

Introduction to Non-linear Longitudinal Beam Dynamics



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Introduction to Accelerator Physics

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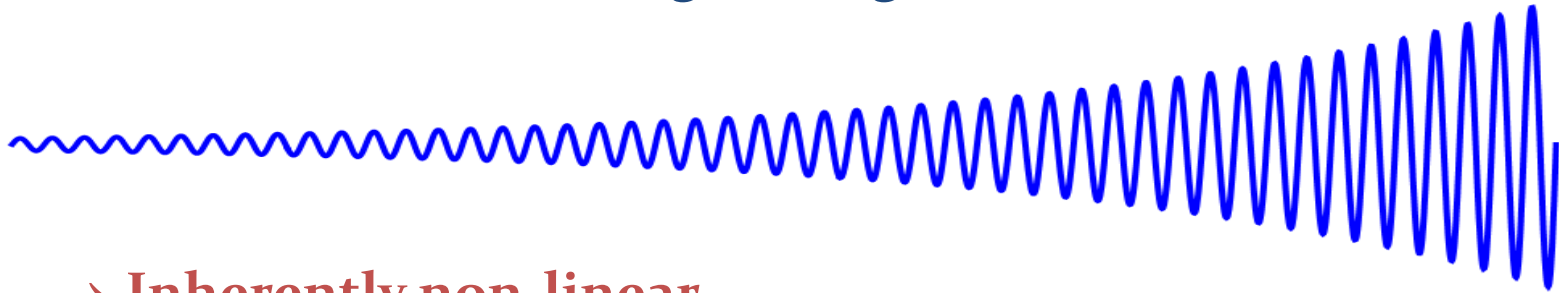
Outline

- **Introduction**
- **Linear and non-linear longitudinal dynamics**
 - Equations of motion, Hamiltonian, RF potential
- **Longitudinal manipulations**
 - Bunch length and distance control by multiple RF systems
 - Bunch rotation
- **Synchrotron frequency distribution**
 - Effect on longitudinal beam stability
- **Summary**

Introduction

Introduction

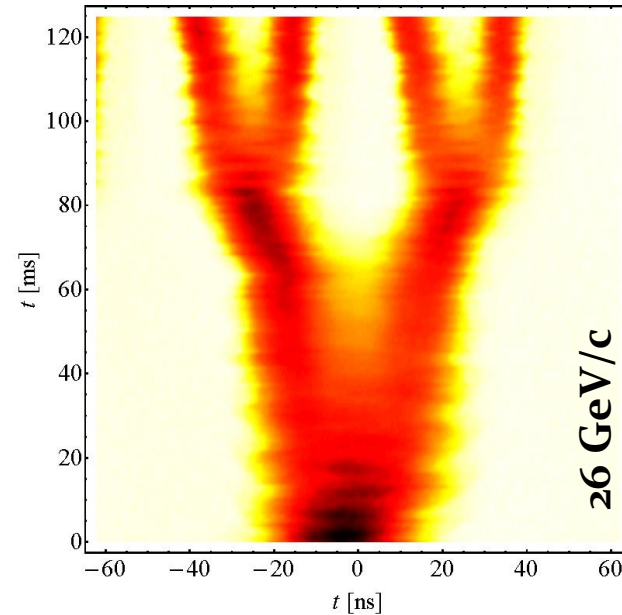
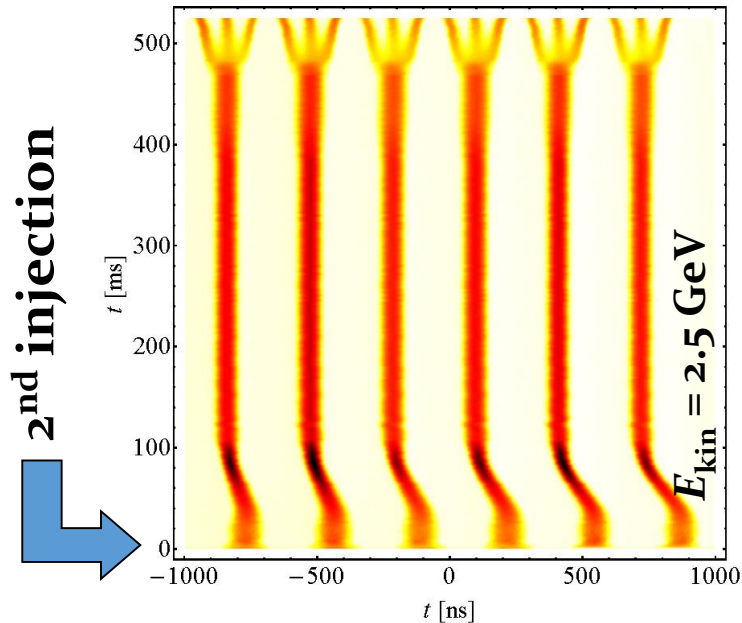
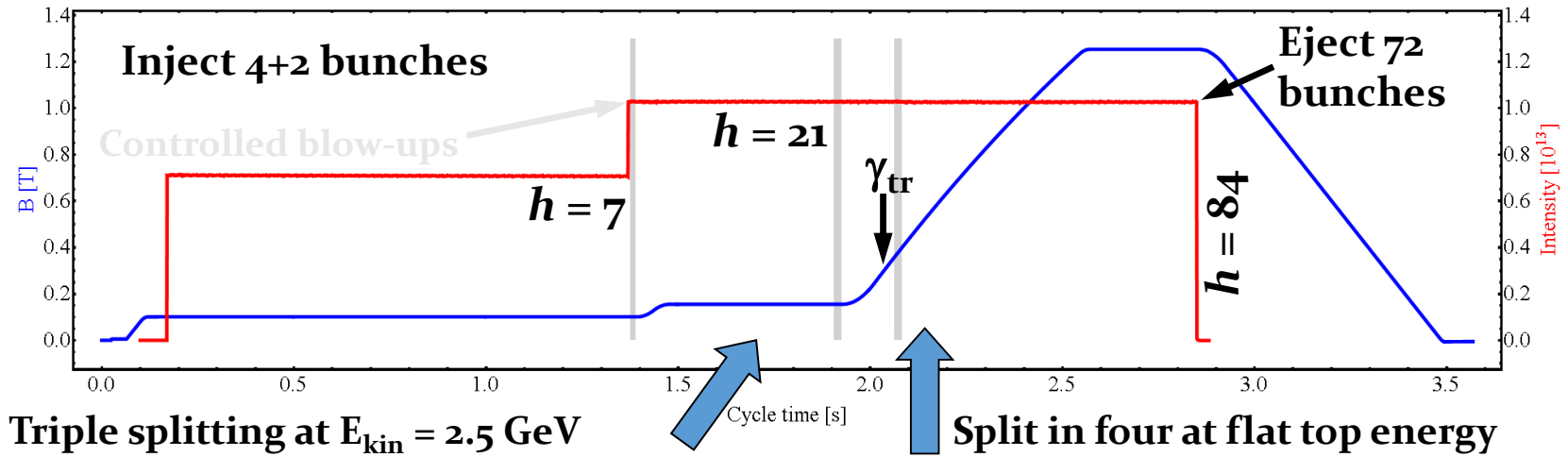
- **Signals generated by radio-frequency systems in particle accelerators are of the form $V \sin(h\omega_{\text{rev}}t)$**
 - **Resonance effect: large voltage with little effort**



- **Inherently non-linear**
- **Linear longitudinal beam dynamics only an approximation**
- **Effect of non-linearity on beam?**
- **Tools to describe and analyse non-linearity**
- **Use non-linearity to improve beam conditions**

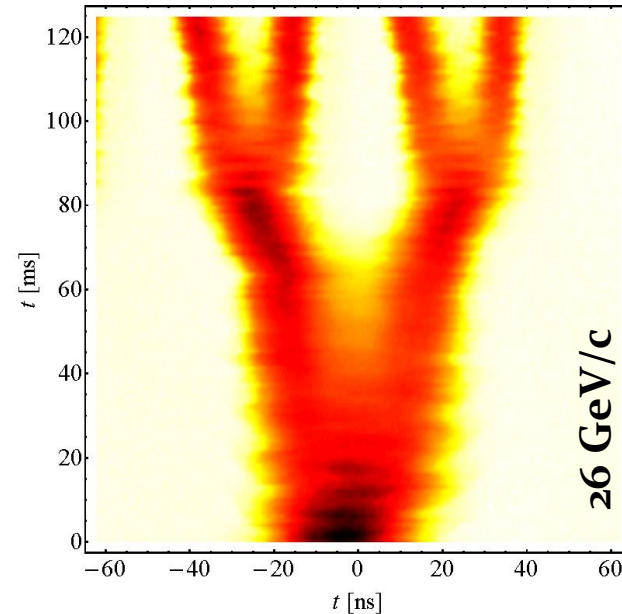
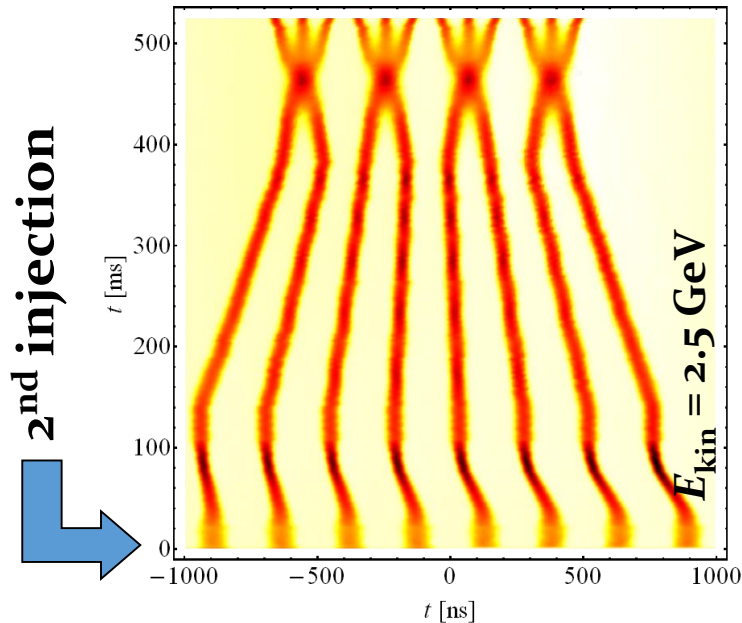
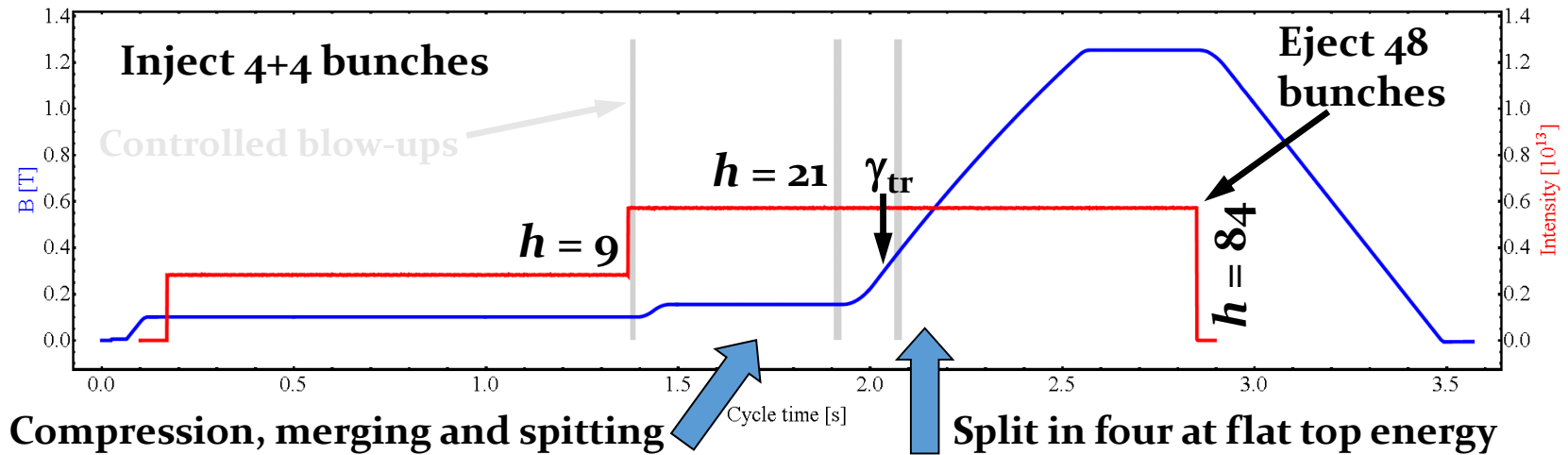
Non-linear longitudinal dynamics

Example: LHC-type beam in the CERN PS



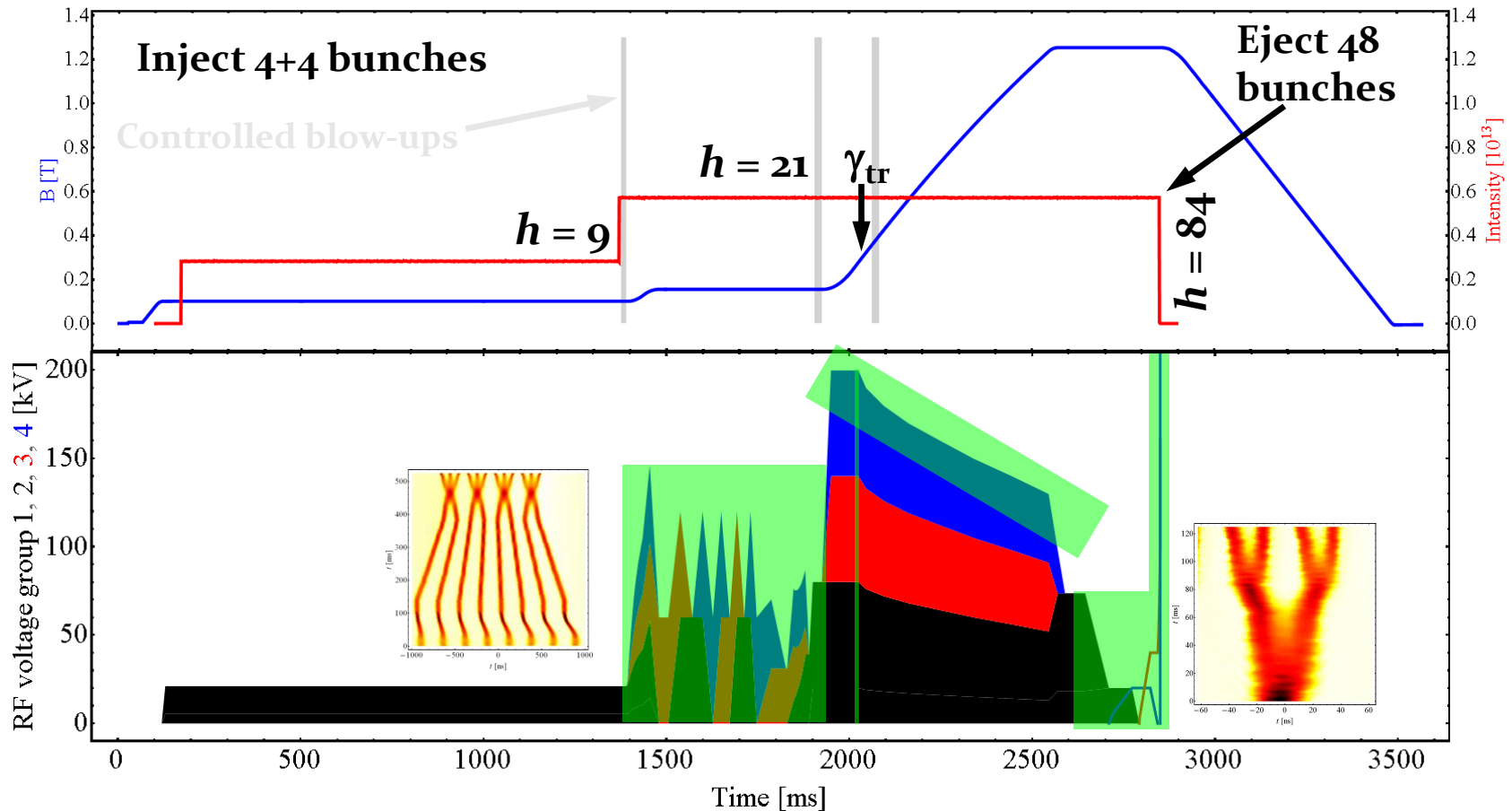
- Non-linear RF allows to control all longitudinal parameters
 → Number of bunches, bunch length and emittance, longitudinal stability

