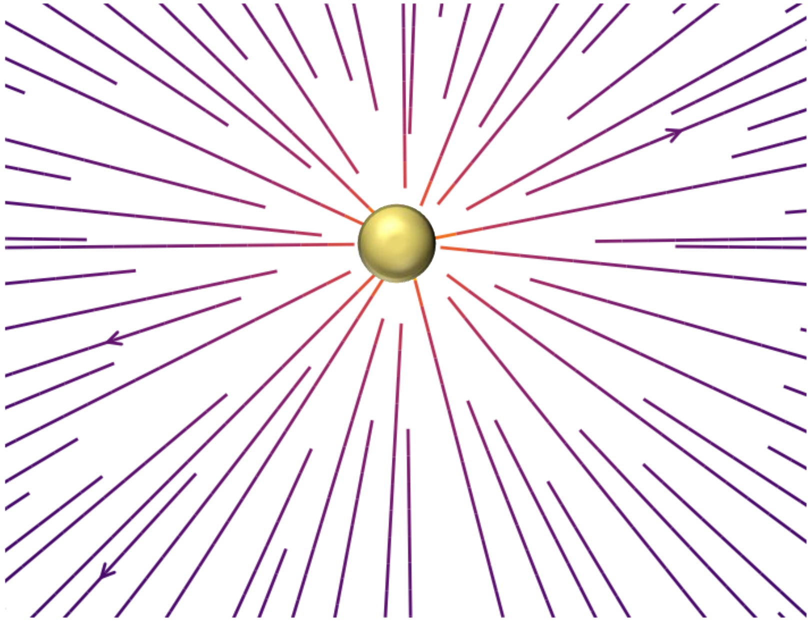
We will now look at the effect of **indirect space charge**. We will briefly look at the **different sources** of indirect space charge, that it can lead to both **incoherent as well as coherent tuneshifts** and how these tune shifts are usually parameterized using the **Laslett coefficients**.

- Part 2: Direct- and indirect space charge, wake fields and impedances
  - Direct space charge – mitigation techniques
  - Indirect space charge
  - From indirect space charge to (resistive) wall wakes
  - Concept of wake fields
  - Longitudinal and transverse wake fields and impedances

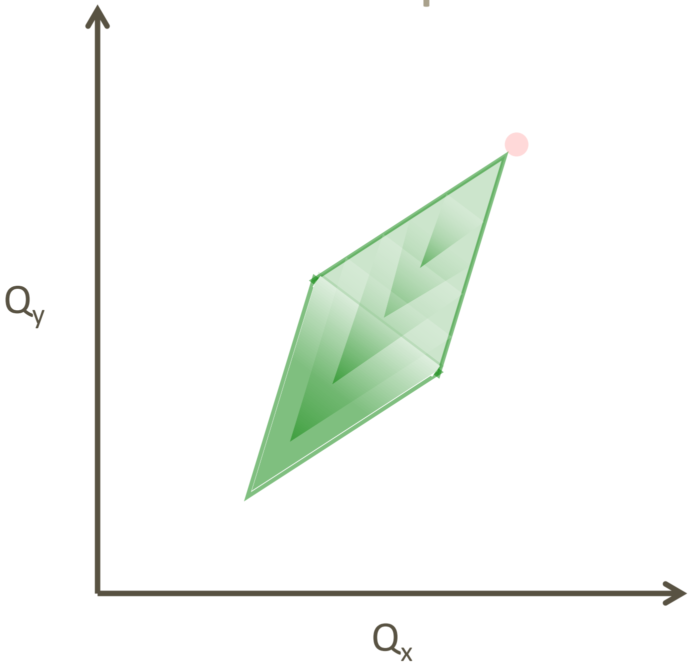
Direct vs. indirect space charge

- Direct space charge – central force
- The space symmetry endowed by free space **eliminates any impact on the centroid** motion

→ Exclusively incoherent tune shifts



Tune footprint - necktie



Direct vs. indirect space charge

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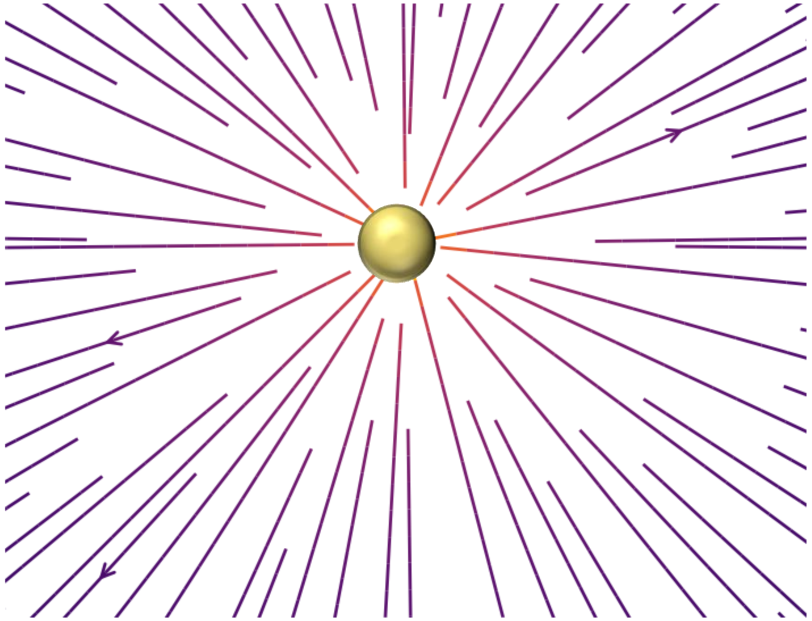
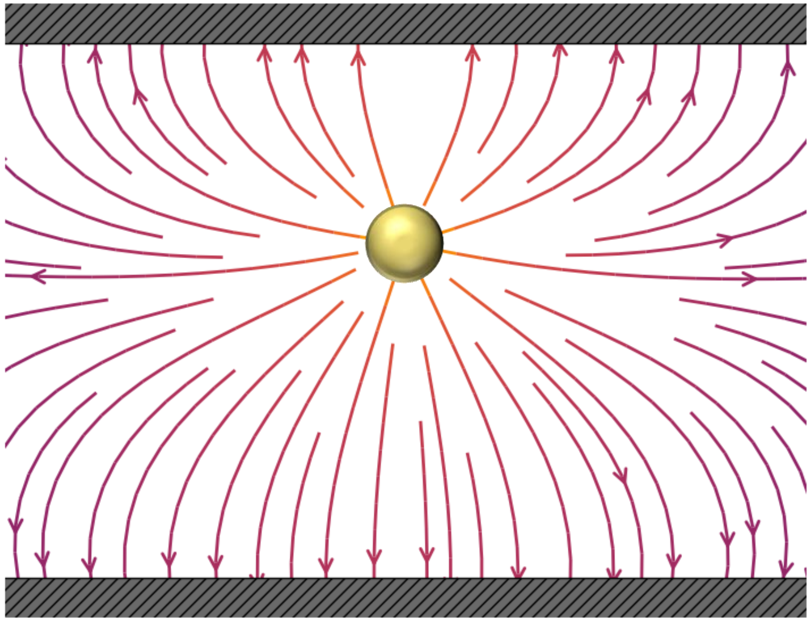
→ **Exclusively incoherent tune shifts**

- Indirect space charge – image charge forces
- The free space symmetry is broken by the geometrical arrangement of the conducting parallel plates; this gives a **net impact on the centroid** motion

→ **Both incoherent and coherent tune shifts**





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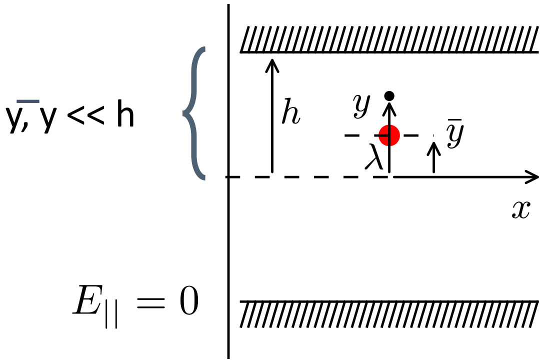

Electro- and magneto-static forces can be computed with the **method of image charges:**

- **We iteratively add charges to satisfy the boundary conditions. The resulting sum of fields converges to the final net field.**

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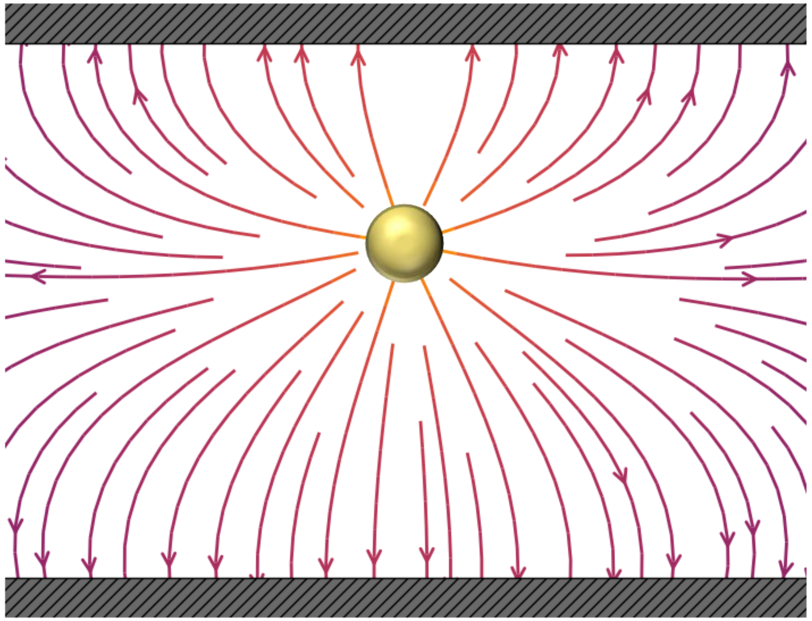

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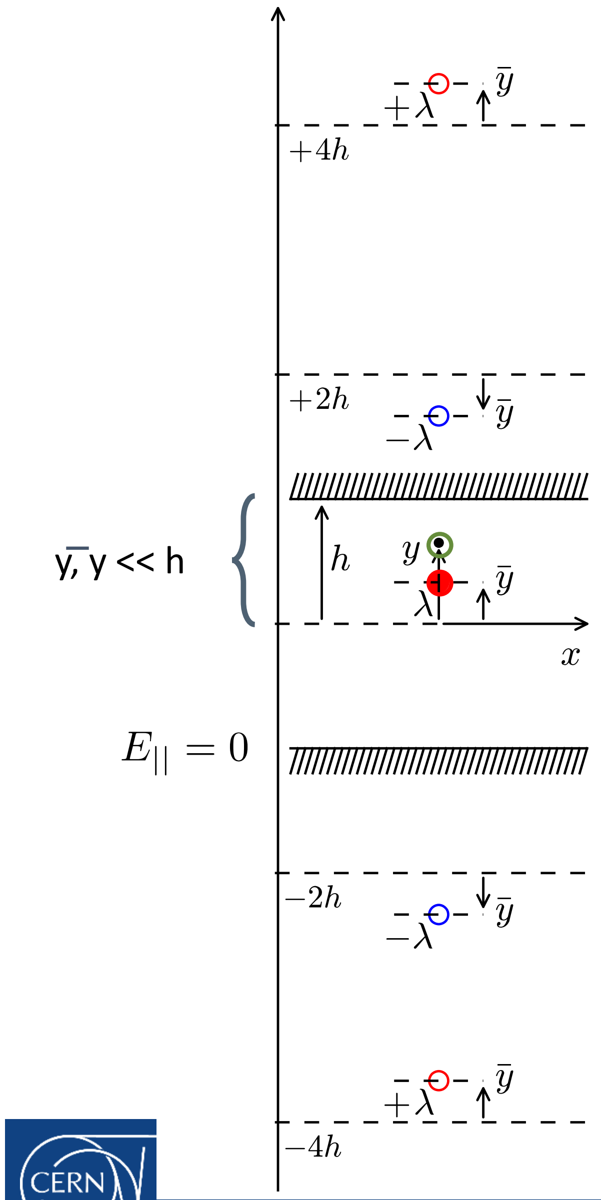
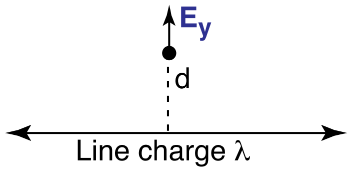


Electro- and magneto-static forces can be computed with the **method of image charges:**