Example: LHC-type beam in the CERN PS



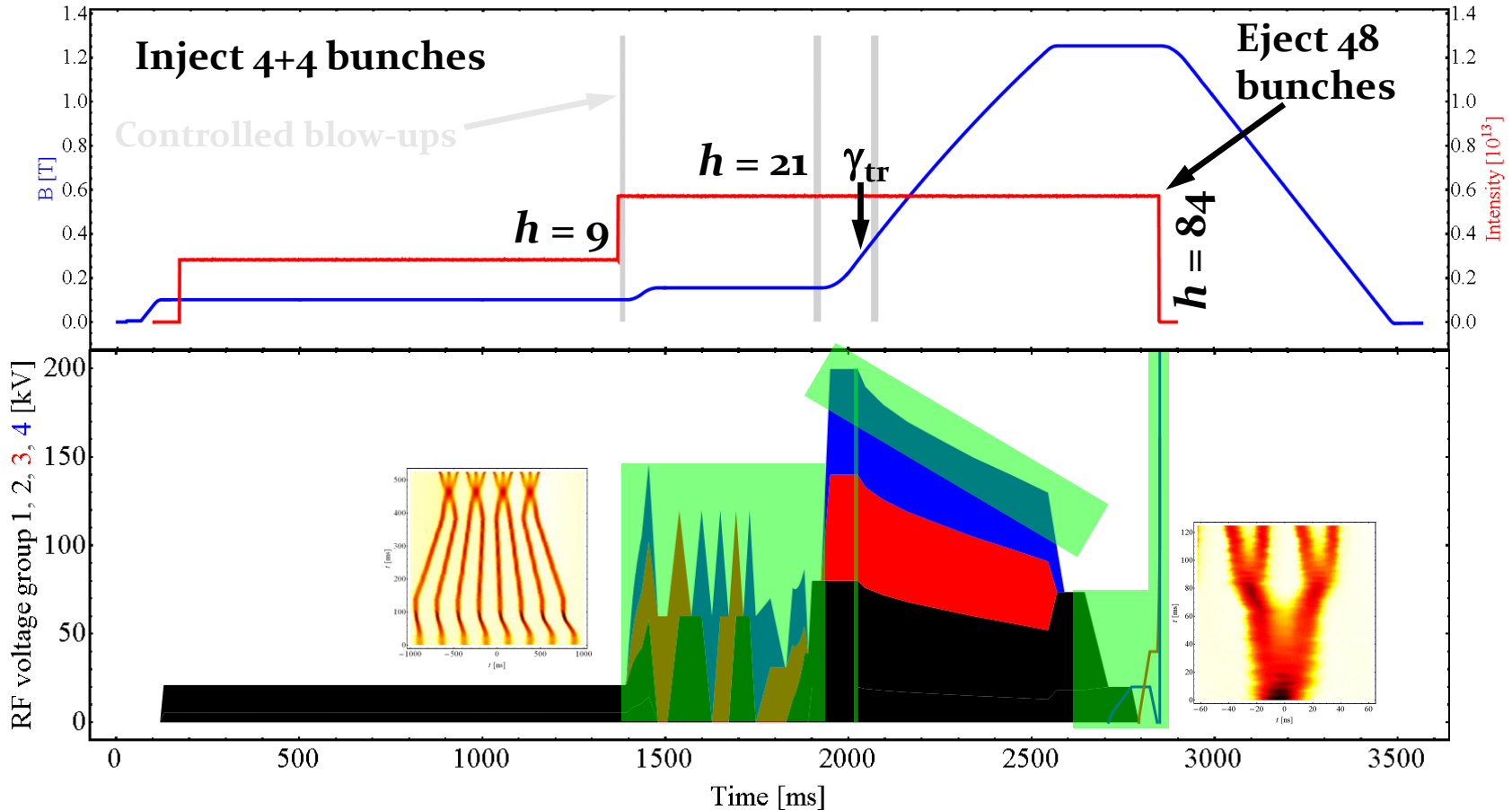
- Non-linear RF allows to control all longitudinal parameters
 → Number of bunches, bunch length and emittance, longitudinal stability

Where profit from non-linear RF?



- RF manipulation from 8 bunches in $h = 9$ to 12 in $h = 21$
- Transition crossing
- RF voltage reduction during acceleration
- Splitting at the flat-top
- Bunch shortening (rotation) before extraction

Where profit from non-linear RF?



- RF manipulation from 8 bunches in $h = 9$ to 12 in $h = 21$
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Applications

- **Introduce extra non-linearity**
 - **Bunch lengthening in double-harmonic RF system to reduce peak current (space charge)**

$$V_1 \sin(h_1 \omega_{\text{rev}} t + \phi_1) + V_2 \sin(h_2 \omega_{\text{rev}} t + \phi_2)$$

- **Short and long bunches with multi-harmonic RF systems**

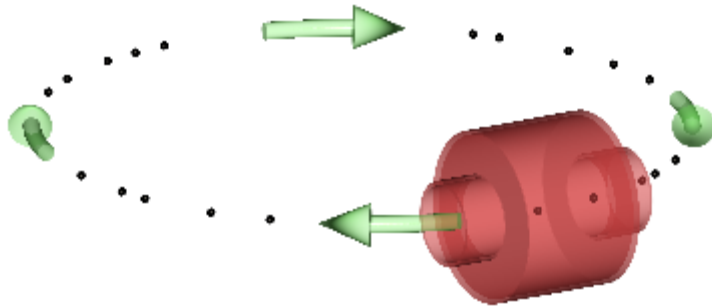
$$\sum_n V_n \sin(h_n \omega_{\text{rev}} t + \phi_n)$$

- **Adapt bunch-to-bunch distance**
- **Profit from non-linearity for beam stabilization**
 - **Stabilize beam using higher-harmonic RF**
 - **Controlled longitudinal emittance blow-up**

Linear longitudinal beam dynamics

Interaction between particles and RF

Simple accelerator model:

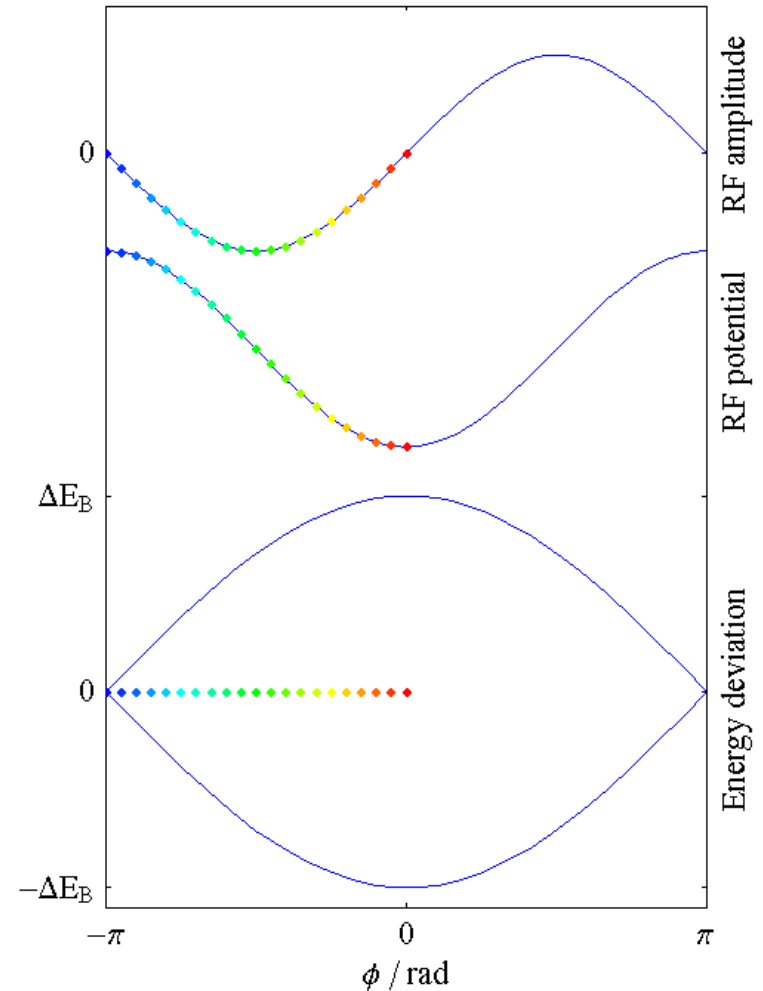


Energy dependent phase advance, ϕ :

$$\phi_{n+1} = \phi_n - 2\pi h\eta/\beta^2 \frac{\Delta E_n}{E_0}, \quad \eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

Phase dependent energy gain, ΔE :

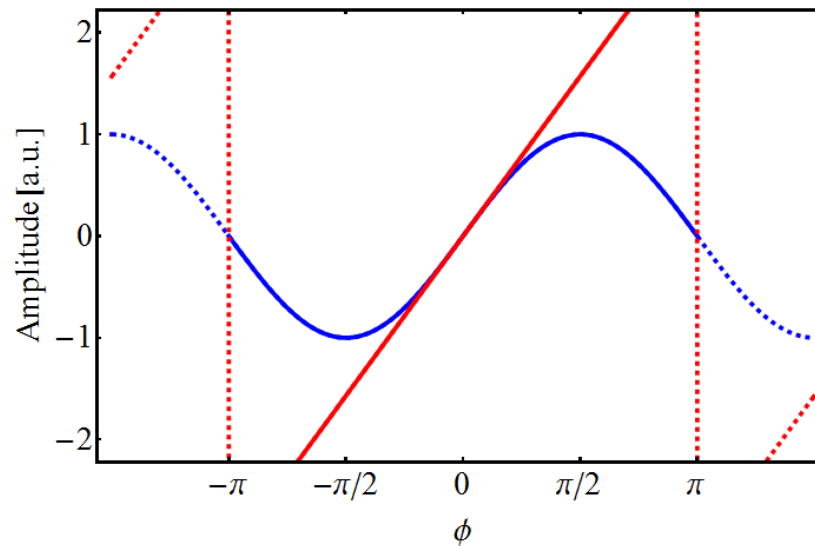
$$\Delta E_{n+1} = \Delta E_n + qVg(\phi_{n+1})$$



Works for arbitrary shape of acceleration amplitude $g(\phi)$

Linear longitudinal beam dynamics

- Usual longitudinal beam dynamics already non-linear, since RF system usually provides **sinusoidal amplitude**
- **Linear** longitudinal beam dynamics?



$$\frac{d}{dt} \phi = -\frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right) = \frac{qV}{2\pi} \phi$$

same structure

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Linear longitudinal beam dynamics

$$\frac{d}{dt}\phi = -\frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right) = \frac{qV}{2\pi} \phi$$



The Hamiltonian from the equations can be written as

$$H \left(\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 - \frac{1}{2} \frac{qV}{2\pi} \phi^2$$

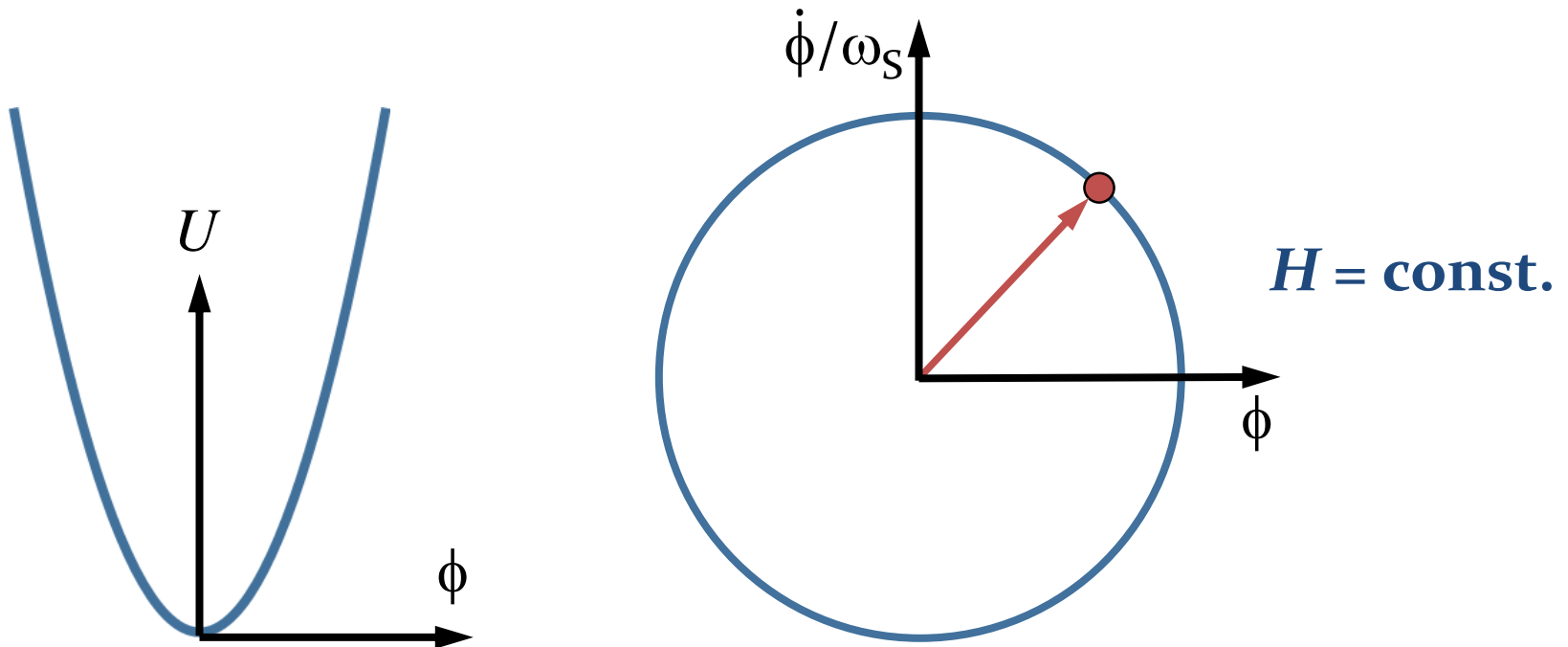
$$= -\frac{1}{2} \frac{pR}{h\eta\omega_{\text{rev}}} \dot{\phi}^2 - \frac{1}{2} \frac{qV}{2\pi} \phi^2$$

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\text{tr}}^2}$$

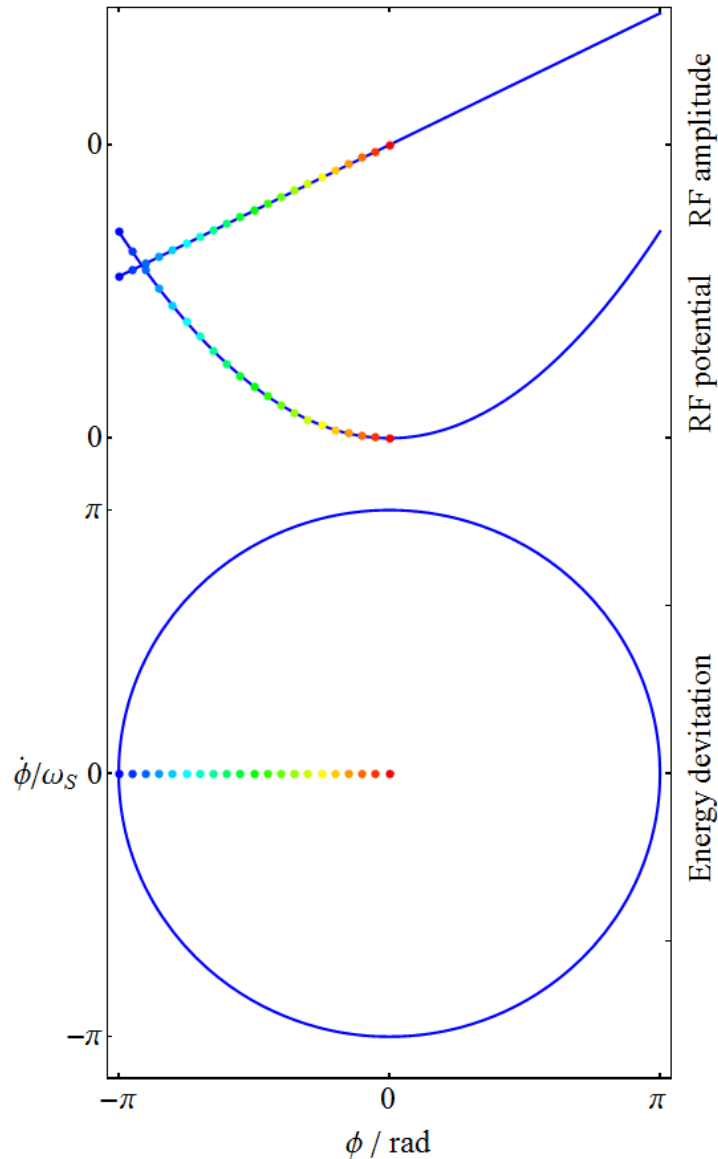
Linear longitudinal beam dynamics

$$H \left(\phi, \frac{\dot{\phi}}{\omega_S} \right) = \left(\frac{\dot{\phi}}{\omega_S} \right)^2 + \phi^2 = T + U$$

- Particles move **on circular trajectories** in ϕ - $\dot{\phi}/\omega_S$ phase space
- RF potential is **parabolic**, $U = W(\phi)$
- **Hamiltonian is constant** on these trajectories



Linear longitudinal phase space



- Simple model
- Circular trajectories
- All particles have same synchrotron frequency
- Normalized bucket area: $A_b = \pi r^2 = \pi^3$

→ Harmonic oscillator

Non-linear longitudinal beam dynamics

Introduce most simple non-linearity

RF amplitude function $V\phi \rightarrow V \sin \phi$

$$\frac{d}{dt} \Delta\phi = -\frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)$$

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right) = \frac{qV}{2\pi} (\sin \phi - \sin \phi_0)$$



$$H \left(\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 + \frac{qV}{2\pi} [\cos \phi - \cos \phi_0 + (\phi - \phi_0) \sin \phi_0]$$

with $\phi = \phi_0 + \Delta\phi$ **this becomes**

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 + \frac{qV}{2\pi} [\cos(\phi_0 + \Delta\phi) - \cos \phi_0 + \Delta\phi \sin \phi_0]$$

→ Standard longitudinal beam dynamics → Lectures F. Tecker

Introduce most simple non-linearity

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 + \frac{qV}{2\pi} [\cos(\phi_0 + \Delta\phi) - \cos \phi_0 + \Delta\phi \sin \phi_0]$$

using $\cos(\phi_0 + \Delta\phi) = \cos \phi_0 \cos \Delta\phi - \sin \phi_0 \sin \Delta\phi$

$$\simeq \cos \phi_0 \left(1 - \frac{1}{2} \Delta\phi^2 \right) - \sin \phi_0 \Delta\phi$$

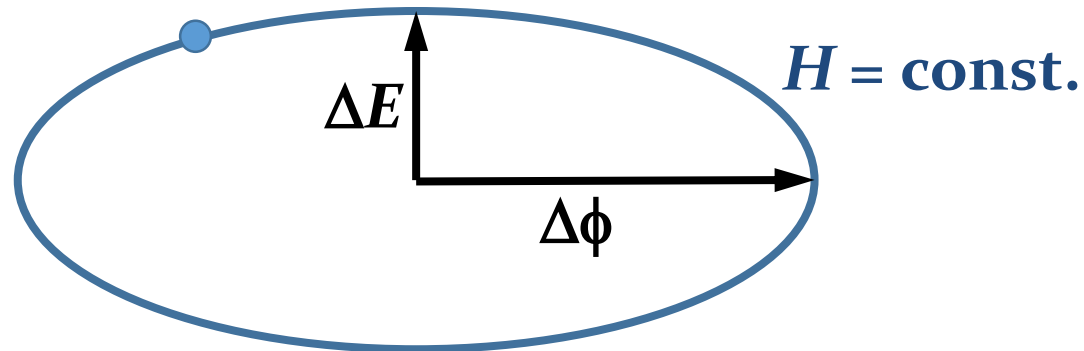
this Hamiltonian simplifies to

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) \simeq -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 - \frac{1}{2} \frac{qV}{2\pi} \Delta\phi^2 \cos \phi_0$$

Linear part of non-linear bucket

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) \simeq -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 - \frac{1}{2} \frac{qV}{2\pi} \Delta\phi^2 \cos \phi_0$$

- In the centre of the bucket, particles move on elliptical trajectories in $\Delta\phi$ - ΔE phase space
- Hamiltonian is constant on these trajectories



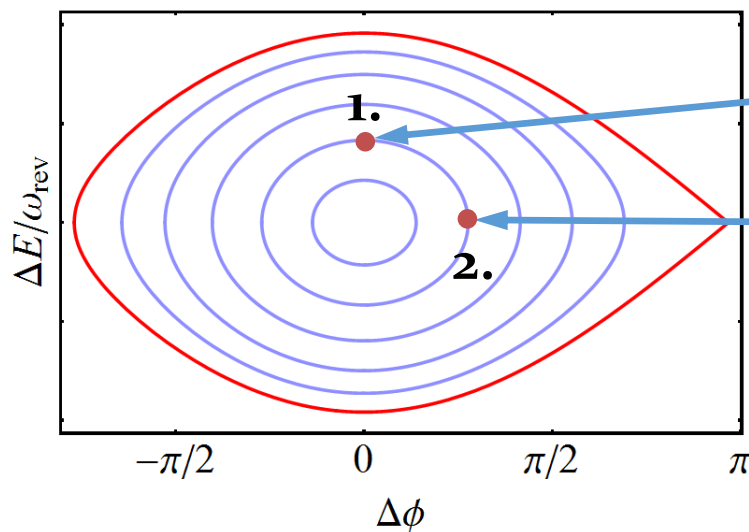
- In the bucket centre, particles oscillate with the synchrotron frequency, $\omega_S = 2\pi f_S$

$$\omega_S^2 = \frac{h\eta\omega_{\text{rev}}qV \cos \phi_0}{2\pi pR}$$

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\text{tr}}^2}$$

Longitudinal emittance

- Compare two particles on the same trajectory
 1. No phase deviation
 2. No energy deviation

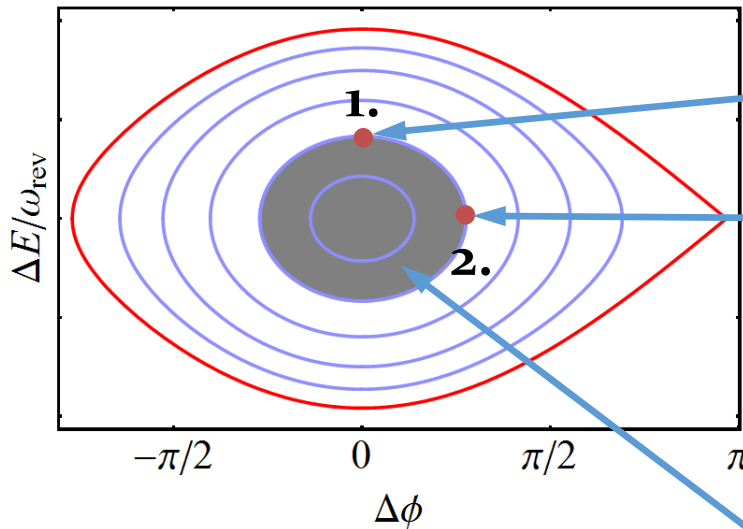


$$H \left(\Delta\phi = 0, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2$$

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} = 0 \right) = -\frac{1}{2} \frac{qV}{2\pi} \Delta\phi^2 \cos\phi_0$$

Longitudinal emittance

- Compare two particles on the same trajectory
 - No phase deviation
 - No energy deviation



$$H \left(\Delta\phi = 0, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2$$

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} = 0 \right) = -\frac{1}{2} \frac{qV}{2\pi} \Delta\phi^2 \cos \phi_0$$

Longitudinal emittance, ε_l

~ Surface occupied by particles in longitudinal phase space

→ Preserved in physical $[\pi\Delta\tau \Delta E] = eVs$

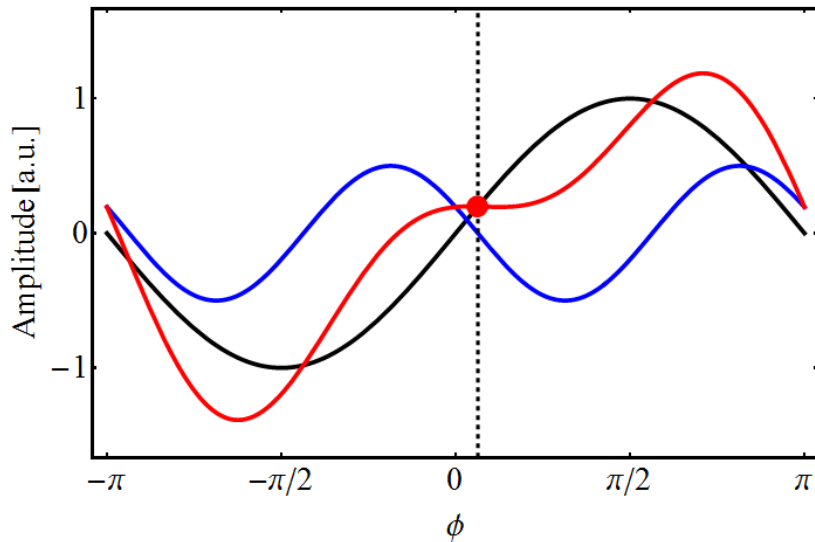
$$\varepsilon_l = \frac{2}{h\omega_{\text{rev}}} \int_{\Delta\phi_i}^{\Delta\phi_f} \Delta E(\Delta\phi) d(\Delta\phi)$$

More non-linearity: multi-harmonic RF

RF amplitude $V \sin \phi \rightarrow V [\sin \phi + r \sin(n\phi + \phi_1)]$

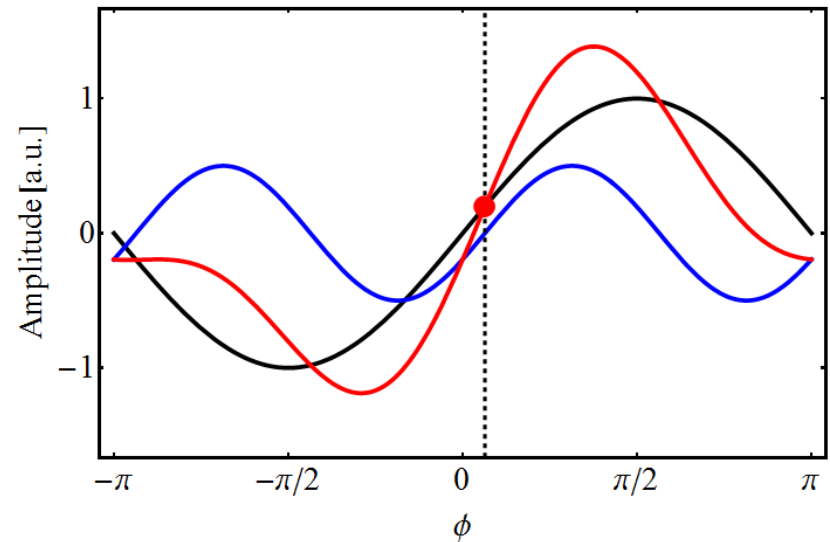
- Example case $n = 2$ and $r = 0.5$

Both RF systems in phase at bunch



- Local voltage gradient **decreased**
- Bunch is stretched
- **Lower** peak current

Both RF systems in counter-phase



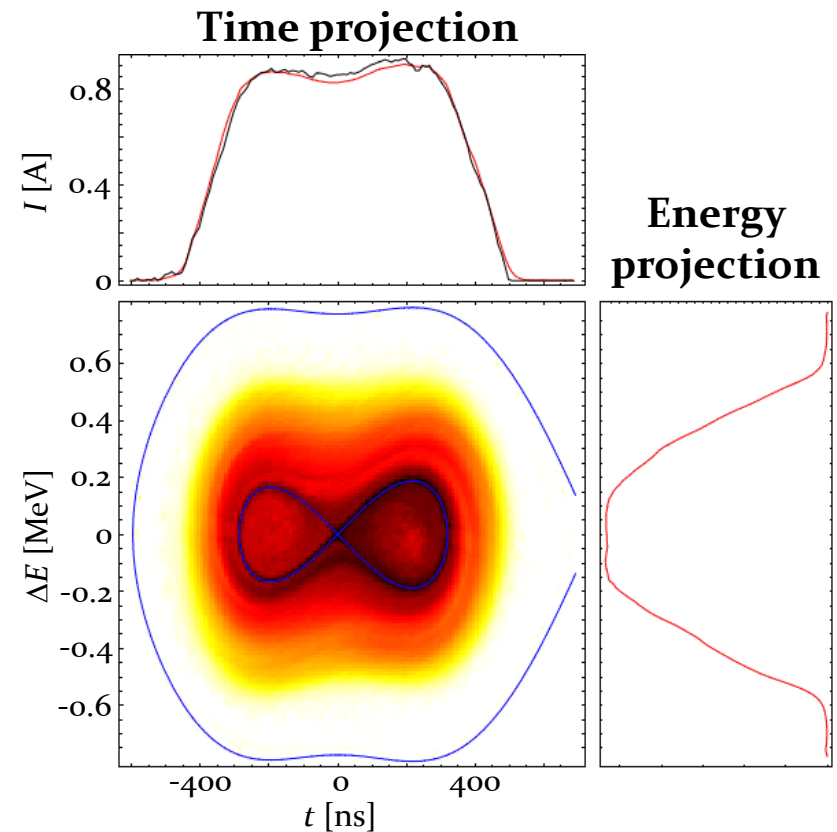
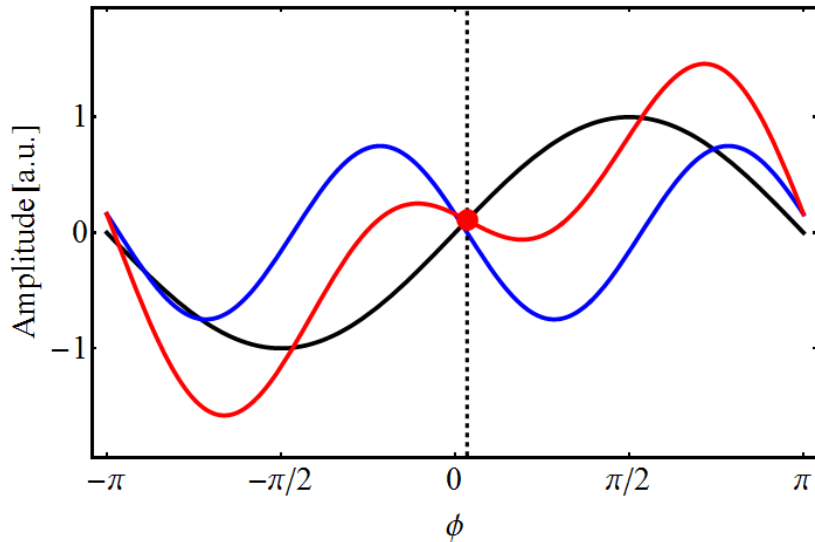
- Local voltage gradient **increased**
- Bunch is compressed
- **Higher** peak current