- **We iteratively add charges to satisfy the boundary conditions. The resulting sum of fields converges to the final net field.**
- **We consider the beam as line charge density λ with infinite length**
- **The electric field at distance d is given by Gauss' law**



$$E_y = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{d}$$

- A flat vacuum chamber can be approximated by two perfect conducting parallel plates placed at vertical positions +h and –h
- Let the **beam centroid be displaced vertically by ȳ**
- The **witness particle is at y**
- The **boundary condition** at the two perfect conducting plates is satisfied by superposing **an infinite number of image line charges** with alternating signs as shown in the sketch
- The resulting electric field as a **function of beam and witness particle offsets** can be computed as

$$E_y(y, \bar{y}) = \frac{\lambda}{\pi\epsilon_0 h^2}\left[(\bar{y}+y)\frac{\pi^2}{32} + (\bar{y}-y)\frac{\pi^2}{96}\right] = \frac{F_y}{e}$$

Image charge forces

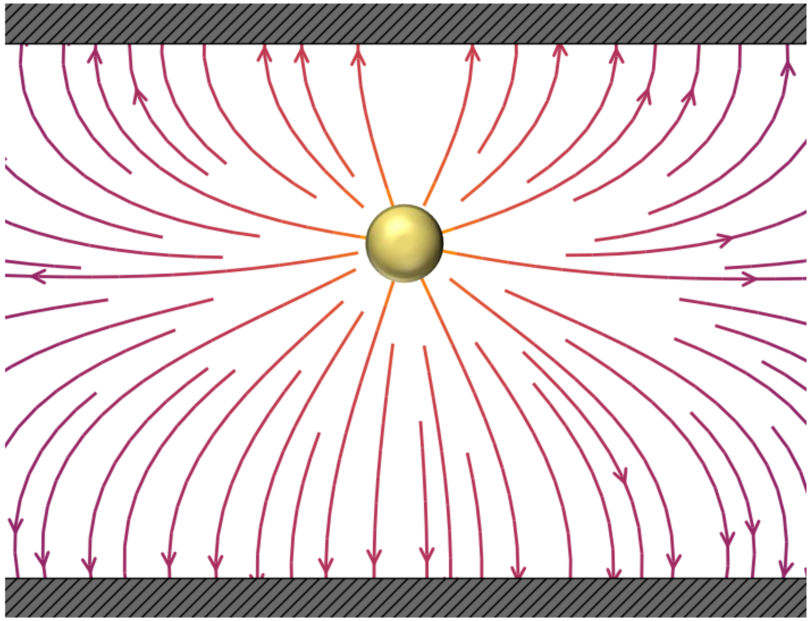
- Static **electric fields vanish** inside a conductor for any finite conductivity, while **static magnetic fields pass through** (unless in case of very high permeability)

- This is **no longer true for time varying fields**, which can penetrate into the material in a region δ_w called skin depth

- The skin depth depends on the **material properties and on frequency**. Fields pass through the conductor wall if the skin depth is larger than the wall thickness Δ_w. This happens at low frequencies. At higher frequencies, for a good conductor $\delta_w << \Delta_w$ and both electric and magnetic fields vanish in the wall

- We have seen how **electric image charge force** are induced by electrostatic fields in the presence of (perfectly) conducting boundaries

- Similarly, there are also **magnetics image current forces** – here we need to **differentiate between AC and DC forces**



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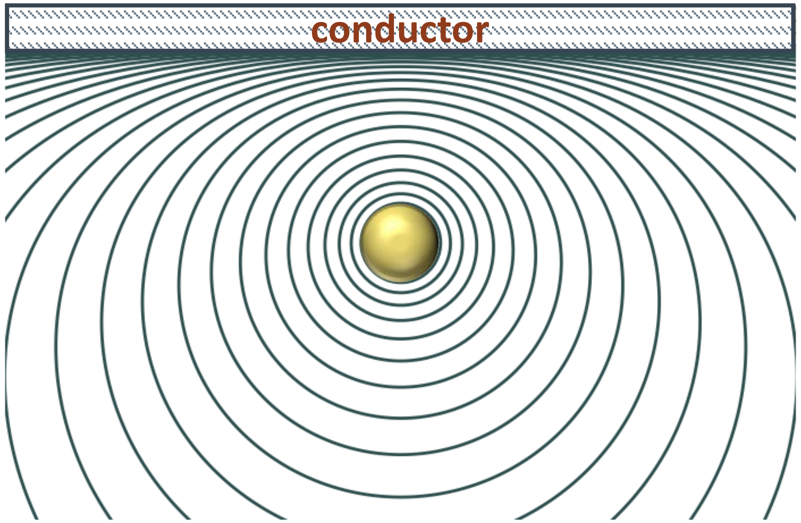
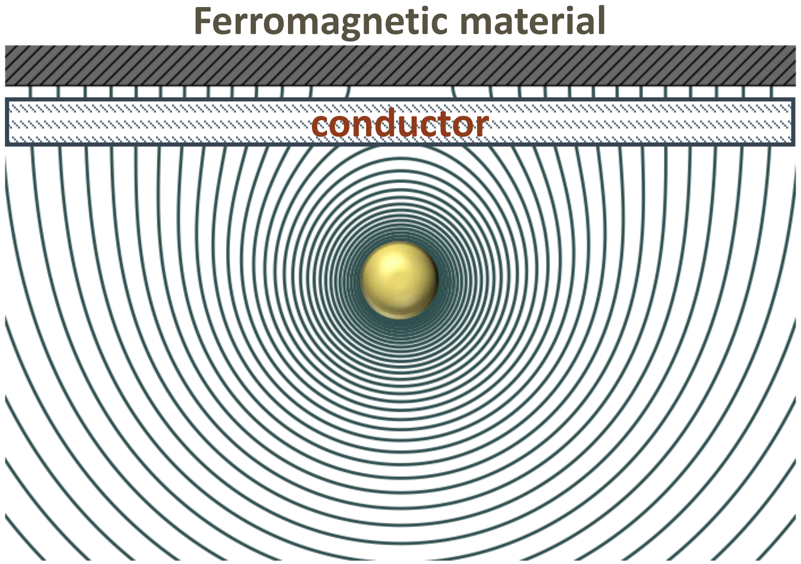


If the skin depth is very small (**rapidly varying fields**), **magnetic fields do not penetrate** and the field lines are tangent to the surface.



At **low frequencies magnetic fields penetrate** and pass through the vacuum chamber, they can interact with bodies behind the chamber

Image current forces

- In a very similar fashion as done for the electrostatic case we can also here deploy the **method of image currents** to solve for the magnetostatic fields and forces
- We obtain forces of the form

$$\frac{F_y}{e} = -\frac{\lambda\beta^2}{\pi\epsilon_0 h^2}\left[(\bar{y}+y)\,\frac{\pi^2}{32} + (\bar{y}-y)\,\frac{\pi^2}{96}\right]$$

$$\frac{F_y}{e} = +\frac{\lambda\beta^2}{\pi\epsilon_0 h^2}\left[(\bar{y}+y)\,\frac{\pi^2}{32} - (\bar{y}-y)\,\frac{\pi^2}{96}\right]$$

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- This is no longer true for time varying fields, which can penetrate into the material in a region δw called skin depth
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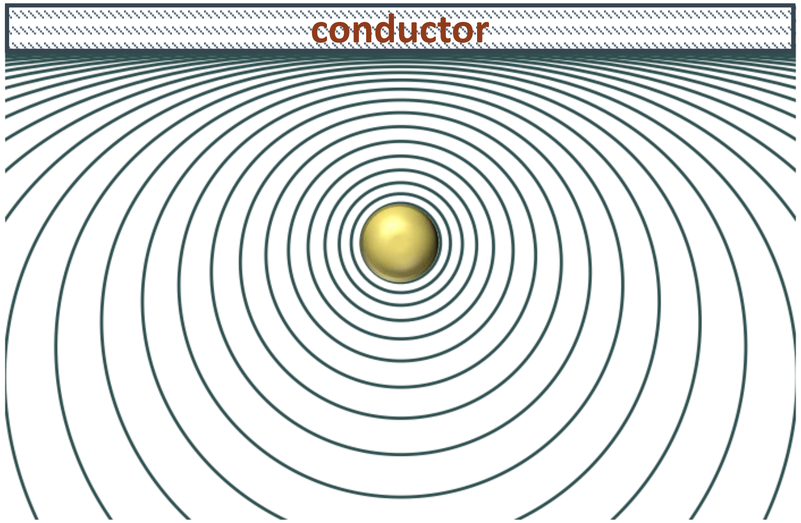

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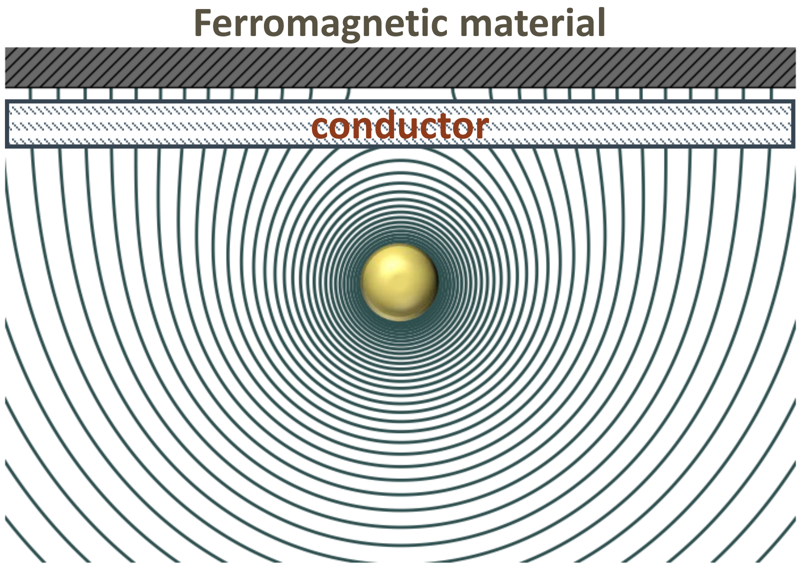

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Tune shifts and Laslett coefficients

- These electric image charge and magnetic ac and dc image current forces have the general similar form

$$F_x \propto \frac{e\lambda}{\pi\epsilon_0 h^2} f(\bar{x}, x) \qquad\qquad F_y \propto \frac{e\lambda}{\pi\epsilon_0 h^2} f(\bar{y}, y)$$

- These forces can lead to **both incoherent as well as coherent tune shifts** – from the forces, the tune shifts can be computed. In fact, it turns out that the **tune shifts can be parameterized** via the **Laslett coefficients**. For example, for electric image charge forces between two perfectly conducting parallel plates, the incoherent and coherent tune shifts can be expressed as:

$$\Delta Q_{x,y}^{\mathrm{inc}} = -\frac{2\,\langle\beta_{x,y}\rangle\, r_0 R}{e\beta^2\gamma}\,\frac{\varepsilon_1^{x,y}}{h^2} \qquad\qquad \Delta Q_{x,y}^{\mathrm{coh}} = -\frac{2\,\langle\beta_{x,y}\rangle\, r_0 R}{e\beta^2\gamma}\,\frac{\xi_1^{x,y}}{h^2}$$

The Laslett coefficients can be evaluated **for different geometries** and are classified in **incoherent and coherent tune shifts for electric, magnetic ac and magnetic dc** image charges and currents.