Example application: space charge in PSB

RF amplitude $V \sin \phi \rightarrow V [\sin \phi + r \sin(n\phi + \phi_1)]$

→ Space charge \propto instantaneous current

$V_{h=1} = 8$ kV, $V_{h=2} = 6$ kV, counter-phase



- Inverted gradient at bucket centre
- Flattened bunch with reduced peak current → Space charge reduction at low energy

Long and short bunches simultaneously

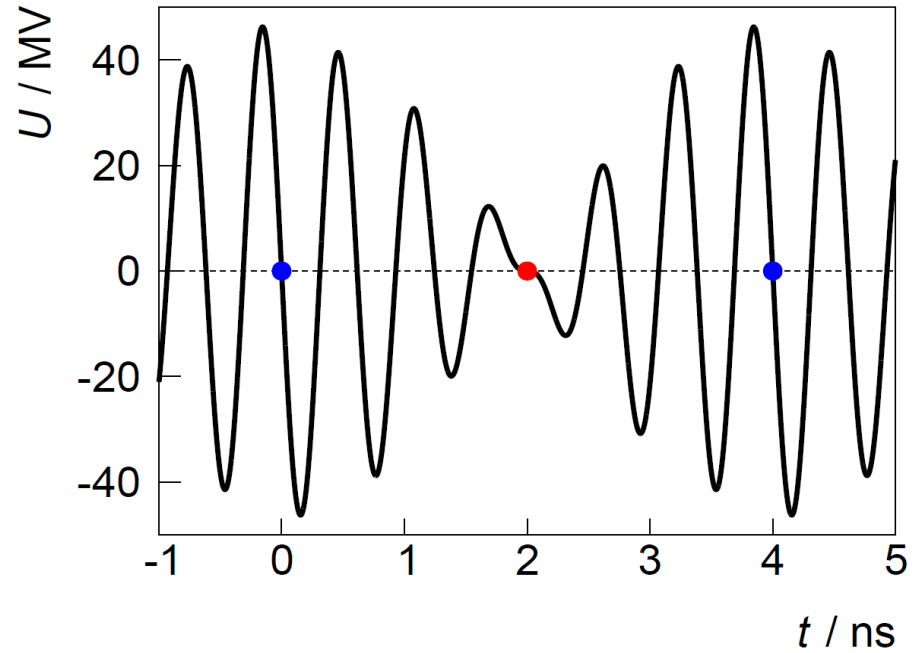
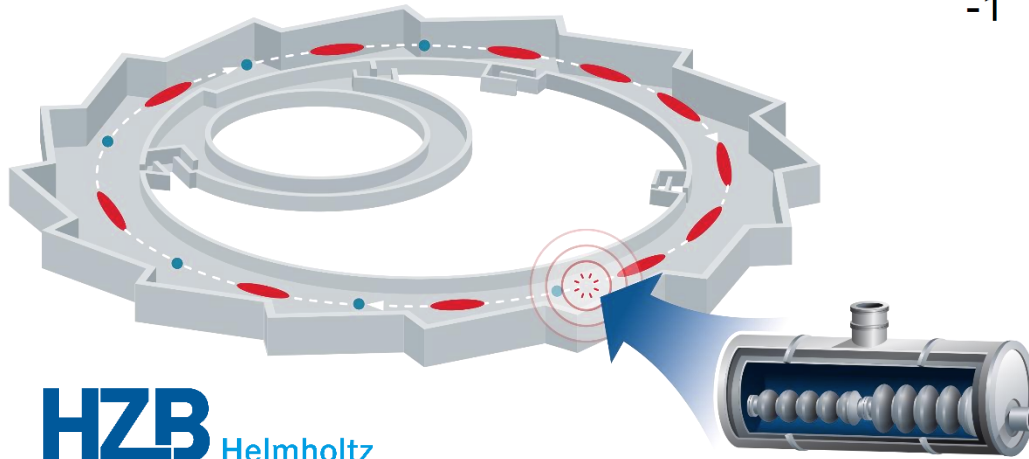
Markus Ries et al.

- Example BESSY VSR

→ Depending on user of synchrotron radiation:
need **long or short** bunches



👉 Do **long and short** bunches simultaneously!

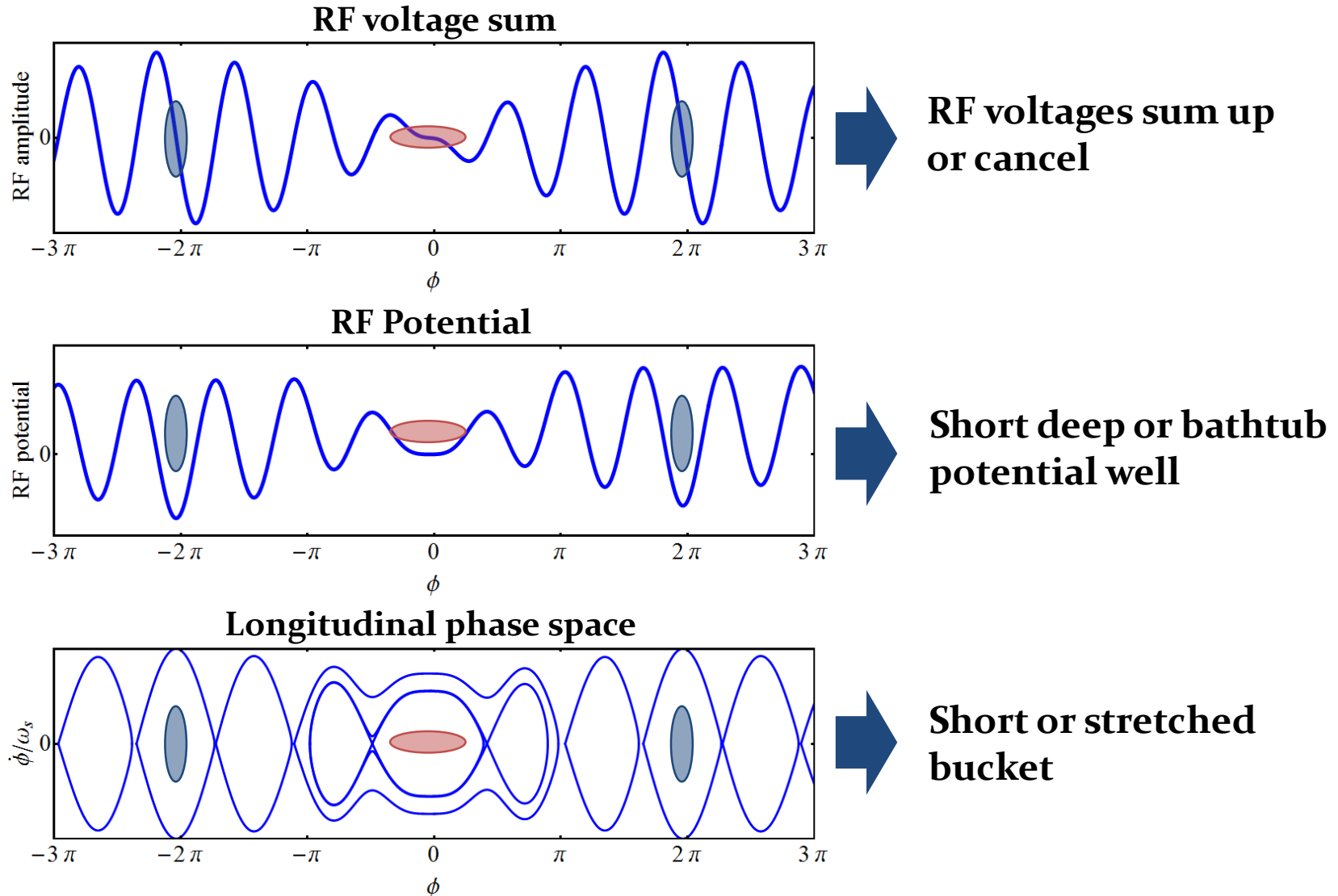


- 4×0.5 GHz NC (existing)
- 4×1.5 GHz supercond.
- 4×1.75 GHz supercond.

Bunch length modulation

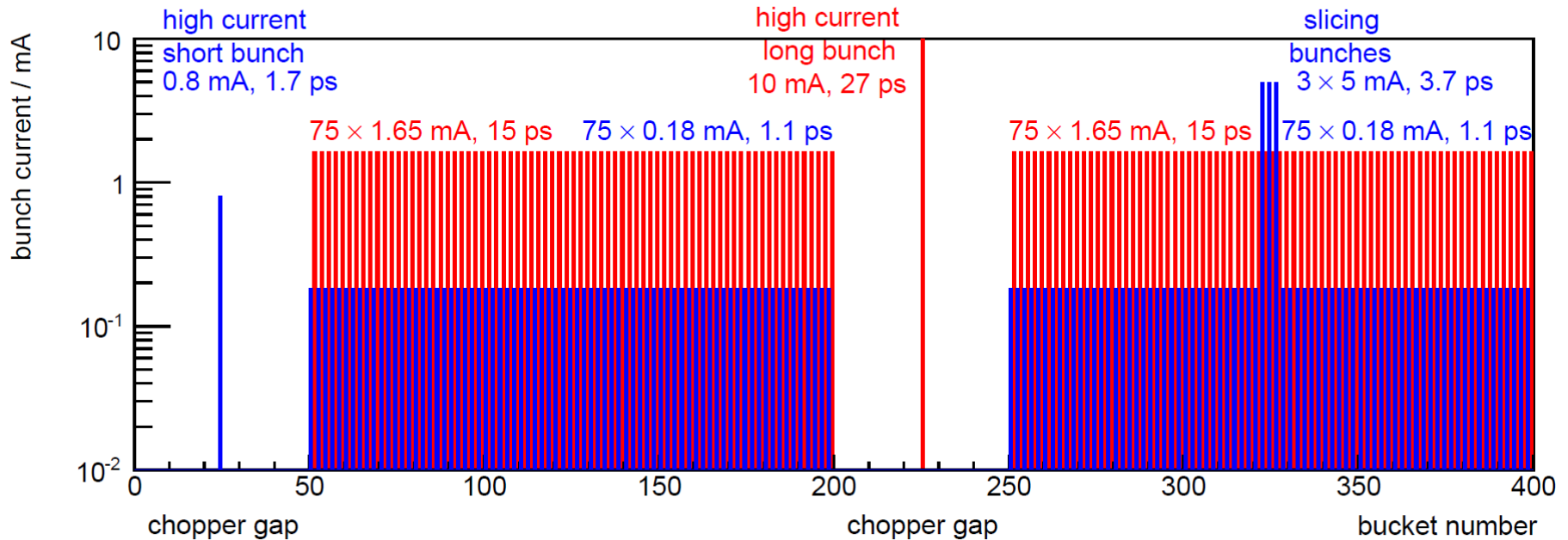
Markus Ries et al.

- Future 3-harmonic RF system for BESSY VSR



Filling pattern

Markus Ries et al.

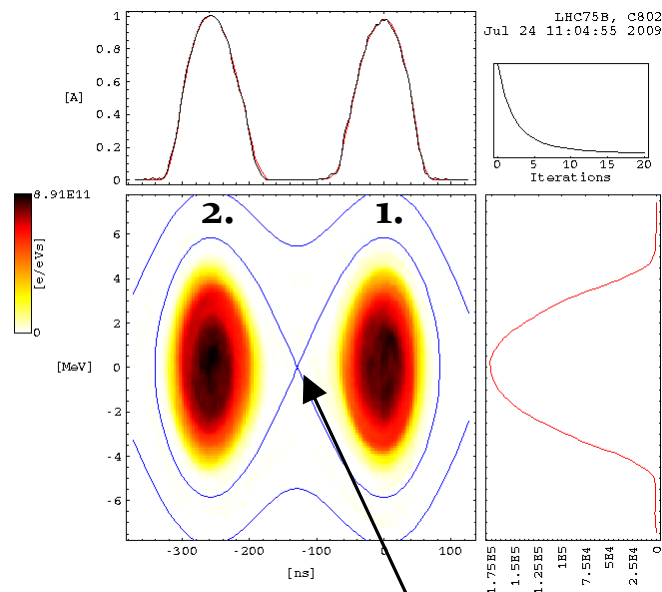


- **300 mA average current**
- **High-current single bunches**
 - short (0.8 mA) & long (10 mA)
- **Special high-current density bunches**
- 👍 **Two electron storage ring in one**

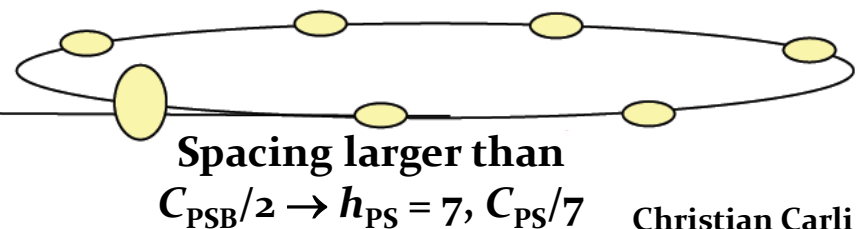
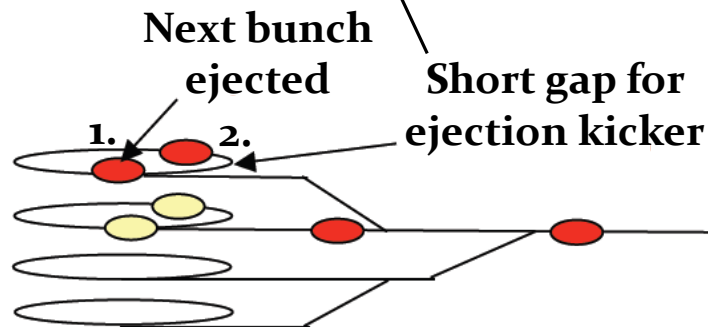
👍 **Thanks to longitudinal beam dynamics trick**

Example: adjust bunch spacing

- Was used at CERN PSB-to-PS to transfer 2 bunches at once
- Circumference ratio $C_{PS}/C_{PSB} = 4$
- Ratio virtually moved to $2/7$: use $h_{RF} = 2 + 1$



1. Add h_1 component such that bunches approach to 245 ns (small spacing) → big spacing becomes **327 ns**
2. Synchronize on h_1 to the PS
3. Trigger extraction kicker in-between the small spacing
4. **Eject two bunches per ring at a distance of 327 ns**

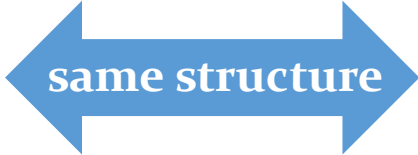


Introduce general non-linearity

Replace $V \sin \phi \rightarrow V g(\phi) \rightarrow$ **arbitrary amplitude**

Equations of motion

$$\begin{aligned} \frac{d\phi}{dt} &= -\frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right) & \frac{dq}{dt} &= \frac{\partial H}{\partial p} \\ \frac{d}{dt} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right) &= \frac{qV}{2\pi} [g(\phi) - g(\phi_0)] & \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}$$

 same structure

The Hamiltonian describing the system becomes

$$\begin{aligned} H \left(\phi, \frac{\Delta E}{\omega_{\text{rev}}} \right) &= -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2 + \frac{qV}{2\pi} \left[g(\phi_0)\phi - \int g(\phi) d\phi \right] \\ &= \text{kinetic} + \text{potential terms} \end{aligned}$$

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{\text{tr}}^2}$$

Arbitrary RF waveform

$$H\left(\phi, \frac{\Delta E}{\omega_{\text{rev}}}\right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}}\right)^2 + \frac{qV}{2\pi} \left[g(\phi_0)\phi - \int g(\phi) d\phi \right]$$