Tune shifts and Laslett coefficients

- Thes
m

Laslett coefficients	Circular ($w = h$)	Elliptical (e.g. $w = 2h$)	Parallel plates ($h/w = 0$)
ε_1^x	0	-0.172	$-\pi^2/48$
ε_1^y	0	$+0.172$	$+\pi^2/48$
ξ_1^x	$+1/2$	0.083	0
ξ_1^y	$+1/2$	0.55	$+\pi^2/16$
ε_2^x	$-\pi^2/24$	$-\pi^2/24$	$-\pi^2/24$
ε_2^y	$+\pi^2/24$	$+\pi^2/24$	$+\pi^2/24$
ξ_2^x	0	0	0
ξ_2^y	$+\pi^2/16$	$+\pi^2/16$	$+\pi^2/16$

Assuming parallel plates for the ferro-magnetic boundary for all geometries …

* L. J. Laslett, LBL Document PUB-616, 1987, vol III

Note: they are always defined with respect to the *vertical half gap h or g*

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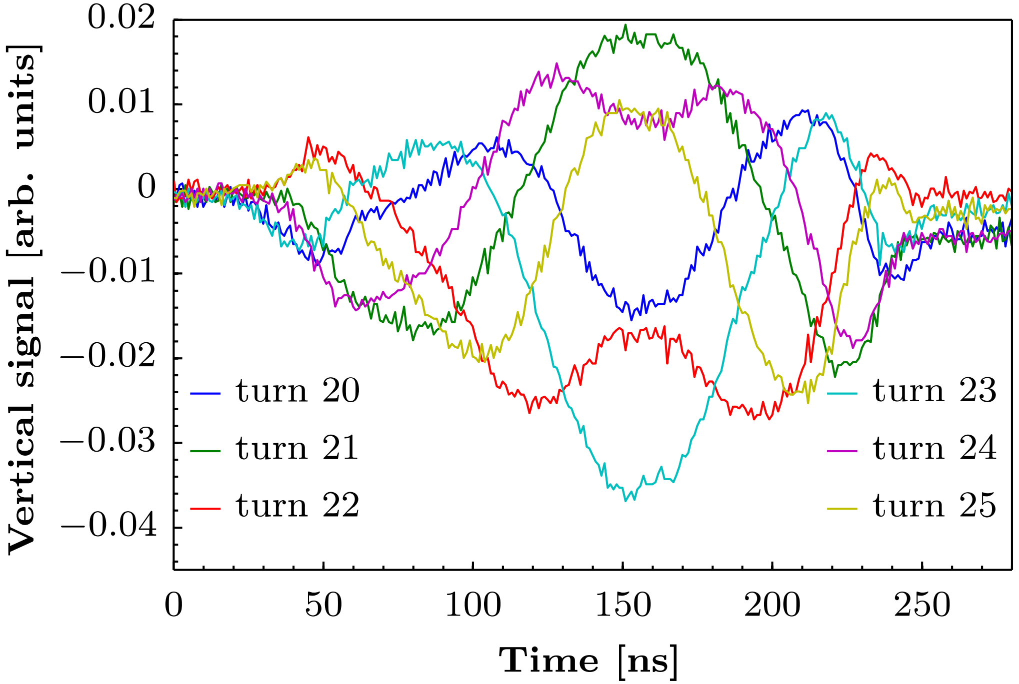
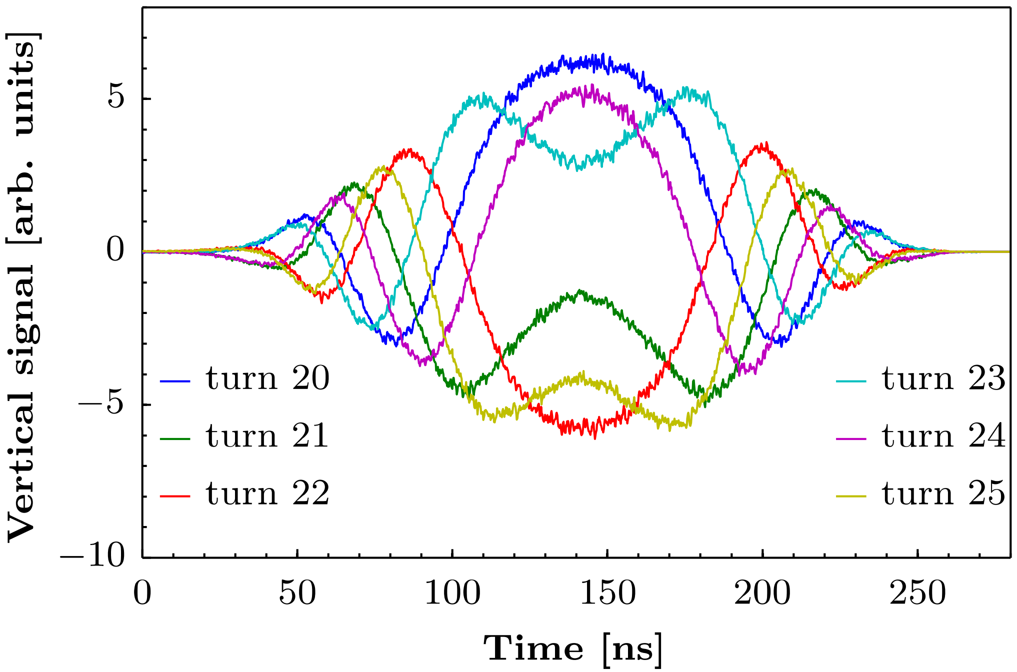
Example: PS injection oscillations

- The observed intra-bunch motion was reproduced with an amazing precision with multi-particle simulations (HEADTAIL code) **including the indirect space charge effect** taking

measurement



simulation



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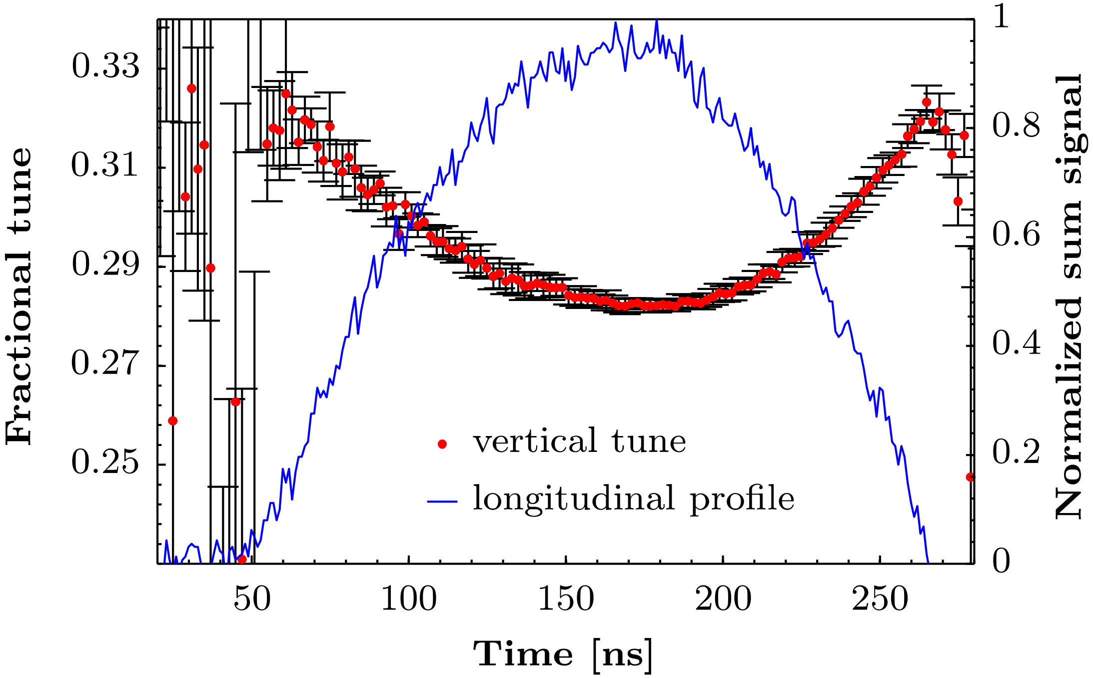
Example: PS injection oscillations

- The observed intra-bunch motion was reproduced with an amazing precision with multi-particle simulations (HEADTAIL code) **including the indirect space charge effect** taking
- It was understood from simulations that the observed intra-bunch motion was induced by the beam injected **off-center in combination with the indirect space charge effect**, which causes a tune shift along the bunch proportional to the local charge density

measurement



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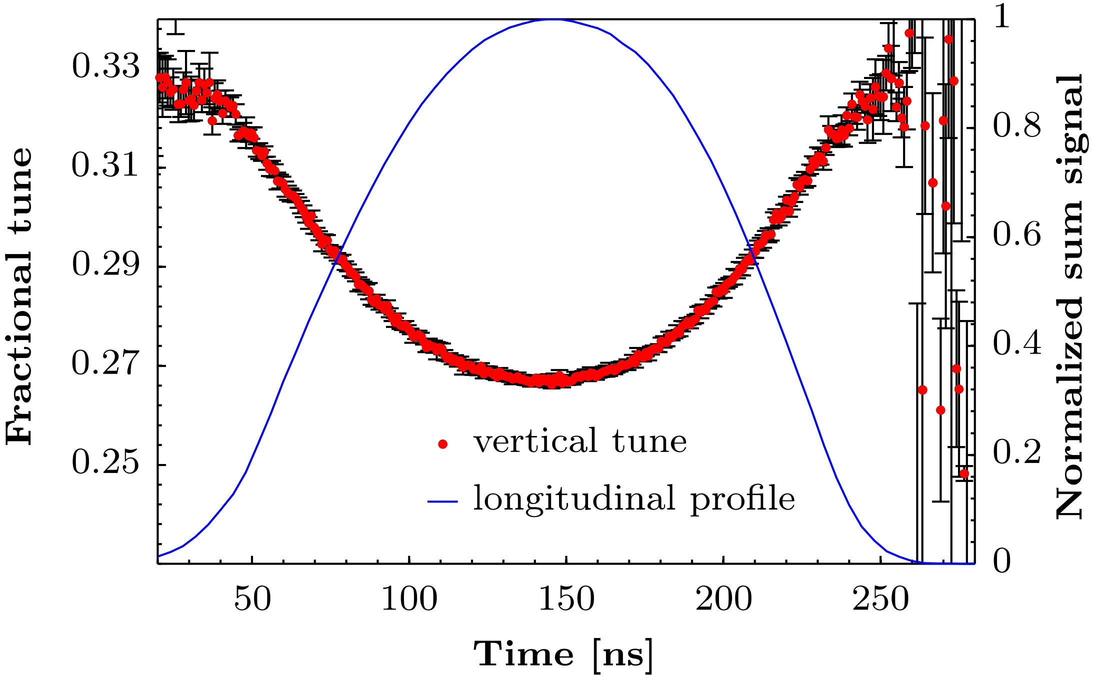

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Remarks

- Direct space charge
  - interaction of the bunch particles with the self induced electro-magnetic fields in free space
  - results in an incoherent tune shift (or spread)
  - the space charge force along a bunch is modulated with the local line density along the bunch and this results in an additional tune spread
  - decreases with energy like $\beta^{-1}\gamma^{-2}$
  - is a typical performance limitation for low energy machines
- Indirect space charge
  - interaction with image charges and currents induced in perfect conducting walls and ferromagnetic materials close to the beam pipe
  - results in incoherent and coherent tune shifts (or spreads), some of which are proportional to the average line density and others to the local line density
  - the contributions to the coherent and incoherent tune shifts for different standard geometries are expressed in terms of Laslett coefficients
  - decreases with energy like $\beta^{-2}\gamma^{-1}$

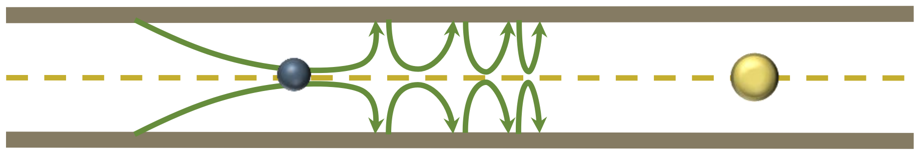
So far, we have introduced **direct and indirect space charge as collective effects**. The corresponding forces were not externally given but **dependent on the actual particle distribution** within the beam (remember, we looked at single particles as well as uniform and Gaussian distributions). The forces led to **incoherent and coherent tune shifts**.

We will now go a step further and investigate more complicated structures. We will try to find a smart way to deal with these structures. In the course of this, we will generalize and extend the direct and indirect space charge effects towards the **concept of wake fields and impedances**.

- Part 2: Direct- and indirect space charge, wake fields and impedances
    - ○ Direct space charge – mitigation techniques
    - ○ Indirect space charge
    - ○ From indirect space charge to (resistive) wall wakes
    - ○ Concept of wake fields

Electromagnetic fields of different sources

- Direct space charge
  - **Free space:** probe particles are effected directly by the source particles via the Lorentz force.



- Indirect space charge/**resistive wall wake**
  - **Smooth boundaries:** probe particles are effected by the source particles' induced image charges and currents.

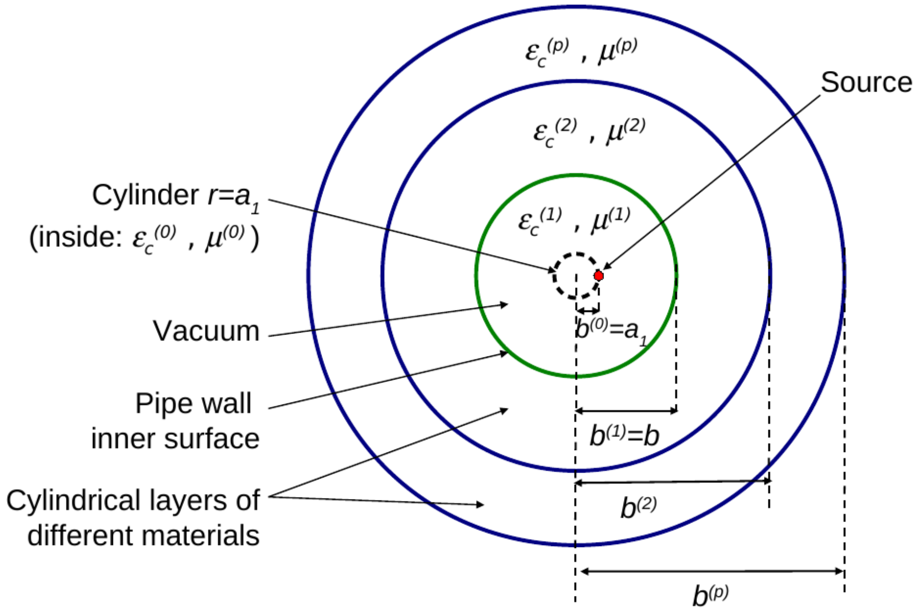
- Resonator wake fields
  - Discontinuities in boundaries: fields are excited by the source particles' distribution and can keep ringing, thus effecting trailing probe particles

From (in-)direct space charge to resistive walls…

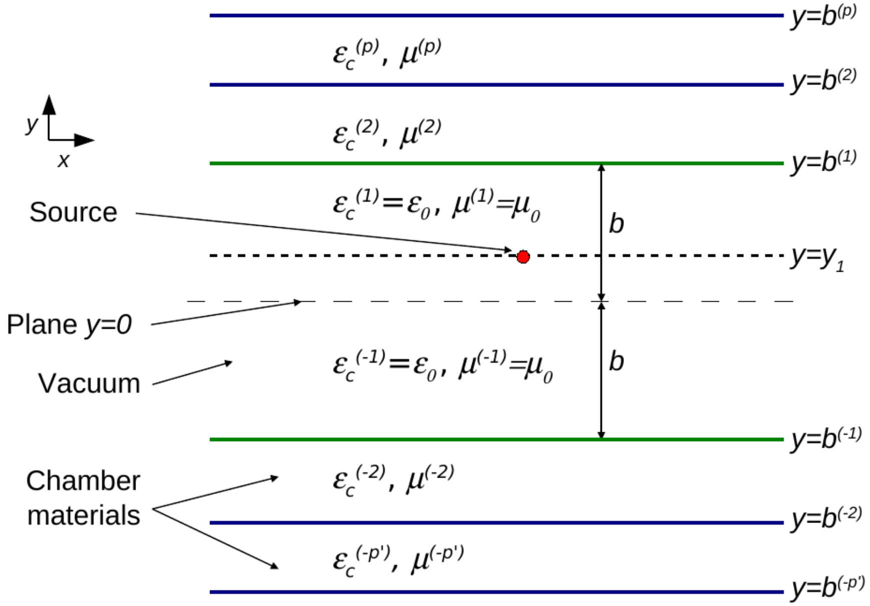
From (in-)direct space charge to resistive wall

- We consider a smooth multilayer structure, **longitudinally translation invariant and transversely bounded**.
- We consider a charged point particle traveling through the smooth multilayered structure.
- The **induced electromagnetic fields can be computed** for certain geometries by means of Maxwell's equations (longitudinal and transverse electromagnetic fields) with **field matching at the boundaries**. Two examples of such geometries are shown below.

Cylindrical



Flat



From (in-)direct space charge to resistive wall

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- The **induced electromagnetic fields can be computed** for certain geometries by means of Maxwell's equations (longitudinal and transverse electromagnetic fields) with **field matching at the boundaries**. Two examples of such geometries are shown below.
- It turns out, that the **resulting electromagnetic fields** can be decomposed into 3 components:

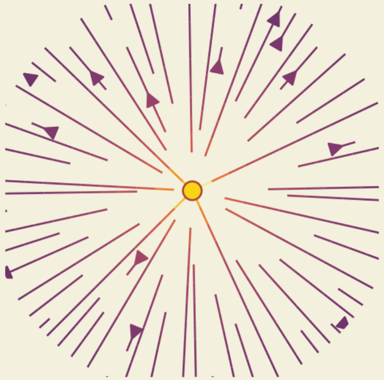
$$\vec{K}_{\text{Total}} = \vec{K}_{\text{direct}} + \vec{K}_{\text{boundaries}}$$

$$\big\downarrow \; \sigma \to \infty \qquad \qquad \big\downarrow \; \beta = 1$$

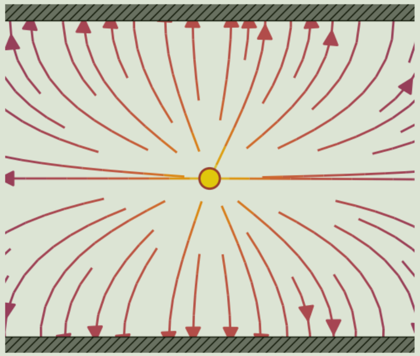
$$= \vec{K}_{\text{direct}} + \vec{K}_{\text{indirect}} + \vec{K}_{\text{resistive wall}}$$

From (in-)direct space charge to resistive wall

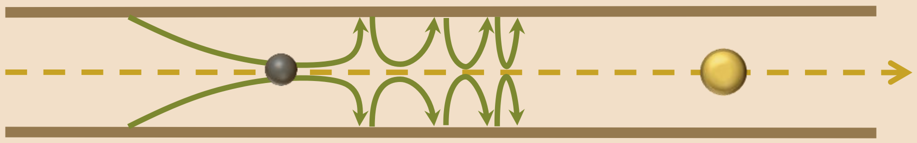
- A component which is **independent of the surrounding boundaries → direct space charge**



- A component which is independent of the surrounding material properties and purely **depended on the surrounding geometry → indirect space charge**



- A component which is **dependent on the surrounding material's electromagnetic properties → resistive wall**



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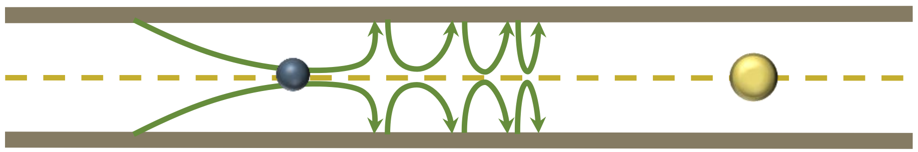
$$\sigma \to \infty \qquad \beta = 1$$

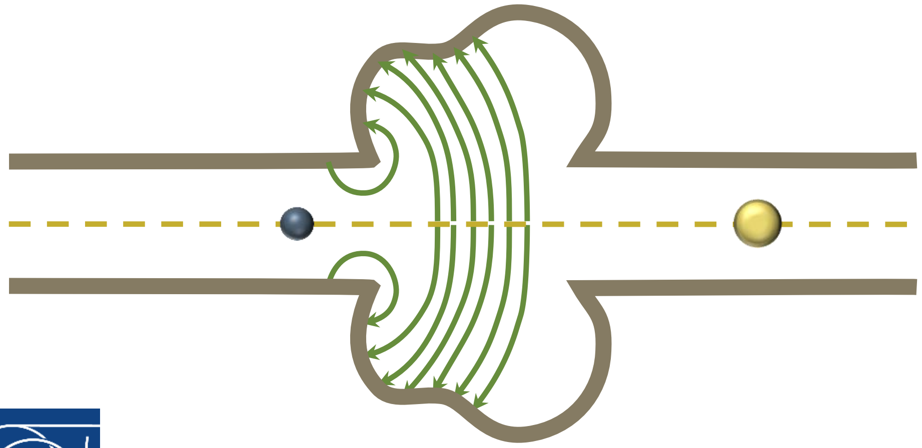
$$= \boxed{\vec{K}_{\text{direct}}} + \boxed{\vec{K}_{\text{indirect}}} + \boxed{\vec{K}_{\text{resistive wall}}}$$

Electromagnetic fields in different types of structures

- Direct space charge
  - **Free space:** probe particles are effected directly by the source particles via the Lorentz force.



- Indirect space charge/**resistive wall wake**
  - **Smooth boundaries:** probe particles are effected by the source particles’ induced image charges and currents.



- Resonator wake fields
  - **Discontinuities in boundaries:** fields are excited by the source particles’ distribution and can keep ringing, thus effecting trailing probe particles

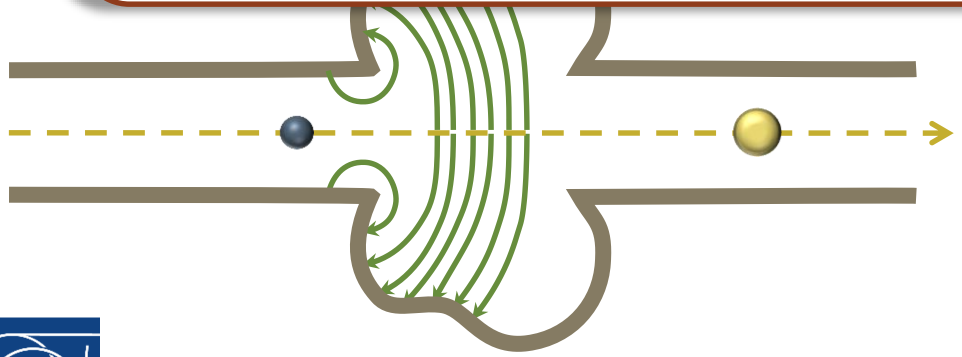
Electromagnetic fields in different types of structures

- Direct space charge
  - **Free space:** probe particles are effected directly by the source particles via the Lorentz force.

For more complicated geometries the (semi-) analytic methods no longer work. One must rely on **computational methods** to evaluate the induced electromagnetic fields (FDTD, FEM etc.).

Some examples:

…

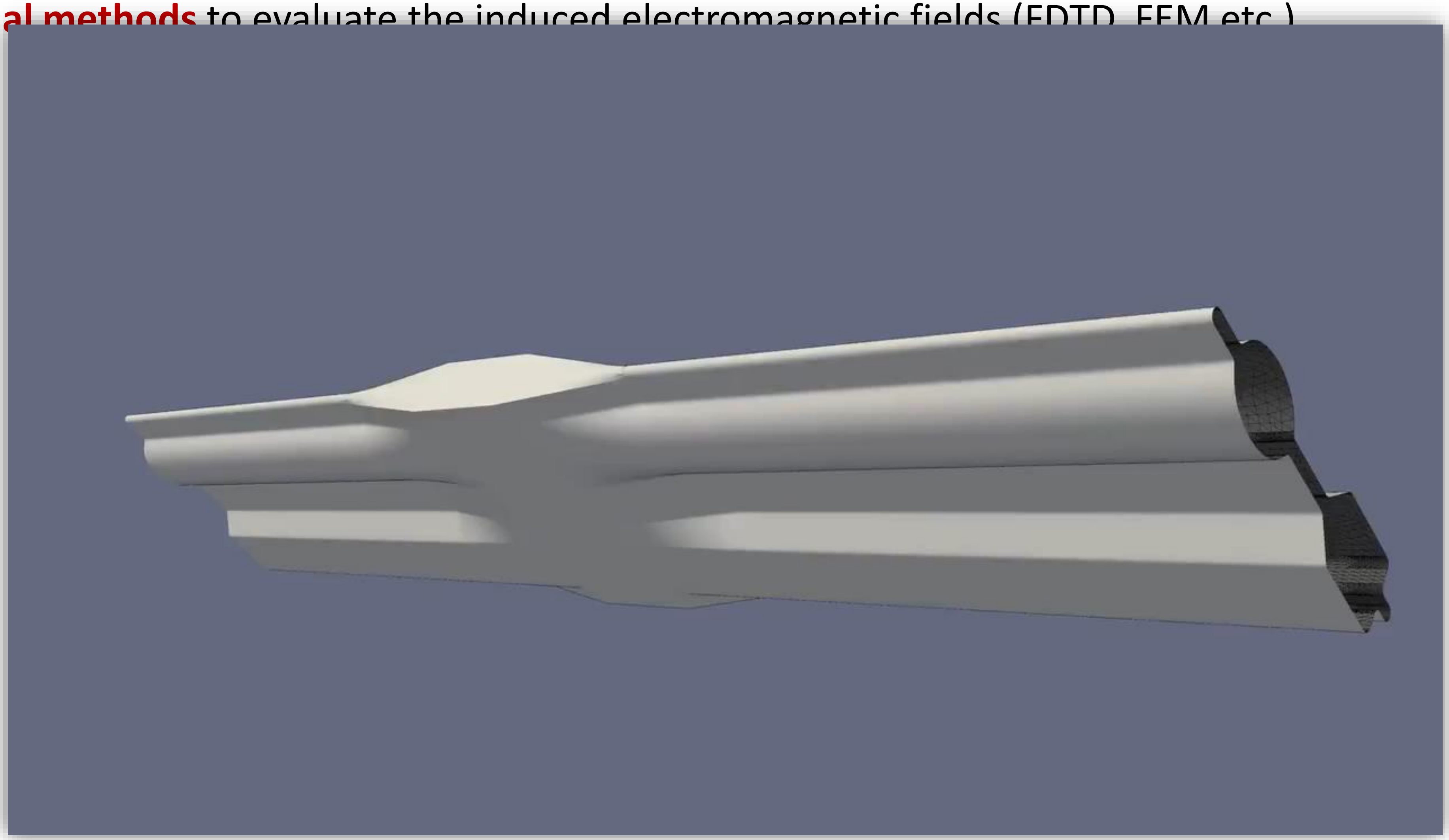


- Resonator wake fields
  - **Discontinuities in boundaries:** fields are excited by the source particles' distribution and can keep ringing, thus effecting trailing probe particles

Electromagnetic fields in complex structures

For more complicated geometries the (semi-) analytic methods no longer work. One must rely on **computational methods** to evaluate the induced electromagnetic fields (FDTD, FEM etc.)