Using
$$\dot{\phi} = -\frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}}\right)$$

The Hamiltonian can be rewritten, with RF potential $W(\phi)$

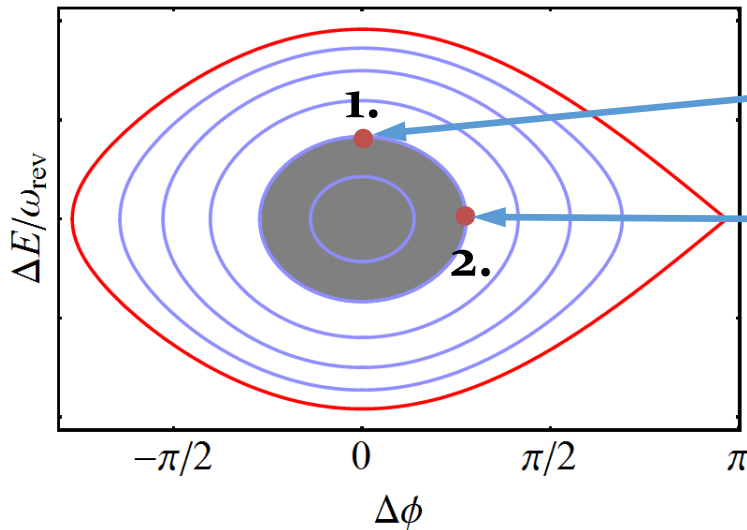
$$H(\phi, \dot{\phi}) = \frac{1}{2} \left(\frac{\dot{\phi}}{\omega_S}\right)^2 + W(\phi)$$

$$W(\phi) = \frac{1}{\cos \phi_0} \left[\int g(\phi) d\phi - g(\phi_0)\phi \right]$$

Longitudinal beam manipulations using non-linearity

Change RF voltage to change bunch length? ³¹

→ Calculate aspect ratio of bucket trajectories



$$H \left(\Delta\phi = 0, \frac{\Delta E}{\omega_{\text{rev}}} \right) = -\frac{1}{2} \frac{h\eta\omega_{\text{rev}}}{pR} \left(\frac{\Delta E}{\omega_{\text{rev}}} \right)^2$$

$$H \left(\Delta\phi, \frac{\Delta E}{\omega_{\text{rev}}} = 0 \right) = -\frac{1}{2} \frac{qV}{2\pi} \Delta\phi^2 \cos \phi_0$$

Equating both sides gives

$$\left(\frac{\Delta E}{\Delta\tau} \right)^2 = \frac{qV}{2\pi} E\beta^2 h\omega_{\text{rev}}^2 \frac{\cos \phi_0}{\eta}$$

with emittance as $\varepsilon_l = \pi\Delta\tau\Delta E = \text{const.}$ →

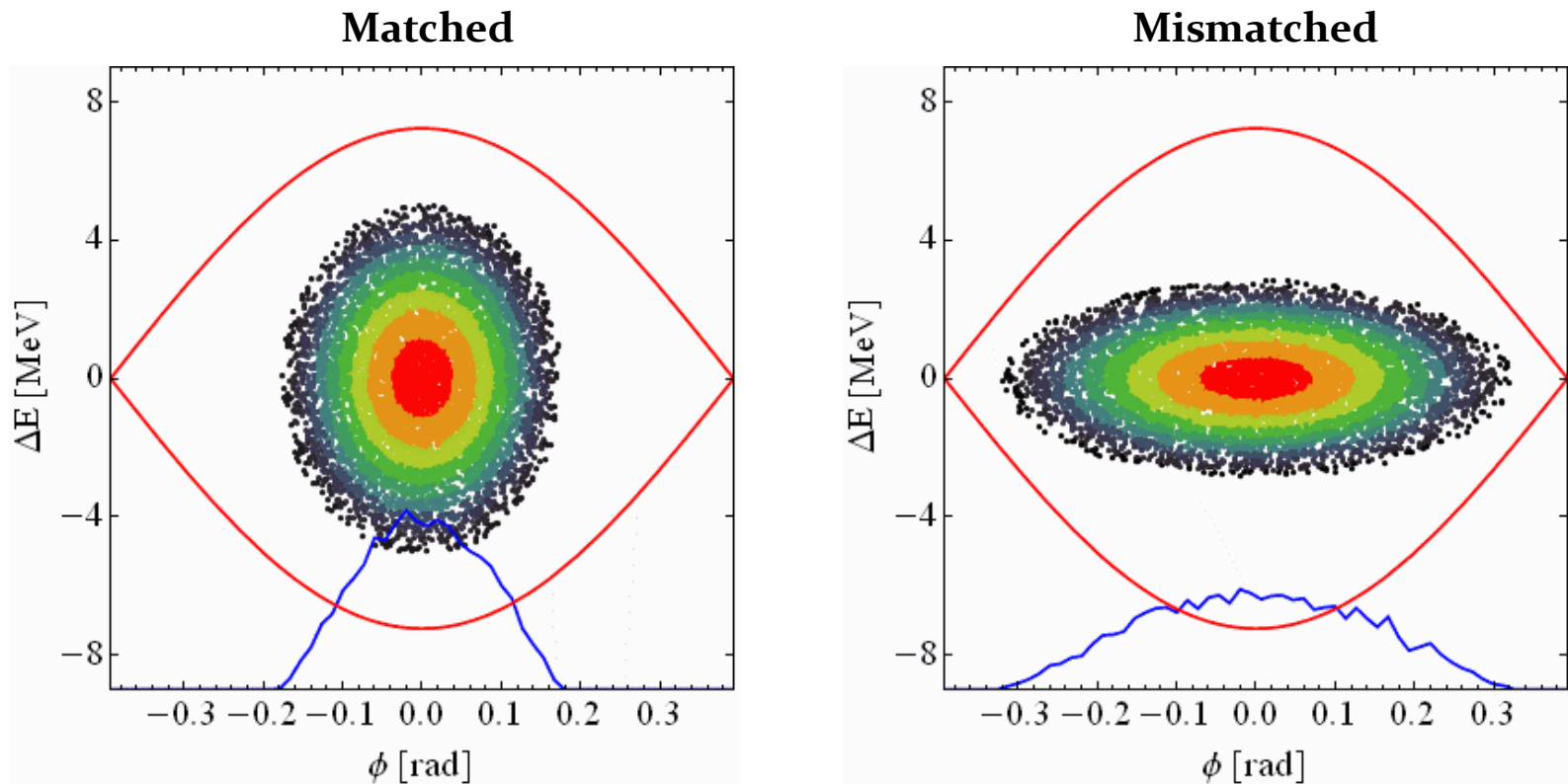
$$\Delta\tau \propto \frac{1}{\sqrt[4]{V}}$$

→ **Not efficient at all**

→ **16 times more RF voltage needed to cut bunch length in half**

Abrupt change of RF voltage

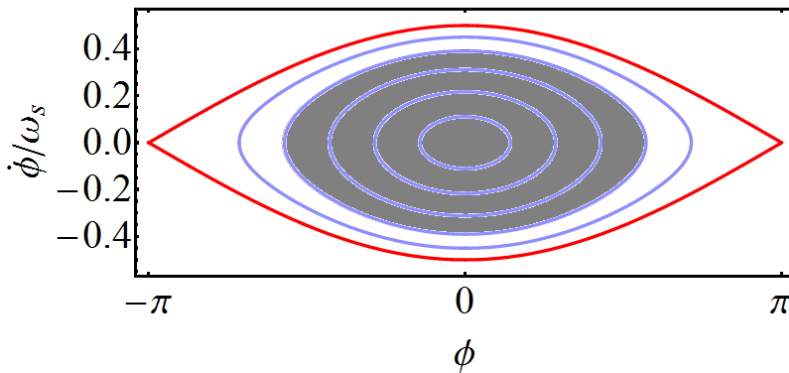
- Individual particles in matched bunch oscillate **but no macroscopic motion**
- Abruptly changing the RF voltage flips **particles to new trajectories**



- The bunch distribution seems to rotate
- Exchange of bunch length and momentum spread

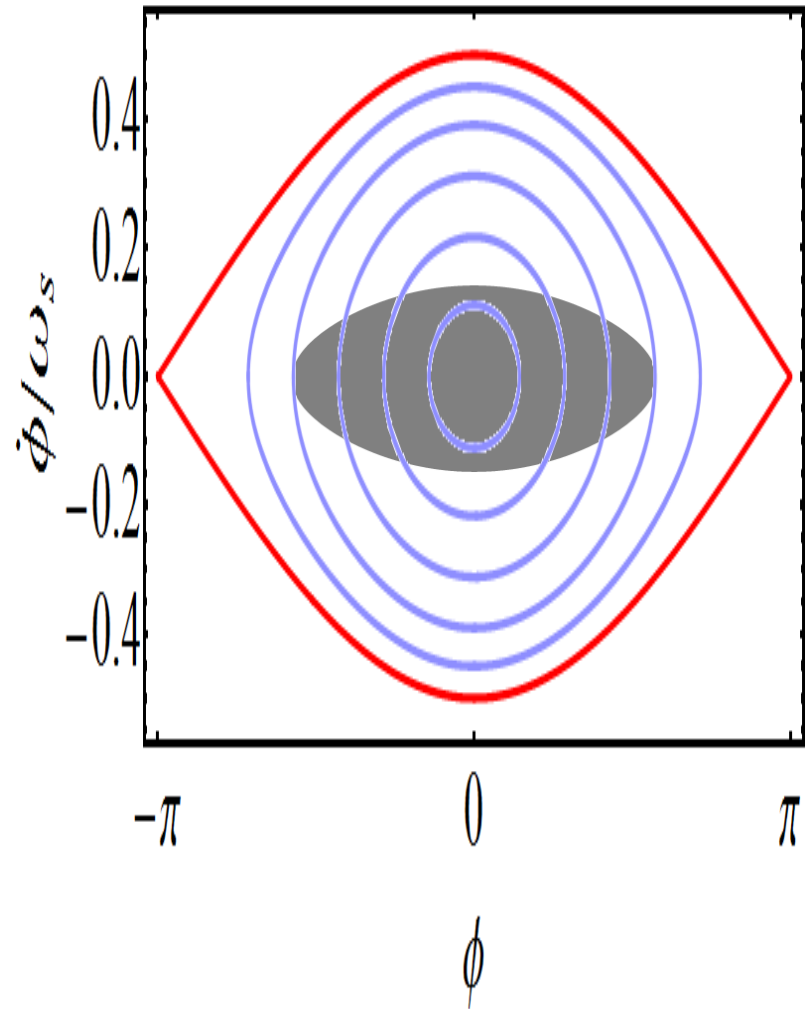
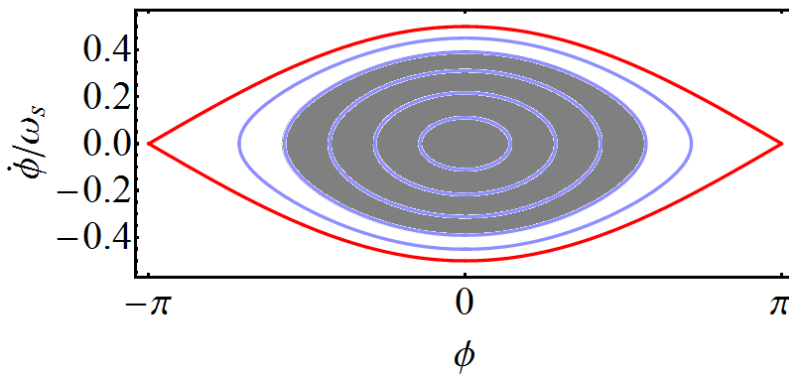
Introduce sudden change: bunch rotation

- Quickly exchange longitudinal phase space behind bunch
- Increase RF voltage much faster than period of f_s



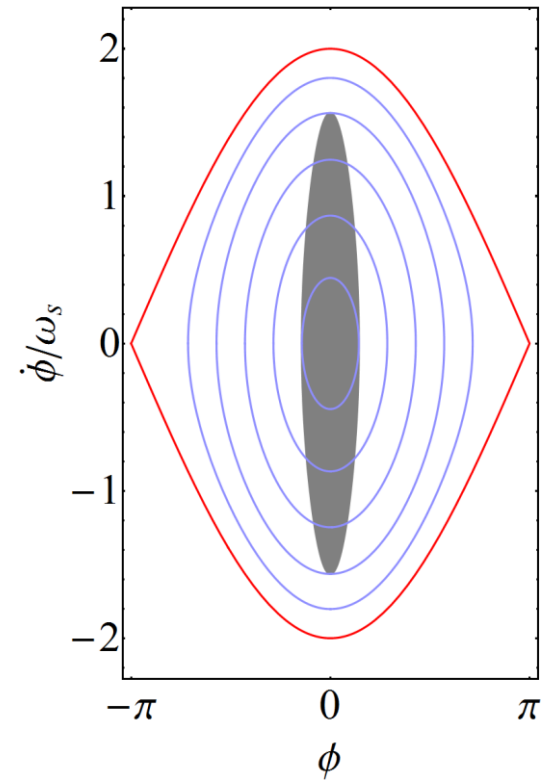
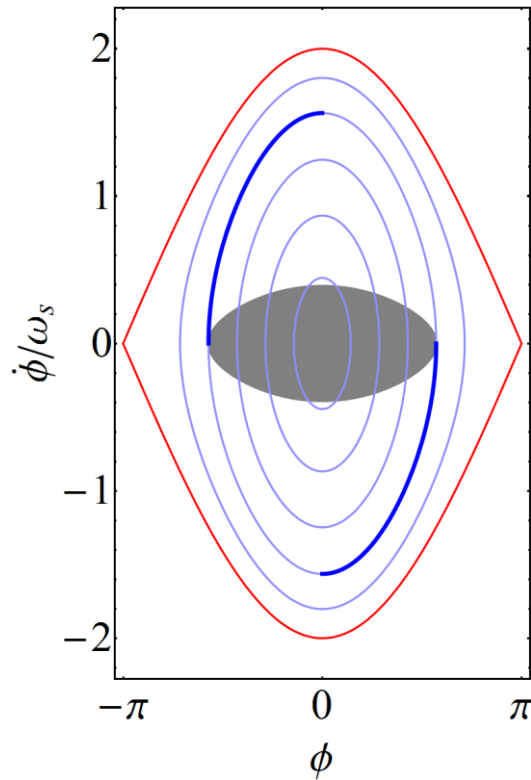
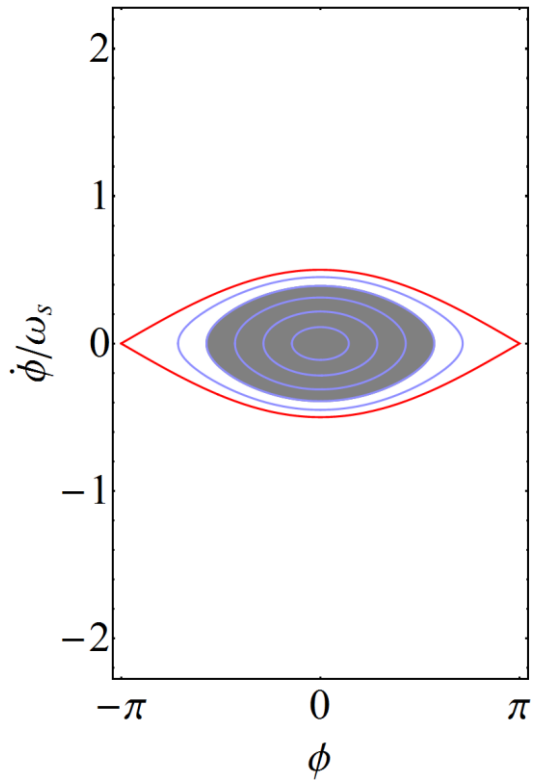
Introduce sudden change: bunch rotation

- Quickly exchange longitudinal phase space behind bunch
- Increase RF voltage much faster than period of f_s



Introduce sudden change: bunch rotation

→ Switch RF voltage much faster than period of f_s



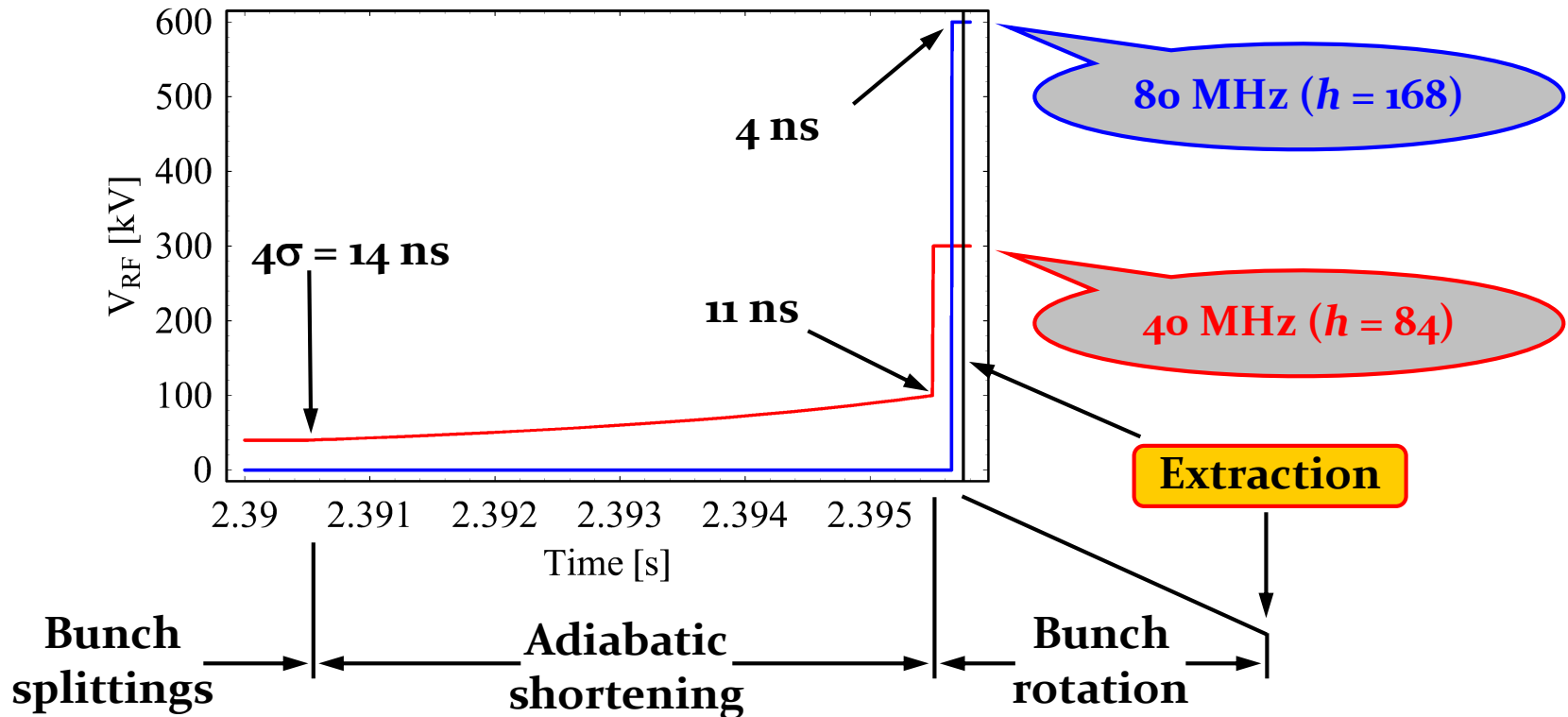
$$V_i \propto \left(\frac{\Delta E_i}{\Delta \tau_i} \right)^2$$

$$V_f \propto \left(\frac{\Delta E_f}{\Delta \tau_f} \right)^2$$

$$\frac{\Delta \tau_f}{\Delta \tau_i} = \frac{\Delta E_i}{\Delta E_f} = \sqrt{\frac{V_i}{V_f}}$$

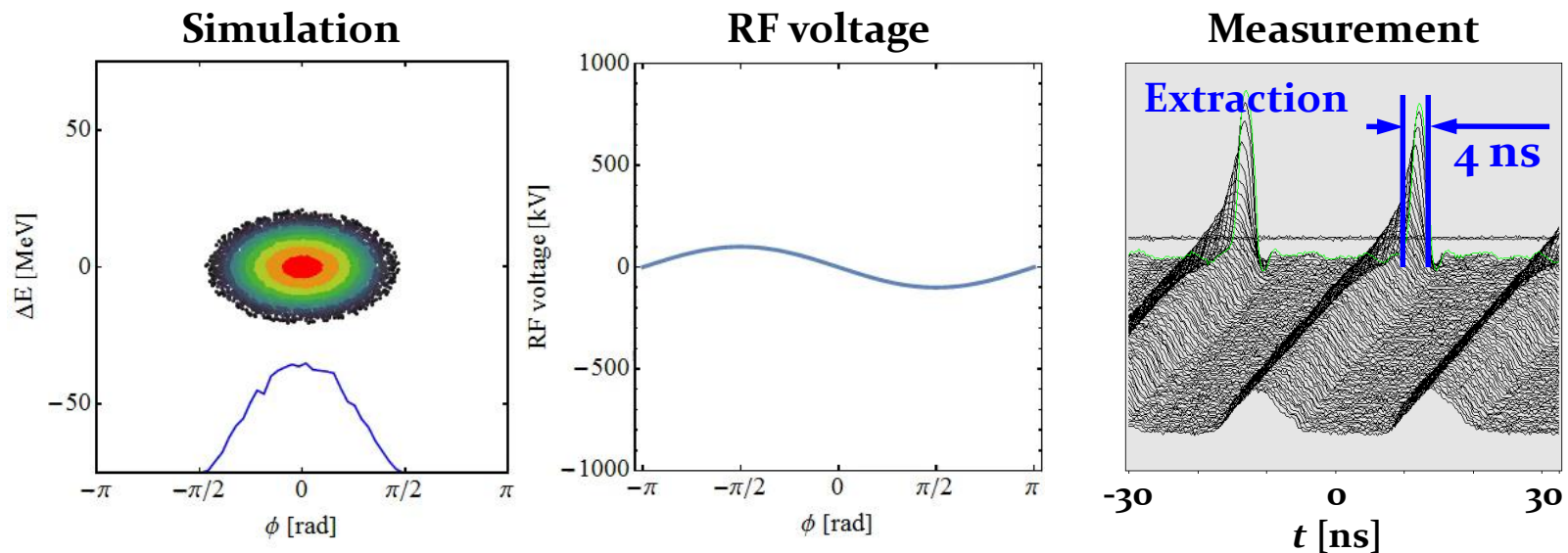
Example: PS to SPS transfer at CERN

- Fit 14 ns long bunches into 5 ns long buckets in the SPS
 → **Double-step bunch rotation**



Example: rotation at PS-SPS transfer

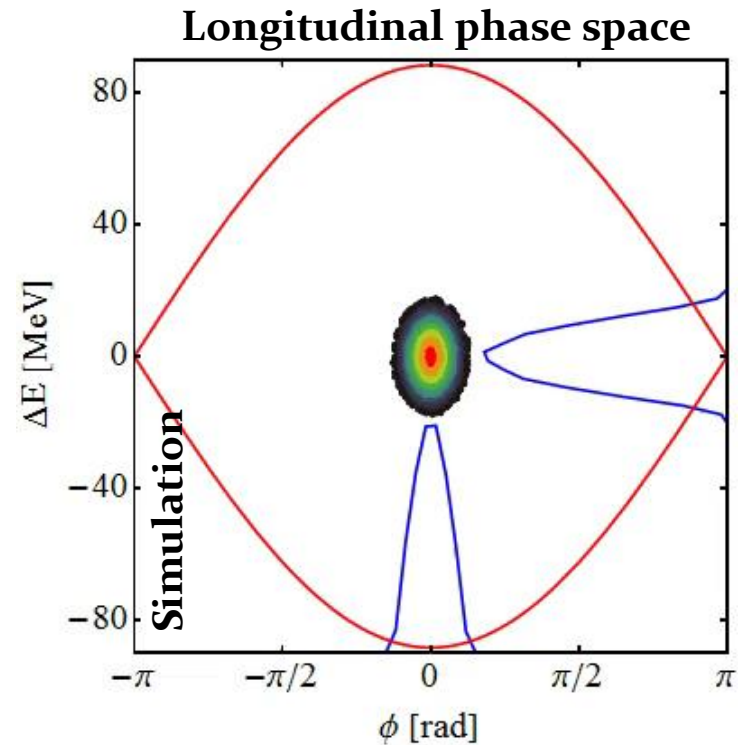
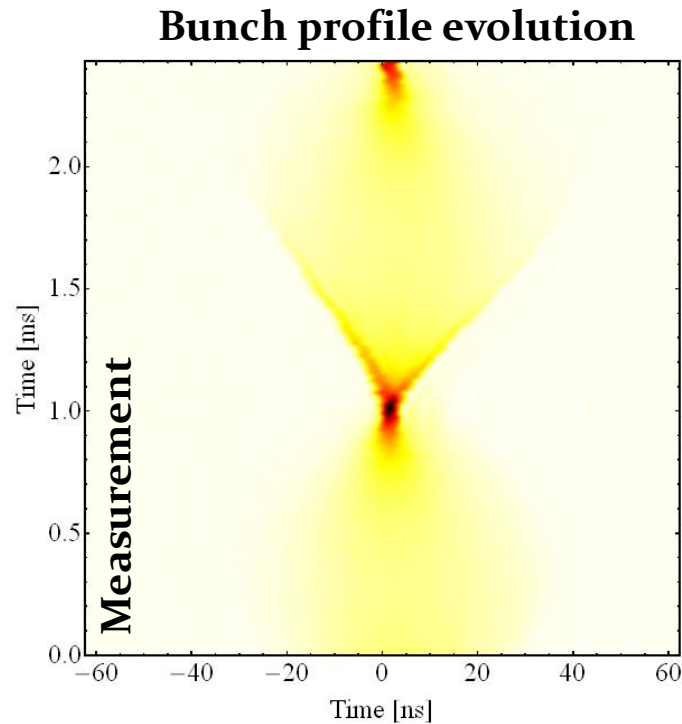
- Bunch length now proportional to \sqrt{V} and not $\sqrt[4]{V}$
- Can save enormous RF voltage
- Bunch shortening from 14 ns to 4 ns (ratio ~ 3.5)
- Starting from 100 kV at 40 MHz
- Slow shortening would require $100 \text{ kV} \cdot 3.5^4 \sim 15 \text{ MV}$
- Installed RF voltage is only about 1.2 MV



Profiting from the non-linear rotation

Need large momentum spread for slow extraction

1. **Jump RF phase such that bunch at unstable fixed point**
2. **Jump back**
3. **Let bunch rotate, switch RF off at large momentum spread**

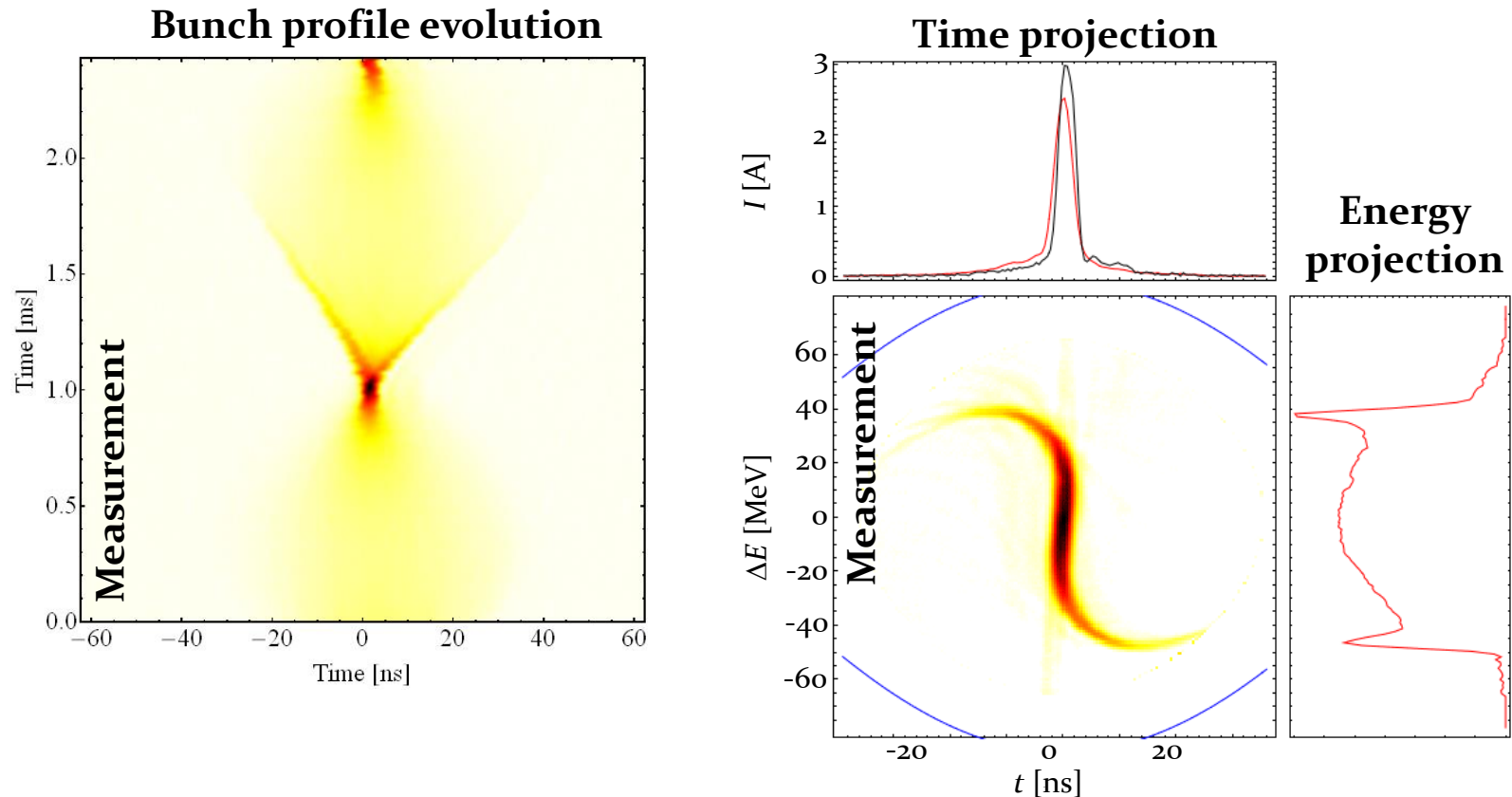


→ **Non-linearly of bunch rotation helps**

Example: using the non-linearity

Need large momentum spread for slow extraction

1. Jump RF phase such that bunch at unstable fixed point
2. Jump back
3. Let bunch rotate, switch RF off at large momentum spread