ERL vacuum chamber



Electromagnetic fields in complex structures

For more complic

computation

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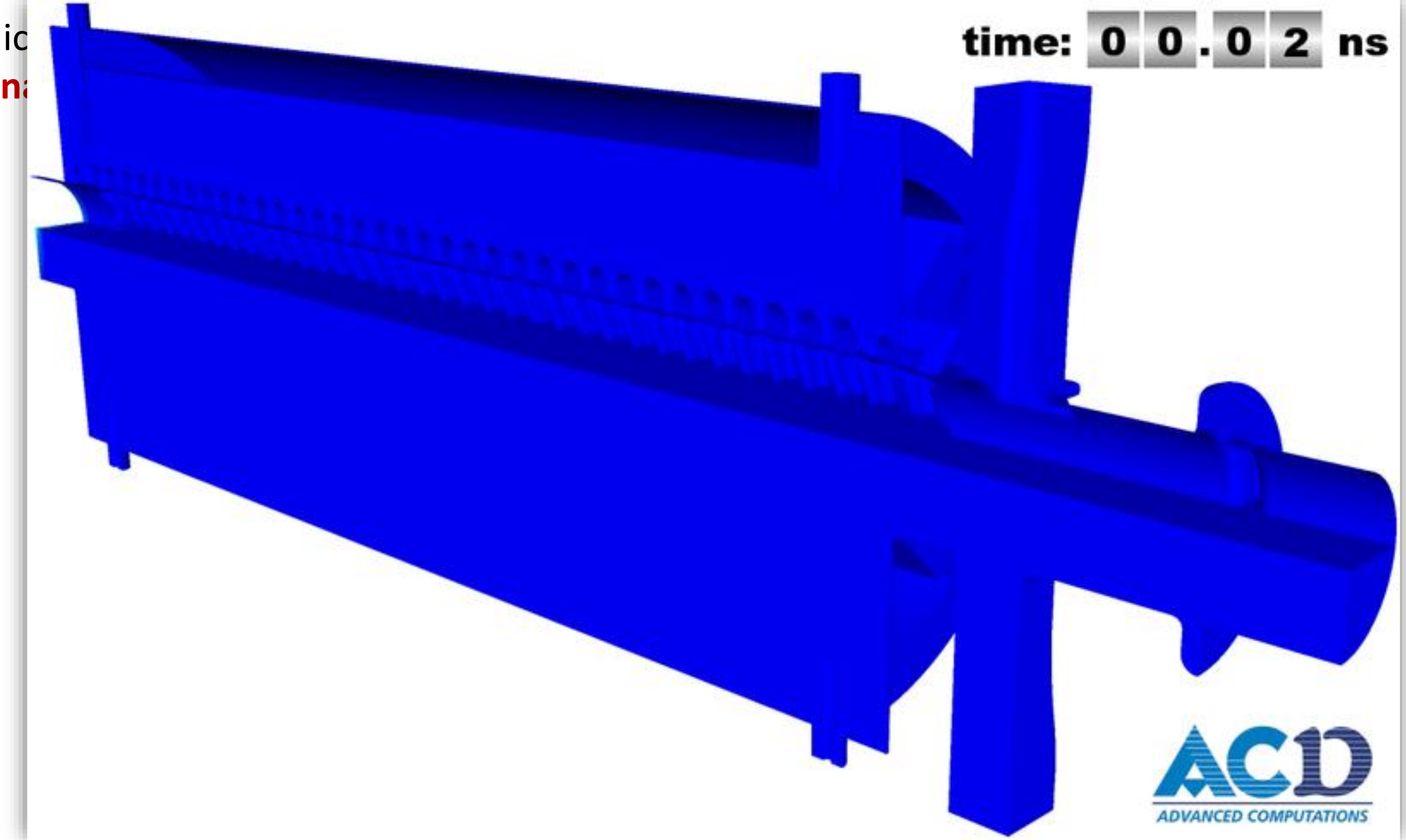


Electromagnetic fields in complex structures

For more complicated geometries the (semi-) analytic methods no longer work. One must rely on **computational methods** to evaluate the induced electromagnetic fields (FDTD, FEM etc.).

In principle, we need to solve the **full set of Maxwell's equations at every time step** to obtain the electromagnetic fields at every location within the structure in order to evaluate to forces at the probe particle's locations. This becomes very tedious and **virtually impossible for a 27km ring such as the LHC**.

How can we treat these phenomena more effectively in our models?

We normally **use a set of assumptions** to simplify the problem:

- **Rigid beam approximation:** the beam traverses the discontinuity of the vacuum chamber rigidly
- **Impulse approximation:** what the beam really cares about is the integrated impulse as it completes the traversal of the discontinuity

We have shown on the example of **multilayer structures** that one can in principle solve Maxwell's equations to obtain the induced electromagnetic fields by a given charge distribution. We saw that it is **possible to decompose these fields**. We were able to identify the part that is dependent on the electromagnetic properties of the surrounding material as the **wall wake** which already led to the idea of wake fields.

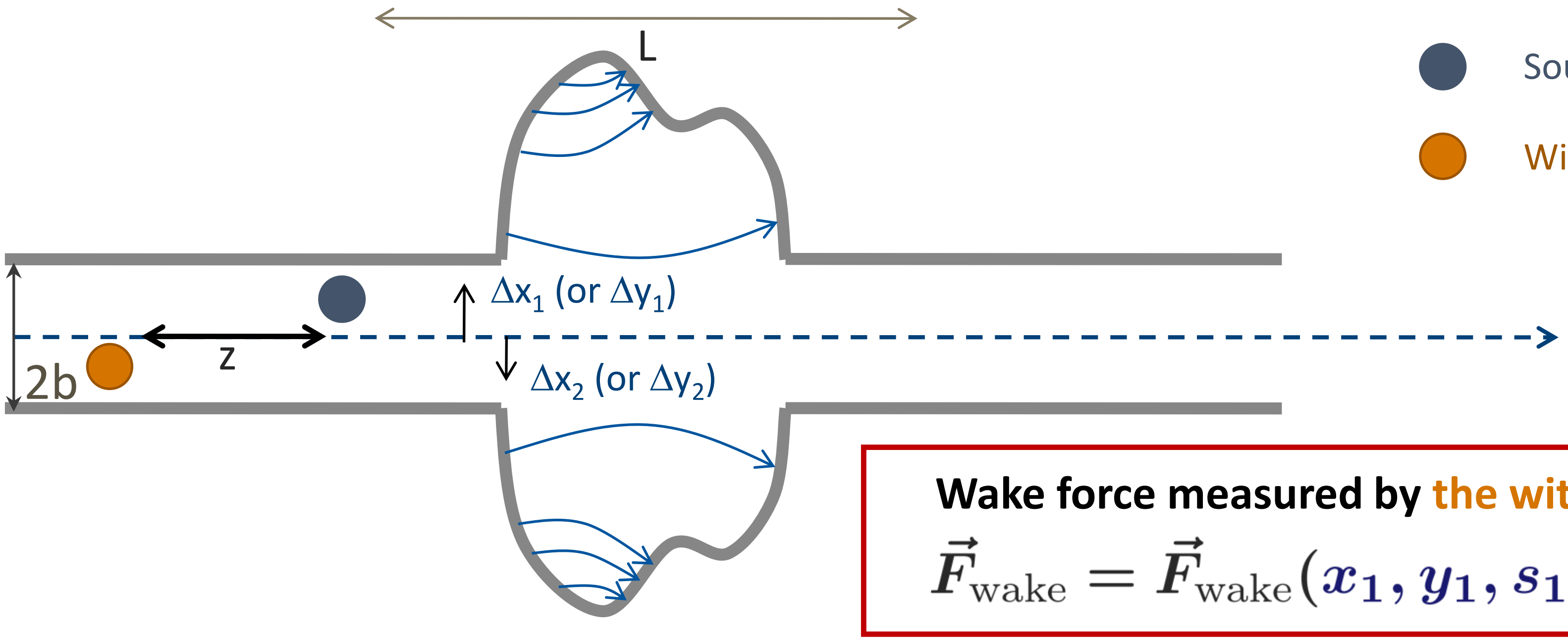
We have seen examples of induced electromagnetic fields within complex structures. We will now look at how we can deal with these types of fields more practically by introducing the **concept of wake fields**.

- Part 2: Direct- and indirect space charge, wake fields and impedances
  - Direct space charge – mitigation techniques
  - Indirect space charge
  - From indirect space charge to (resistive) wall wakes
  - Concept of wake fields

Wake function – general definition

How can we treat these phenomena effectively in our models?

- We will use a little trick: we consider **two point particles** – one source, one probe
- The forces will be a function of the locations of both the source and the probe particles:



● Source, q_1

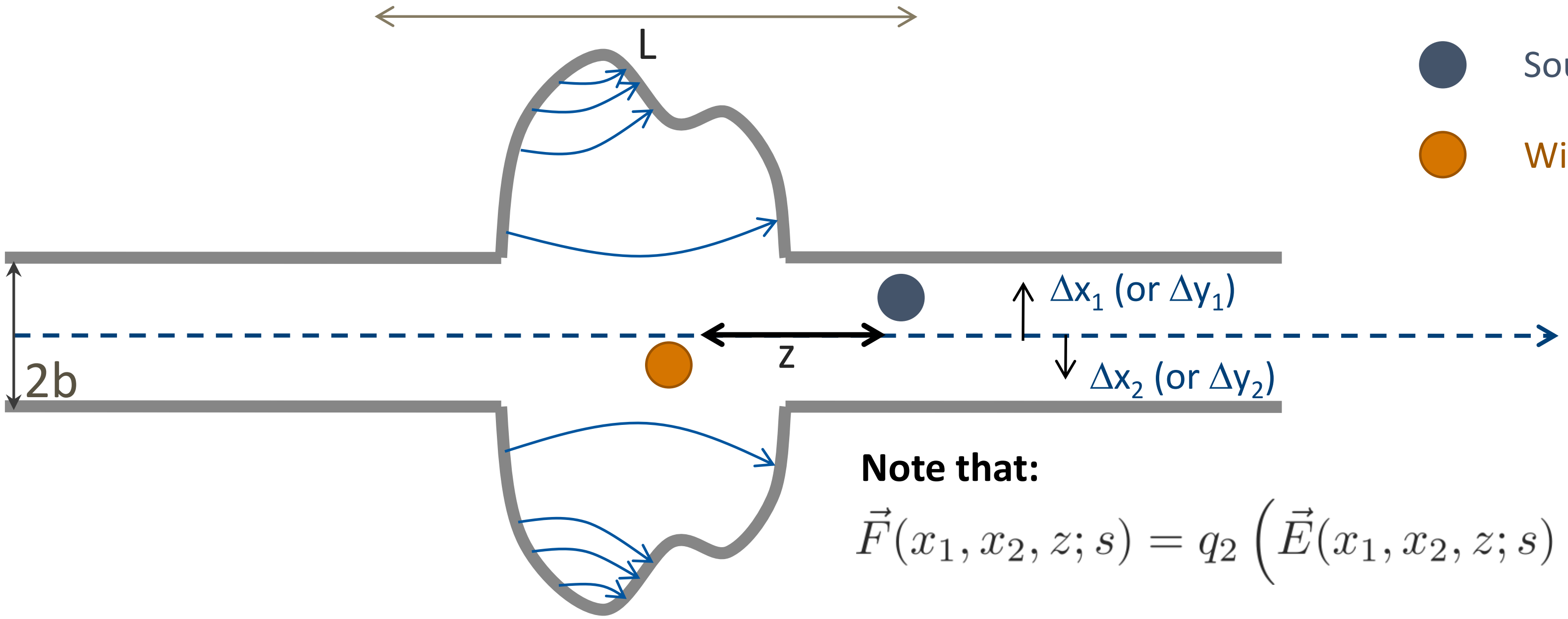
● Witness, q_2

Wake force measured by the witness particle

$$\boldsymbol{\vec{F}_{\text{wake}} = \vec{F}_{\text{wake}}(x_1, y_1, s_1, x_2, y_2, s_2, t)}$$

Note that: $\vec{F}_{\text{wake}} = q_2 \left(\vec{E}(x_1, y_1, s_1, x_2, y_2, s_2, t) + v_z \vec{e}_z \times \vec{B}(x_1, y_1, s_1, x_2, y_2, s_2, t) \right)$

Wake function – general definition



● Source, q_1

● Witness, q_2

Note that:

$$\vec{F}(x_1, x_2, z; s) = q_2 \left(\vec{E}(x_1, x_2, z; s) + v_z \vec{e}_z \times \vec{B}(x_1, x_2, z; s) \right)$$

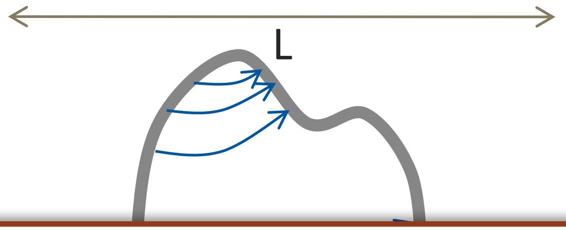
We define the **wake function as the integrated force on the witness particle** (associated to a change in energy):

- Let's focus on the horizontal plane. In general, for two point-like particles, we have

$$\Delta E_2 = \int F(x_1, x_2, z; s)\, ds = -q_1 q_2\, \boldsymbol{w(x_1, x_2, z)}$$

$$z \equiv s_2 - s_1\,, \qquad s \equiv s_1$$

Wake function – general definition



● Source, q_1

● Witness, q_2

w is proportional to **the integrated forces felt by the witness particle**:

- It is a function of **the transverse offsets and the longitudinal separation** of the source and the witness particles.
- It is typically expanded in the transverse offsets of source and witness particles. This then yields the different types of wake fields (dipole, quadrupole, coupling wakes).

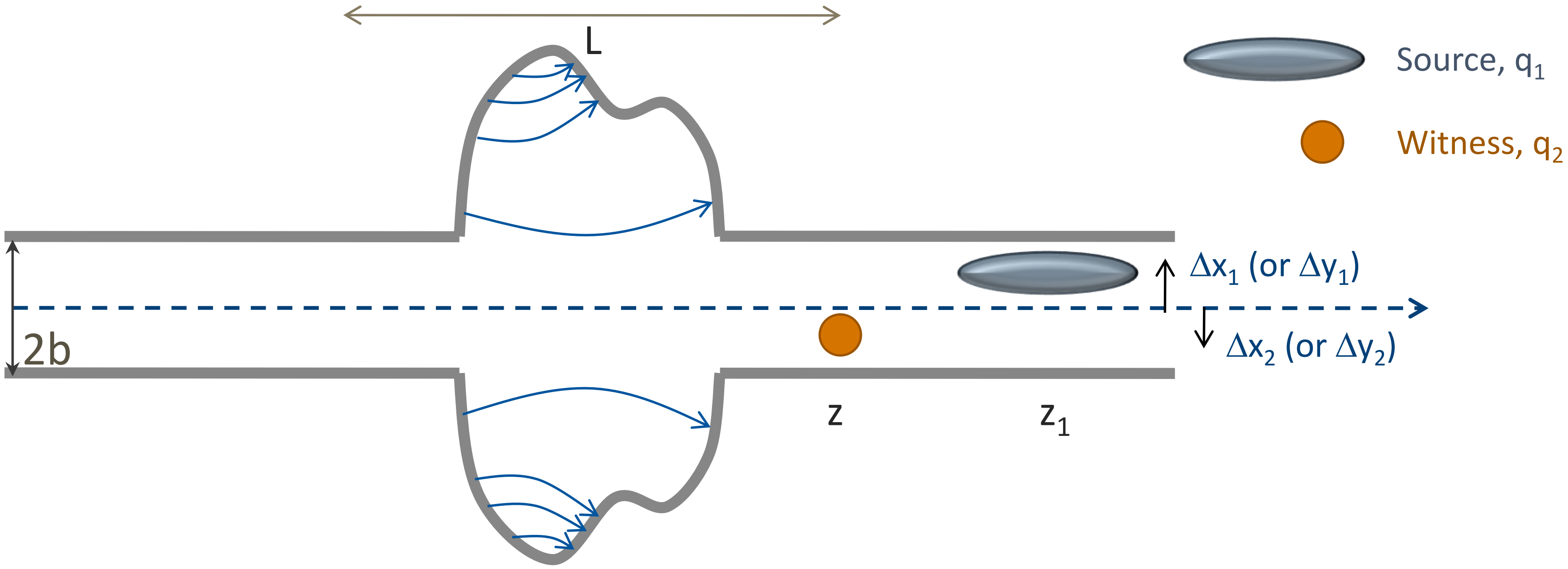
We define the **wake function as the integrated force on the witness particle** (associated to a change in energy):

- Let's focus on the horizontal plane. In general, for two point-like particles, we have

$$\Delta E_2 = \int F(x_1, x_2, z; s)\, ds = -q_1 q_2\, \boldsymbol{w(x_1, x_2, z)}$$

$$z \equiv s_2 - s_1\,, \qquad s \equiv s_1$$

Wake potential for a distribution of particles



We define the wake function as the **integrated force** on the witness particle (associated to a change in energy):

- For an extended particle distribution this becomes (superposition of all source terms)

$$\Delta E_2(z) \propto \int \boxed{\lambda_1(x_1, z_1)} \boldsymbol{w(x_1, x_2, z - z_1)}\, dx_1 dz_1$$

Forces become dependent on the **particle distribution function**

In this lecture we discussed **indirect space charge** and showed that this can lead **to both incoherent as well as coherent tune shifts**. We then moved on the a more general treatment of electromagnetic fields in simple structures where we were able to identify **yet another type of induced fields** originating from the **electromagnetic properties** of the surrounding material – **the wall wake**.

We looked at more general examples of induced fields in complex structures and **then introduced the concept of the wake function**.

Next we will look at some different wake fields and study more the **effect of wake fields and impedances** an the machine and on the beam.

- Part 2: Direct- and indirect space charge, wake fields and impedances
  - Direct space charge – mitigation techniques
  - Indirect space charge
  - From indirect space charge to (resistive) wall wakes
  - Concept of wake fields

End part 2

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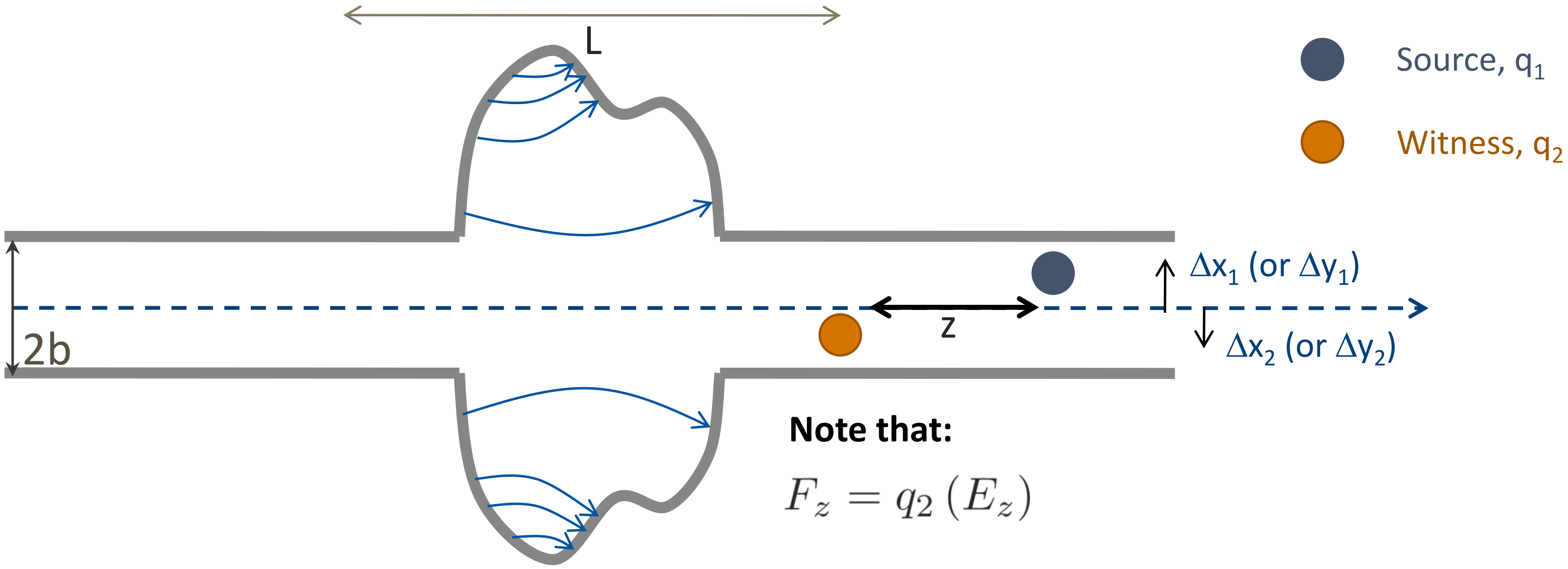
We have seen how we can deal with the induced electromagnetic fields within complex structures by means of the **wake function**. The wake function is the **electromagnetic response** of a structure and is an **intrinsic property** of any such structure.

We will now look how we **handle longitudinal and transverse wake fields** in practice and what are some of the fundamental properties. We will also introduce the impedance.

- Part 2: Direct- and indirect space charge, wake fields and impedances
  - Direct space charge – mitigation techniques
  - Indirect space charge
  - From indirect space charge to (resistive) wall wakes
  - Concept of wake fields
  - Longitudinal and transverse wake fields and impedances

Backup

Longitudinal wake function



Note that:

$$F_z = q_2\left(E_z\right)$$

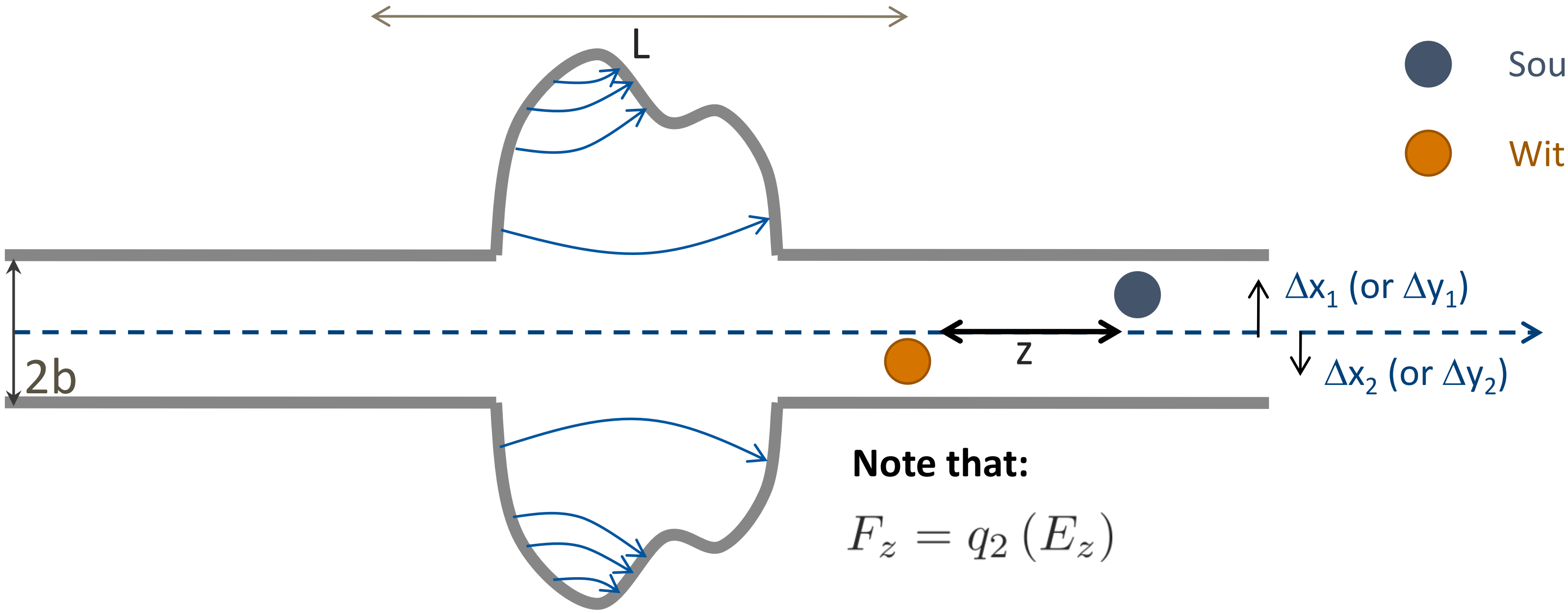
- Longitudinal wake fields

$$\int F_z(\Delta x_1, \Delta x_2, z; s)\, ds = -q_1 q_2 \left(\boxed{W_\parallel(z)} + \boxed{O(\Delta x_1) + O(\Delta x_2)}\right)$$