→ Almost constant momentum distribution after rotation

Synchrotron frequency distribution

General synchrotron frequency

- Synchrotron frequency depends on trajectory
- Calculate average velocity for given trajectories in longitudinal phase space → **Action angle, J**

$$J(H) = \frac{1}{2\pi\omega_S} \oint \dot{\phi}(\phi) d\phi$$

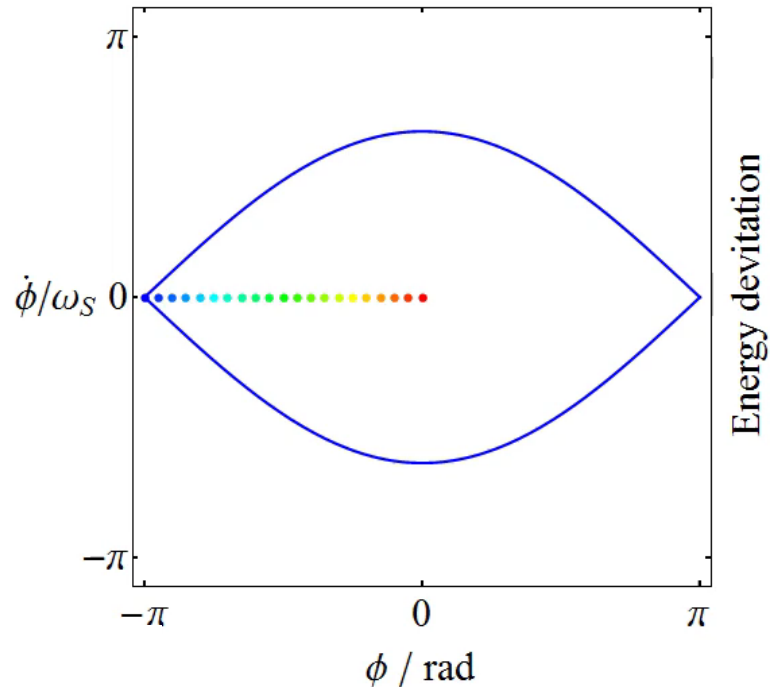
The angular frequency becomes

$$\omega(H) = \frac{d}{dJ} H$$

General expression for ω_S

$$\frac{\omega(H)}{\omega_S} = \frac{\sqrt{2\pi}}{\int_{\phi_l}^{\phi_u} \frac{1}{\sqrt{H/\omega_S^2 - W(\phi)}} d\phi}$$

(for bucket boundaries $\phi_l \rightarrow \phi_u$)



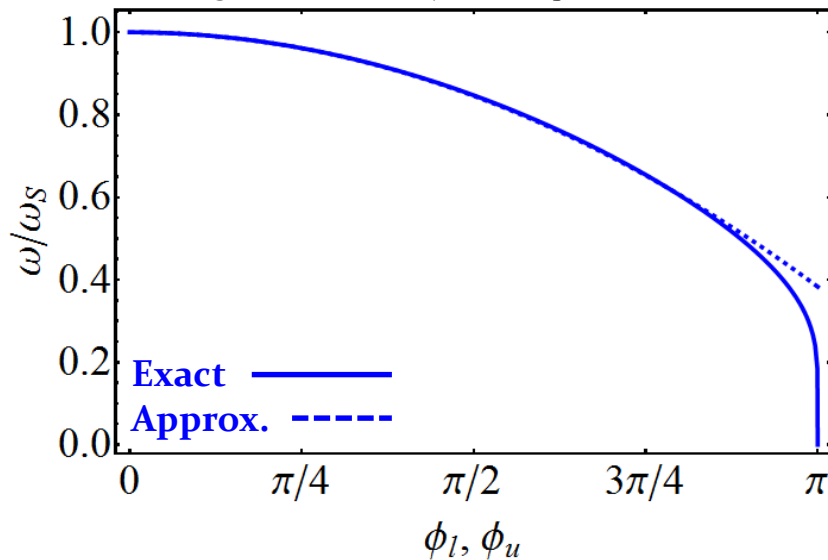
Distribution for stationary bucket

- **Single-harmonic RF in stationary bucket**

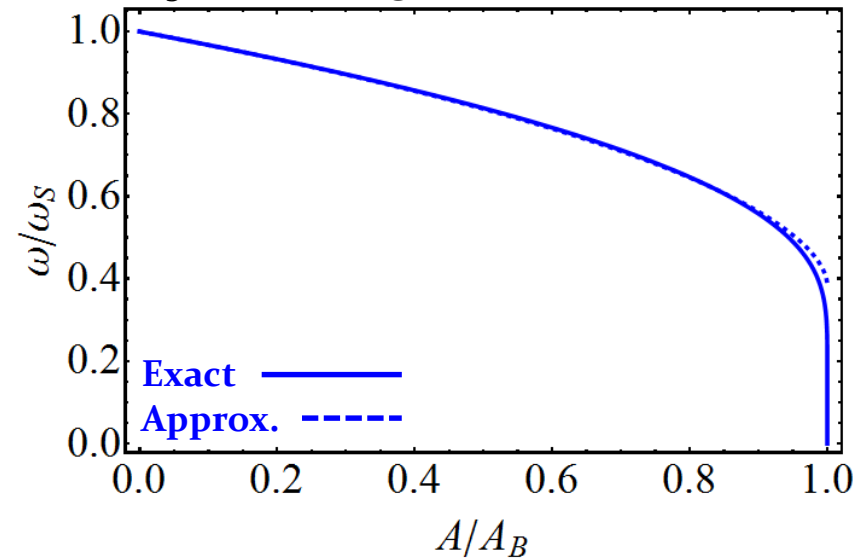
$$\frac{\omega(\Delta\phi_u)}{\omega_S} = \frac{\pi}{2K[\sin(\phi_u/2)]} \simeq 1 - \frac{\phi_u^2}{16}$$

$K(x)$: 1st kind elliptical
integral function

ω_S versus trajectory duration



ω_S versus single particle emittance



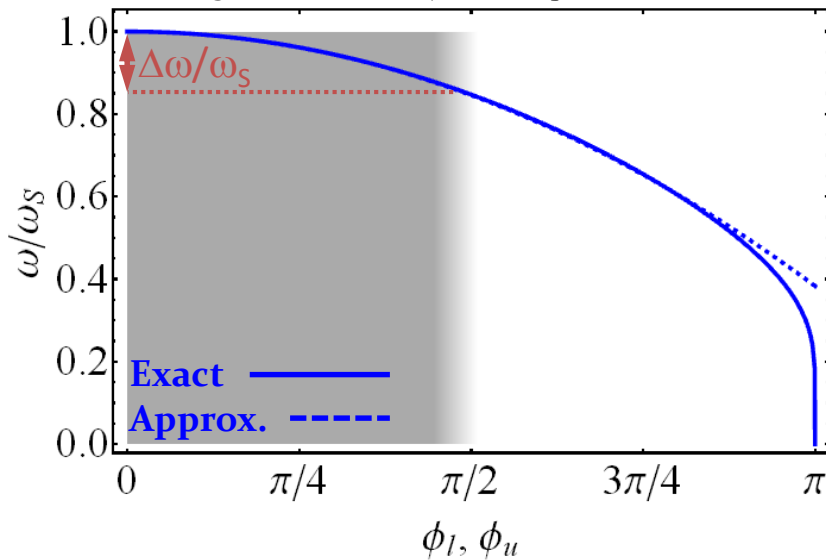
Distribution for stationary bucket

- **Single-harmonic RF in stationary bucket**

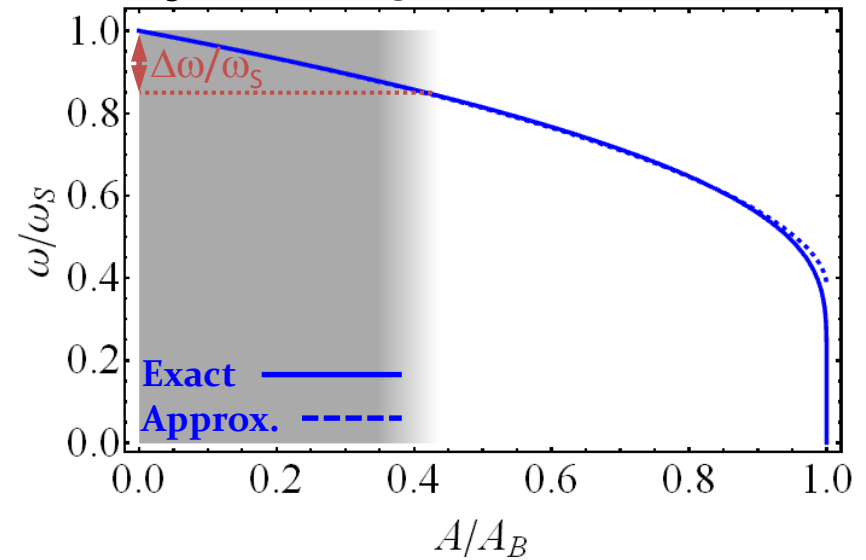
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integral function

ω_S versus trajectory duration



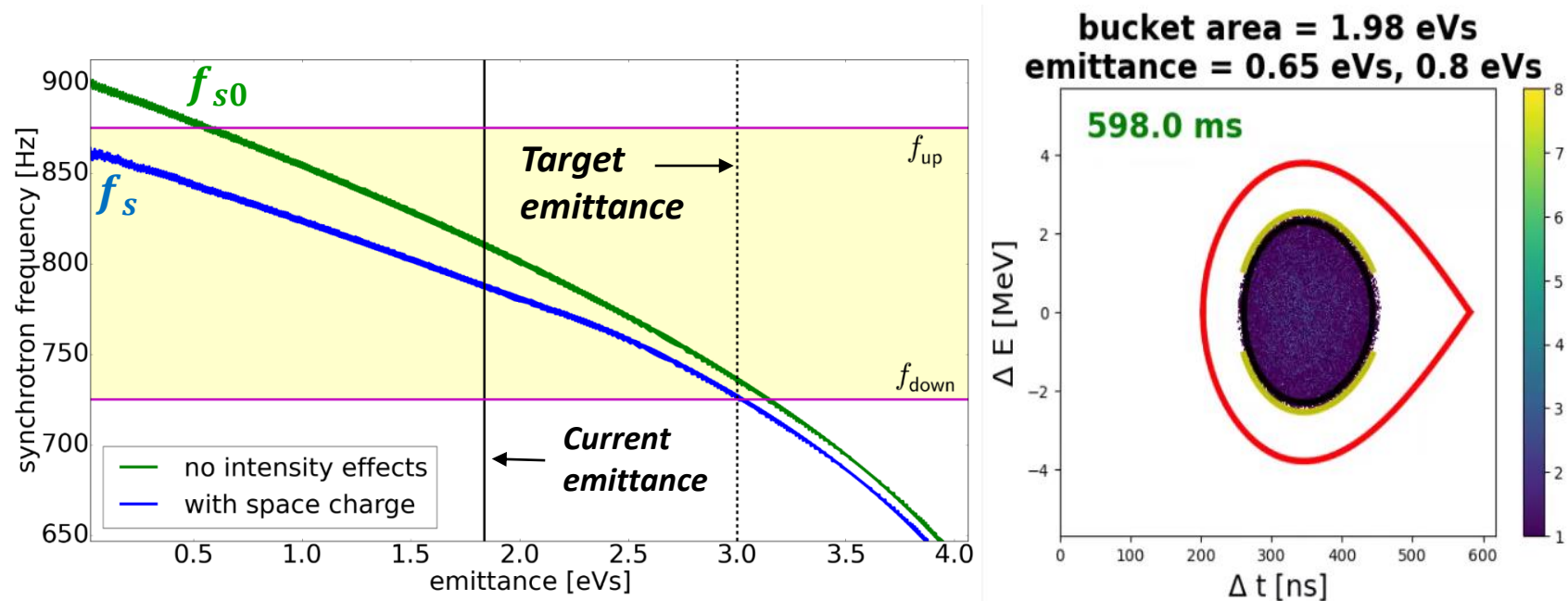
ω_S versus single particle emittance



- Different synchrotron frequencies of particles in bunch
- **Total spread $\Delta\omega/\omega_S$ depends on filling factor of bucket**

Example: Emittance control with noise

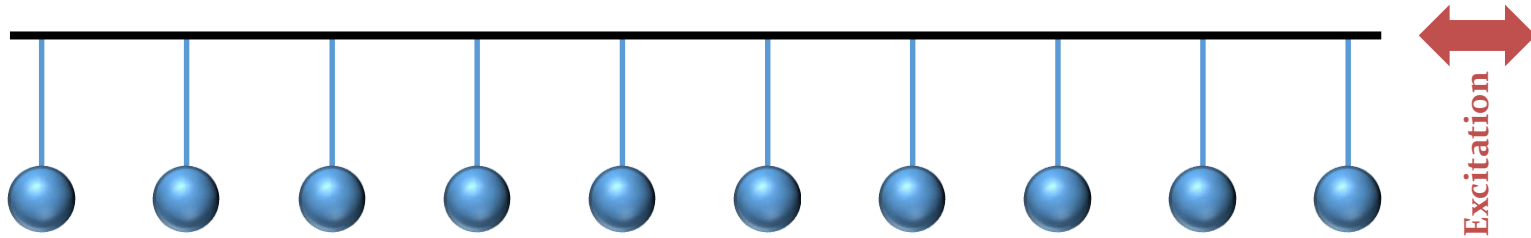
- Noise excitation of bunch by band-width limited noise
- **Controlled longitudinal blow-up in the PSB**



1. Choose upper frequency to cover synchrotron frequency at bunch centre
2. Choose lower frequency to match target emittance
3. Excite

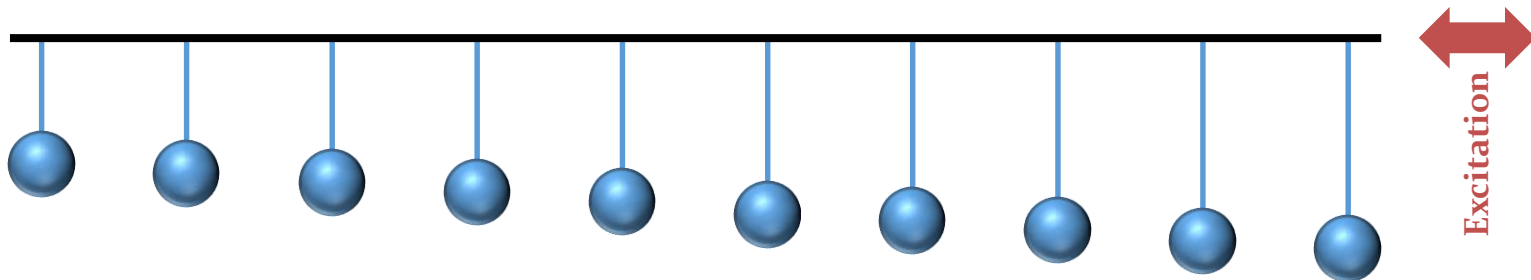
Analogy: pendulums mounted on a bar

- All particles have the same resonance frequency



→ **Easy** to excite macroscopic oscillation

- Resonance frequencies of individual particles varies

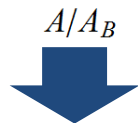
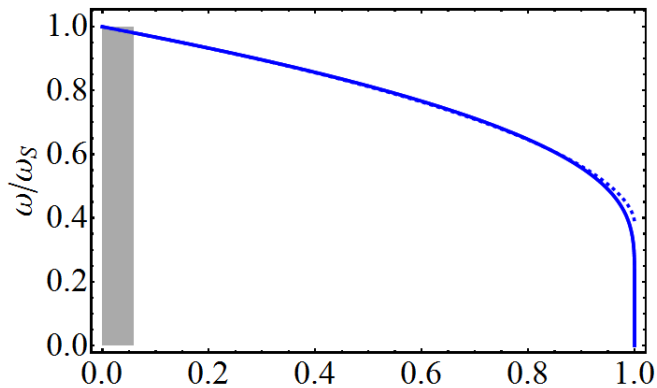
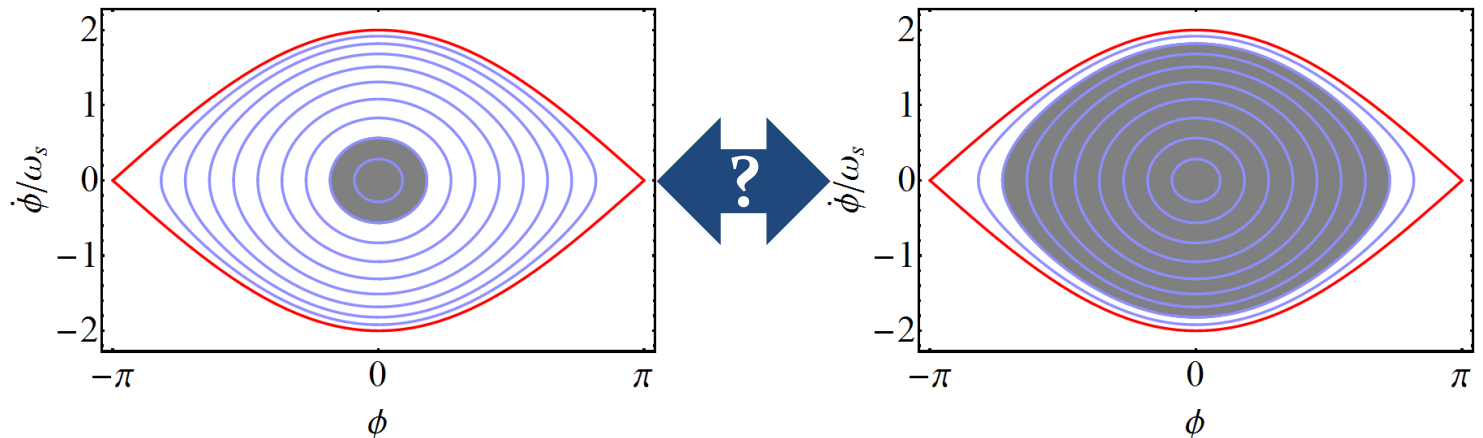