Zeroth order with source and test centred
usually dominant

Higher order terms
Usually negligible for small offsets

Longitudinal wake function



● Source, q_1

● Witness, q_2

Note that:

$$F_z = q_2 \left(E_z \right)$$

- Longitudinal wake fields

$$\Delta E_2 = \int F_z(z;s)\, ds = -q_1 q_2\, W_{||}(z)$$

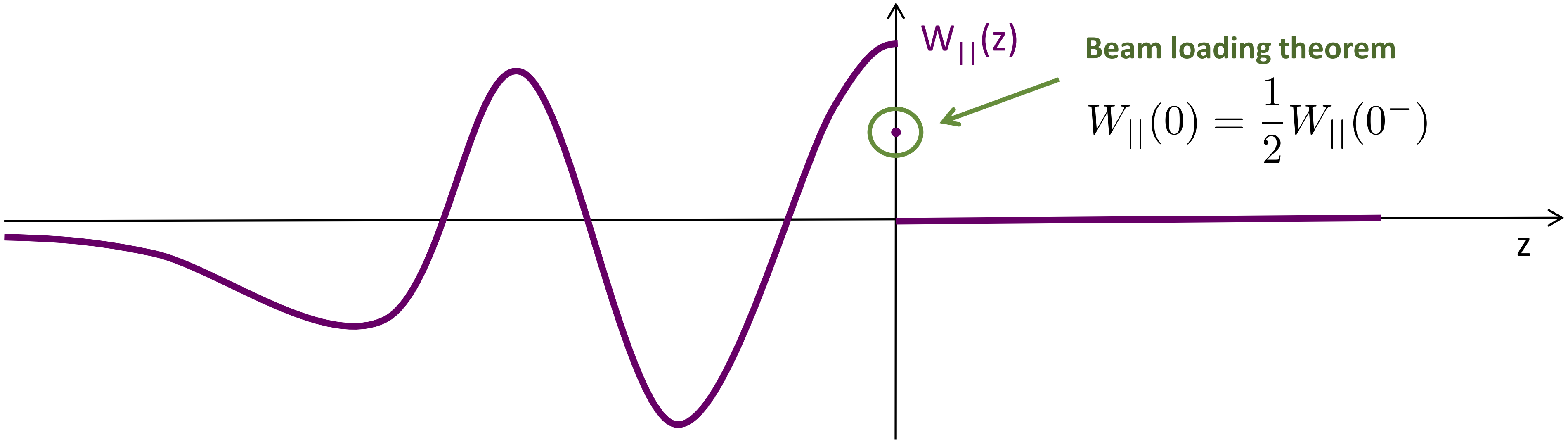
$$\longrightarrow \frac{\Delta E_2}{E_0} = \left(\frac{\gamma^2 - 1}{\gamma} \right) \frac{\Delta p_2}{p_0}$$

Energy kick of the witness particle from longitudinal wakes

Longitudinal wake function

$$W_{\|}(z) = -\frac{\Delta E_2}{q_1 q_2} \quad \xrightarrow[q_2 \to q_1]{z \to 0} \quad W_{\|}(0) = -\frac{\Delta E_1}{q_1^2}$$

- The value of the wake function in z=0 is related to the **energy lost by the source particle** in the creation of the wake
- ***$W_{||}(0)>0$*** since ***$\Delta E_1<0$***
- ***$W_{||}(z)$*** is discontinuous in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation



Beam loading theorem

$$W_{\|}(0) = \frac{1}{2} W_{\|}(0^-)$$

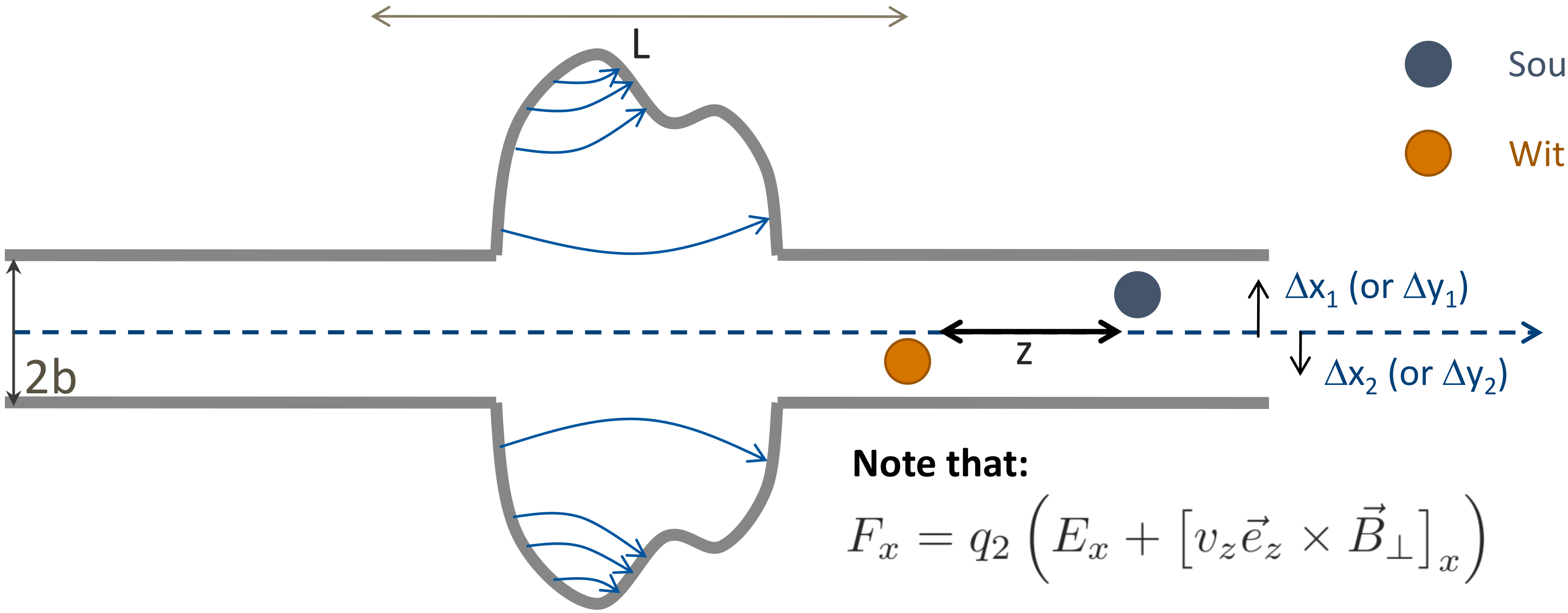
Longitudinal wake function

$$W_{\|}(z) = -\frac{\Delta E_2}{q_1 q_2} \quad \xrightarrow[q_2\to q_1]{z\to 0} \quad W_{\|}(0) = -\frac{\Delta E_1}{q_1^2}$$

- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
  - → Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**
  - → This is the definition of **longitudinal beam coupling impedance** of the element under study

$$\underbrace{\boxed{Z_{\|}(\omega)}}_{[\Omega]} = \int_{-\infty}^{\infty} \underbrace{\boxed{W_{\|}(z)}}_{[\Omega/s]} \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

Transverse wake functions



● Source, q_1

● Witness, q_2

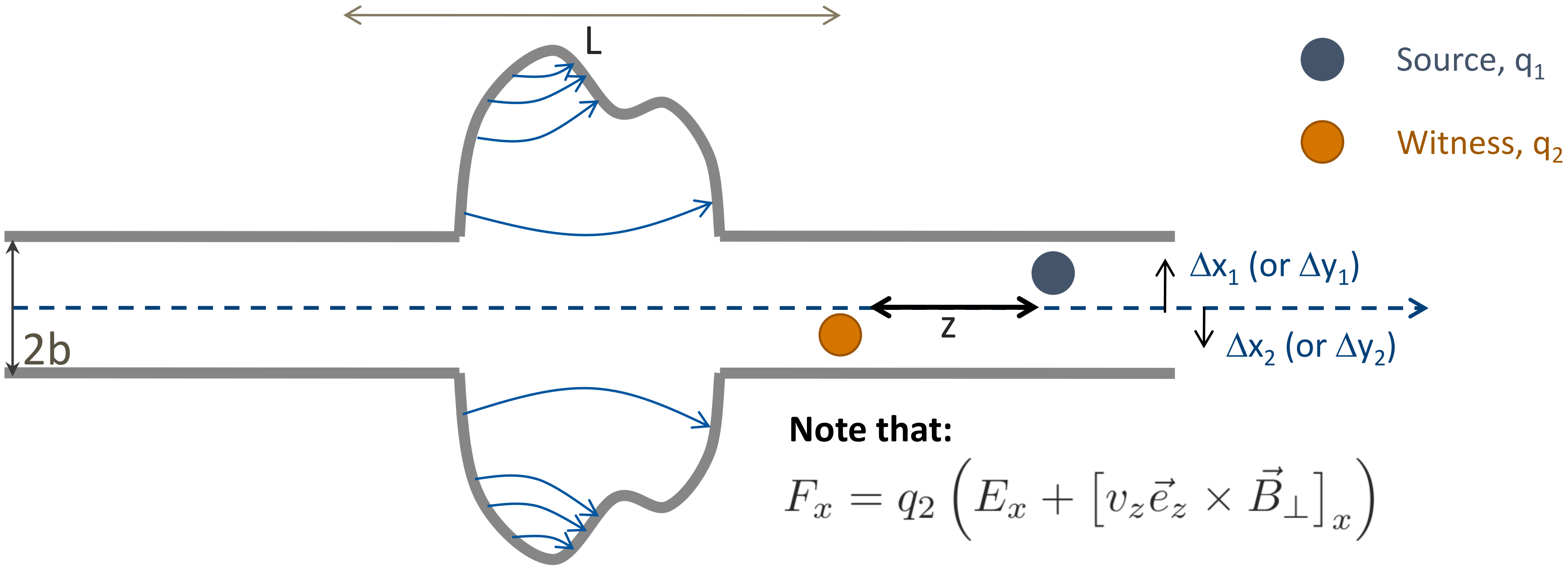
Note that:

$$F_x = q_2 \left(E_x + \left[v_z \vec{e}_z \times \vec{B}_\perp \right]_x \right)$$

- Transverse wake fields

$$\Delta E_{x2} = \int F_x(\Delta x_1, \Delta x_2, z; s)\, ds$$

Transverse wake functions



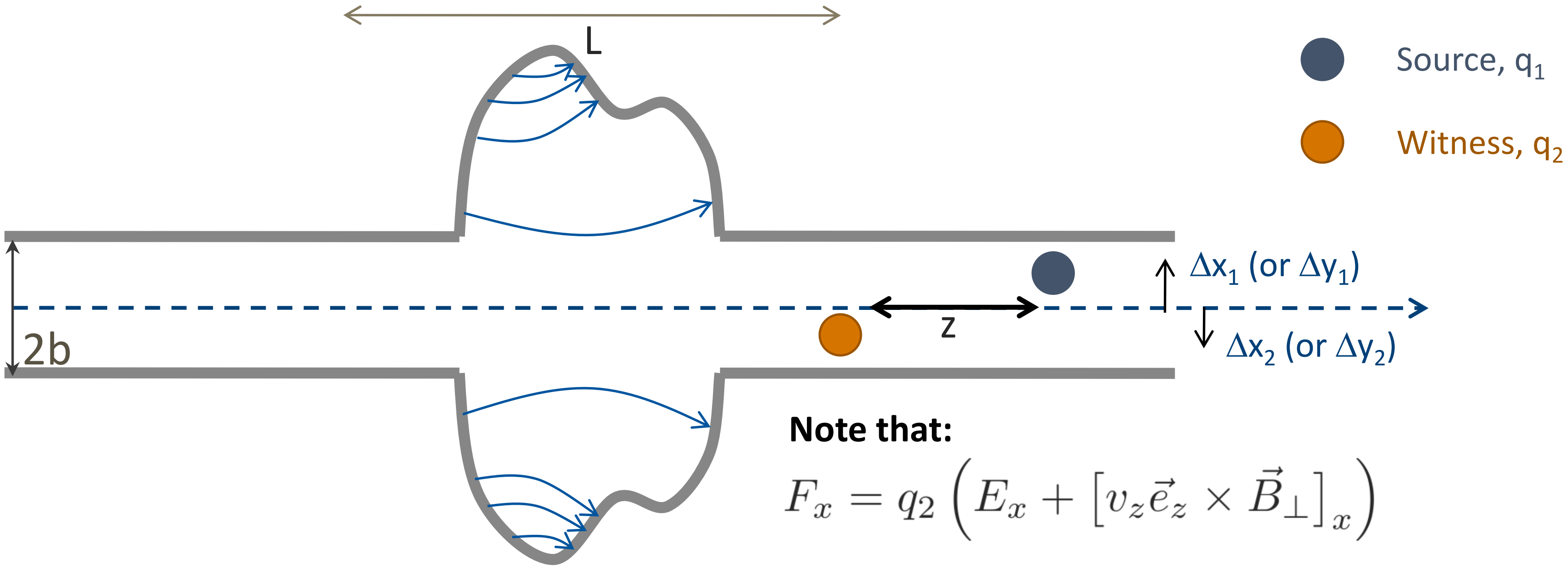
Note that:

$$F_x = q_2 \left(E_x + \left[v_z \vec{e}_z \times \vec{B}_\perp \right]_x \right)$$

- Transverse wake fields

$$\Delta E_{x2} = \int F_x(\Delta x_1, \Delta x_2, z; s)\, ds = -q_1 q_2 \overbrace{\left(W_{C_x}(z) + W_{Dx}(z)\, \Delta x_1 + W_{Q_x}(z)\, \Delta x_2 \right)}^{\text{First order expansion in transverse coordinates of source and witness particles}}$$

Transverse wake functions



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- Transverse wake fields

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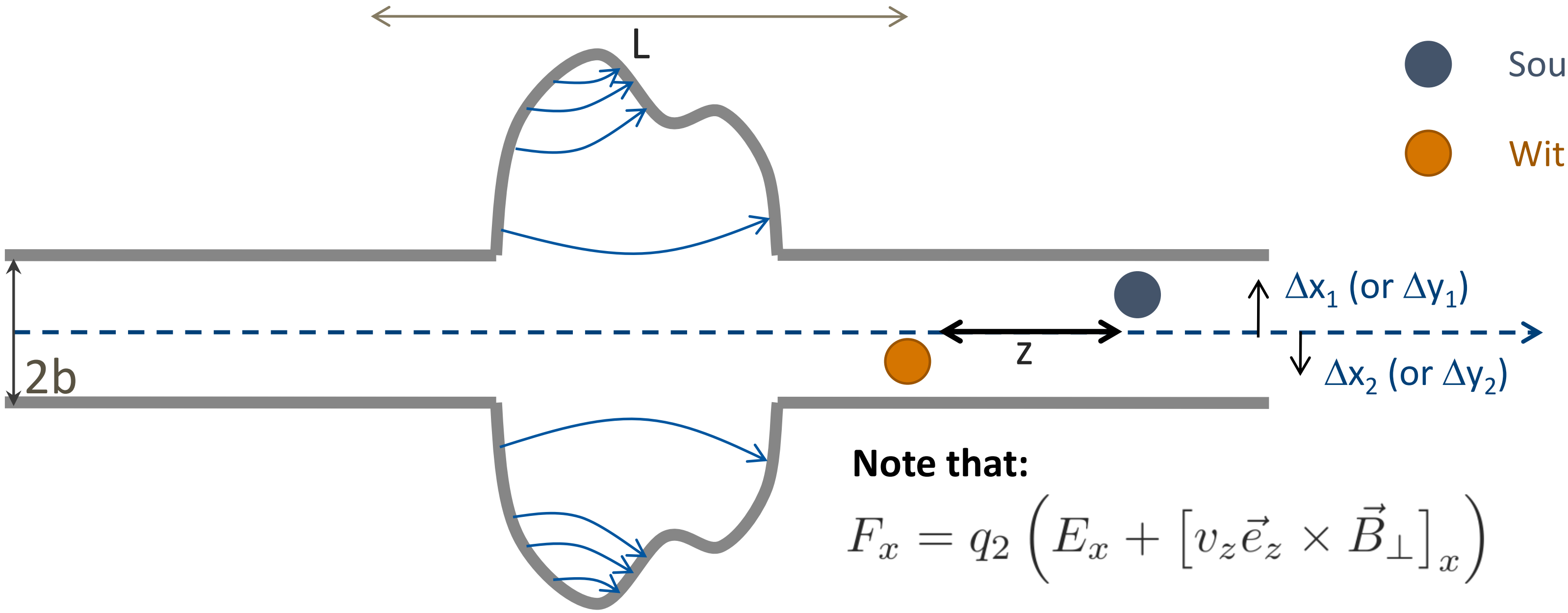
$$\longrightarrow \frac{\Delta E_{x2}}{E_0} = x_2'$$

Transverse deflecting kick of the witness particle from transverse wakes

Transverse wake functions



Note that:

$$F_x = q_2 \left(E_x + \left[v_z \vec{e}_z \times \vec{B}_\perp \right]_x \right)$$

- Transverse wake fields

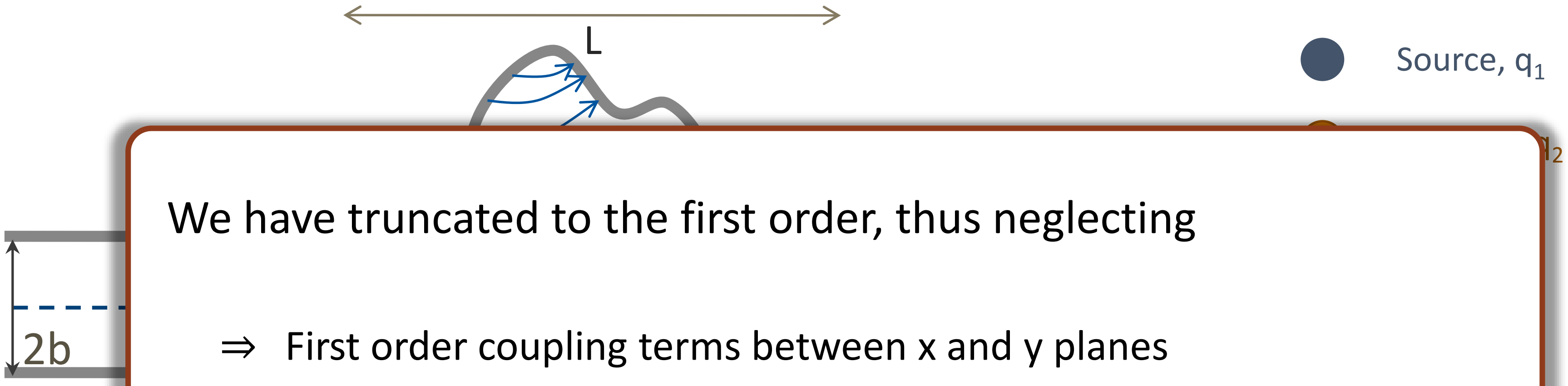
$$\Delta E_{x2} = \int F_x(\Delta x_1, \Delta x_2, z; s)\, ds = -q_1 q_2 \left(\boxed{W_{C_x}(z)} + \boxed{W_{Dx}(z)}\, \Delta x_1 + \boxed{W_{Q_x}(z)}\, \Delta x_2 \right)$$

Zeroth order for asymmetric structures
→ **Orbit offset**

Dipole wakes –
depends on **source particle**
→ **Orbit offset**

Quadrupole wakes –
depends on **witness particle**
→ **Detuning**

Transverse wake functions



We have truncated to the first order, thus neglecting

- ⇒ First order coupling terms between x and y planes
- ⇒ All higher order terms in the wake expansion (including mixed higher order terms with products of the dipolar/quadrupolar offsets)

- Transverse wake fields

$$\Delta E_{x2} = \int F_x(\Delta x_1, \Delta x_2, z; s)\, ds = -q_1 q_2 \left(\boxed{W_{C_x}(z)} + \boxed{W_{Dx}(z)}\,\Delta x_1 + \boxed{W_{Q_x}(z)}\,\Delta x_2\right)$$

Zeroth order for asymmetric structures
→ **Orbit offset**

Dipole wakes – depends on **source particle**
→ **Orbit offset**

Quadrupole wakes – depends on **witness particle**
→ **Detuning**

Transverse impedance

$$W_{D_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_1} \qquad W_{Q_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2}$$

- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
  - → Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**
  - → This is the definition of **transverse beam coupling impedance** of the element under study

Dipolar

$$\boxed{Z_{D_x}(\omega)} = i\int_{-\infty}^{\infty} W_{D_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

Quadrupolar

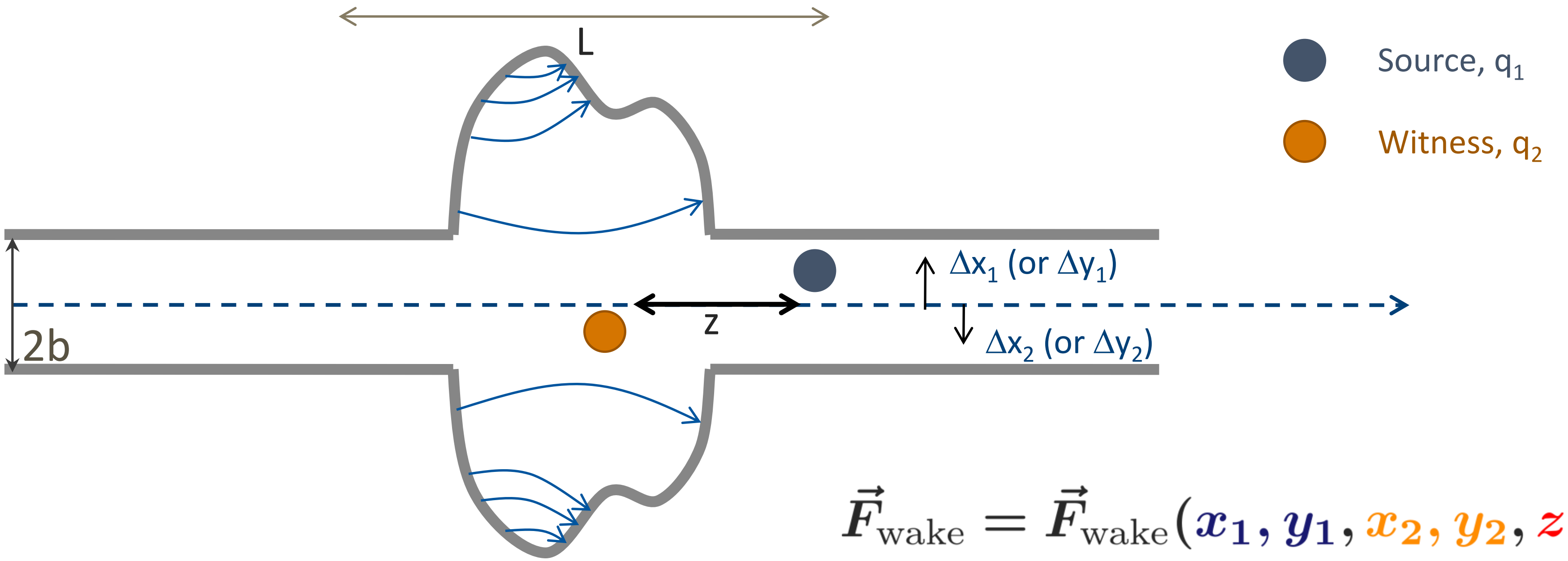
$$\boxed{Z_{Q_x}(\omega)} = i\int_{-\infty}^{\infty} W_{Q_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

$[\Omega/\mathrm{m}]$

Wake function – general definition

How can we treat these phenomena effectively in our models?

- We will use a little trick: we consider **two point particles** – one source, one probe
- The forces will be a function of the locations of both the source and the probe particles:



$$\vec{F}_{\text{wake}} = \vec{F}_{\text{wake}}(x_1, y_1, x_2, y_2, z = z_2 - z_1)$$

Because of translational invariance in z, the forces can only depend on the relative longitudinal distance between source and witness particles