→ **Difficult** to excite macroscopic oscillation

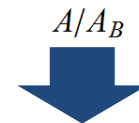
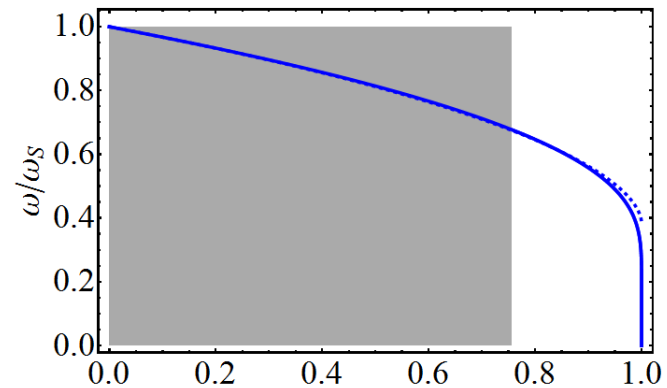
→ Large synchrotron frequency spread increases stability

Bucket filling ratio

Smaller or larger bunch or bucket? What is more stable?



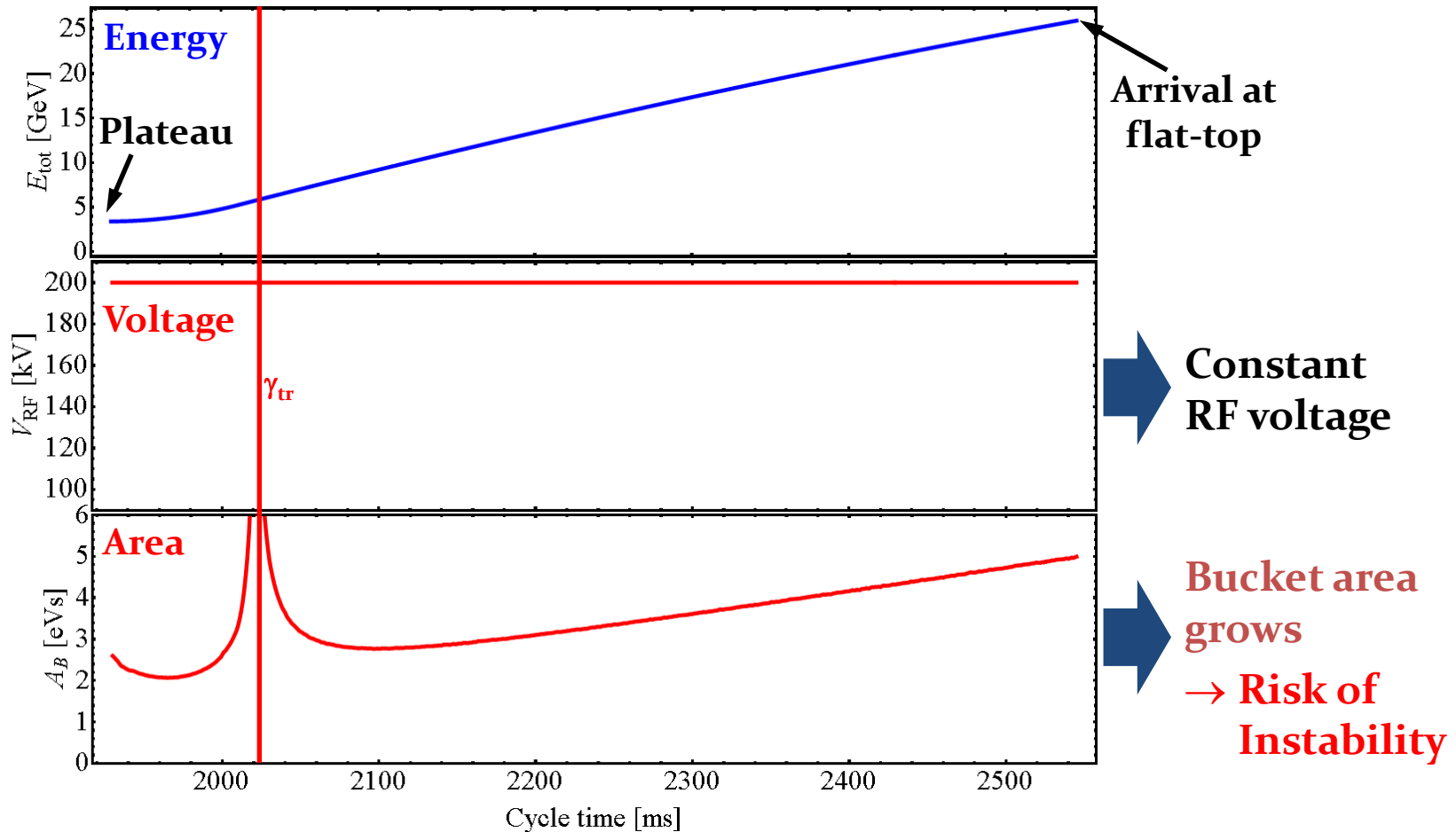
- Easy to excite
- Prone to instability



- Large f_s spread
- More stable

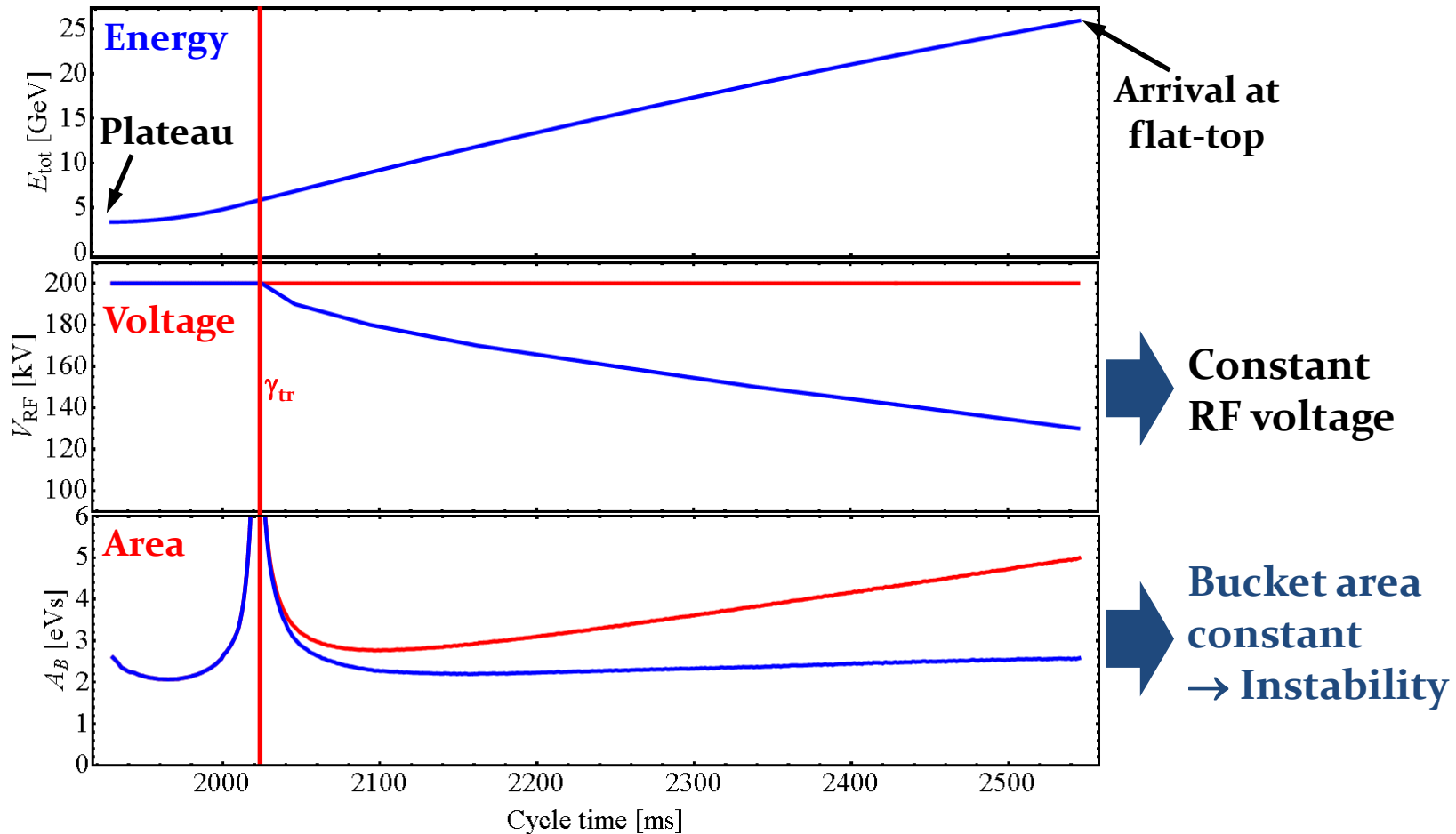
Example: stabilization with lower voltage

→ Acceleration of protons in the CERN PS (3.4 → 26 GeV total)



Example: stabilization with lower voltage

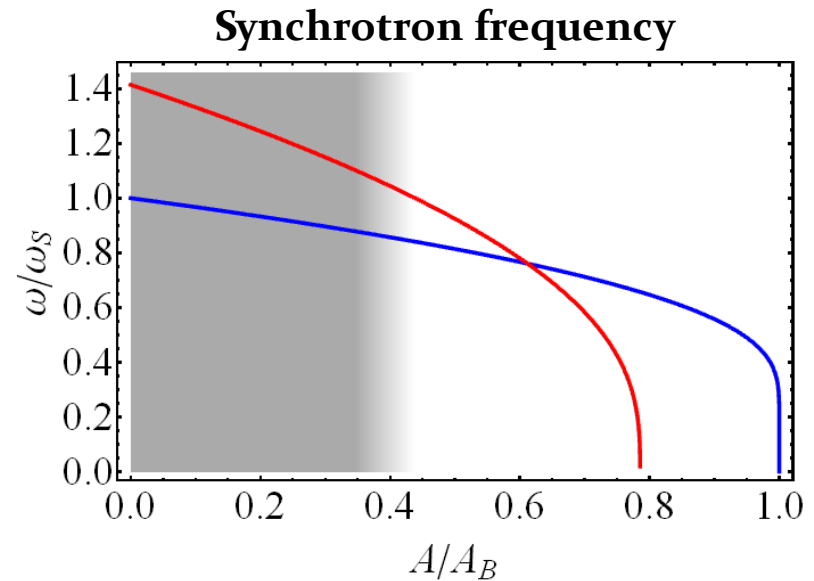
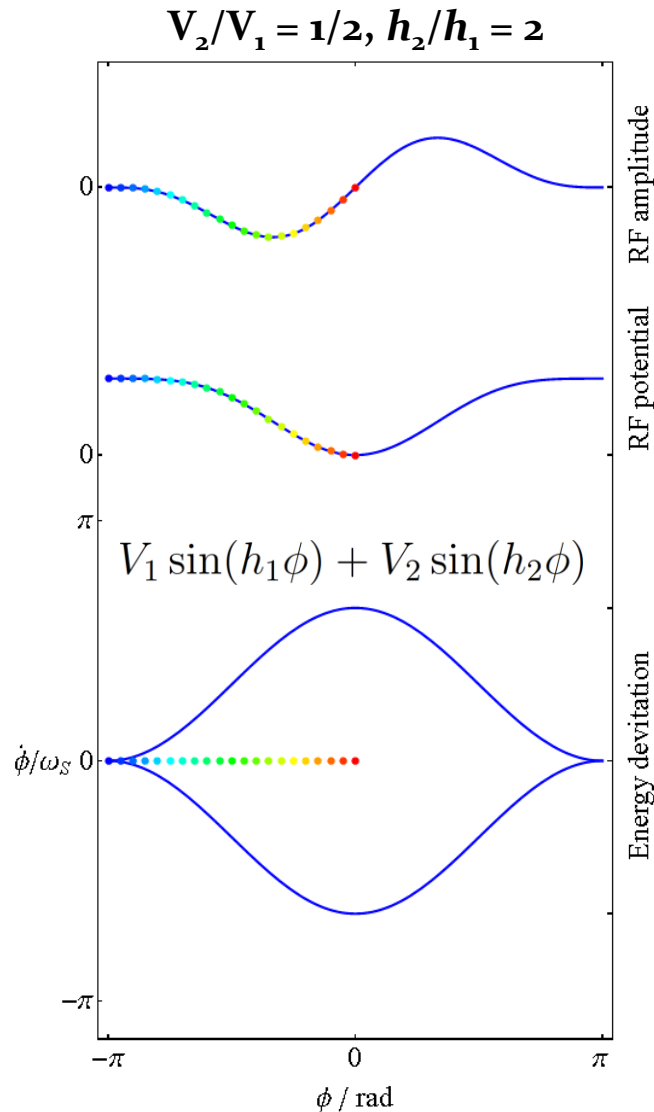
→ Acceleration of protons in the CERN PS (3.4 → 26 GeV total)



- Same principle also applied in SPS and LHC
- Prevent bucket filling to decrease

Additional non-linearity by double RF

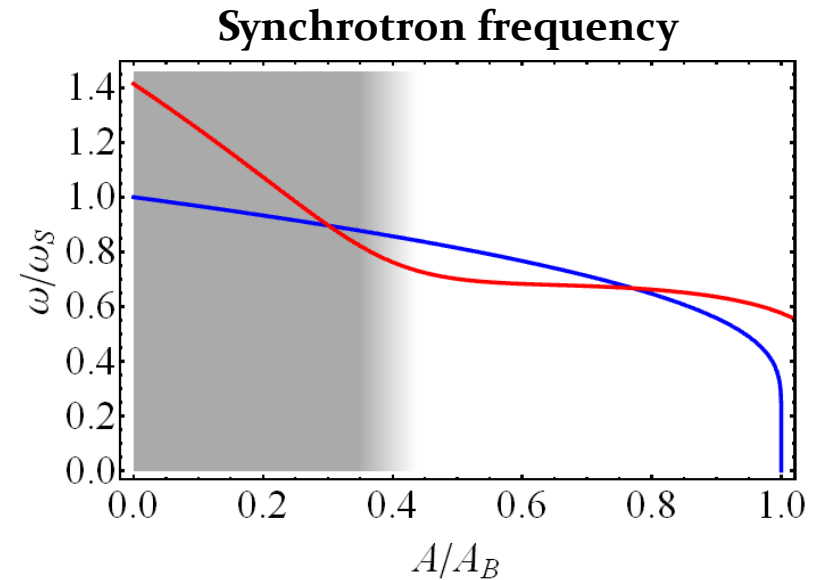
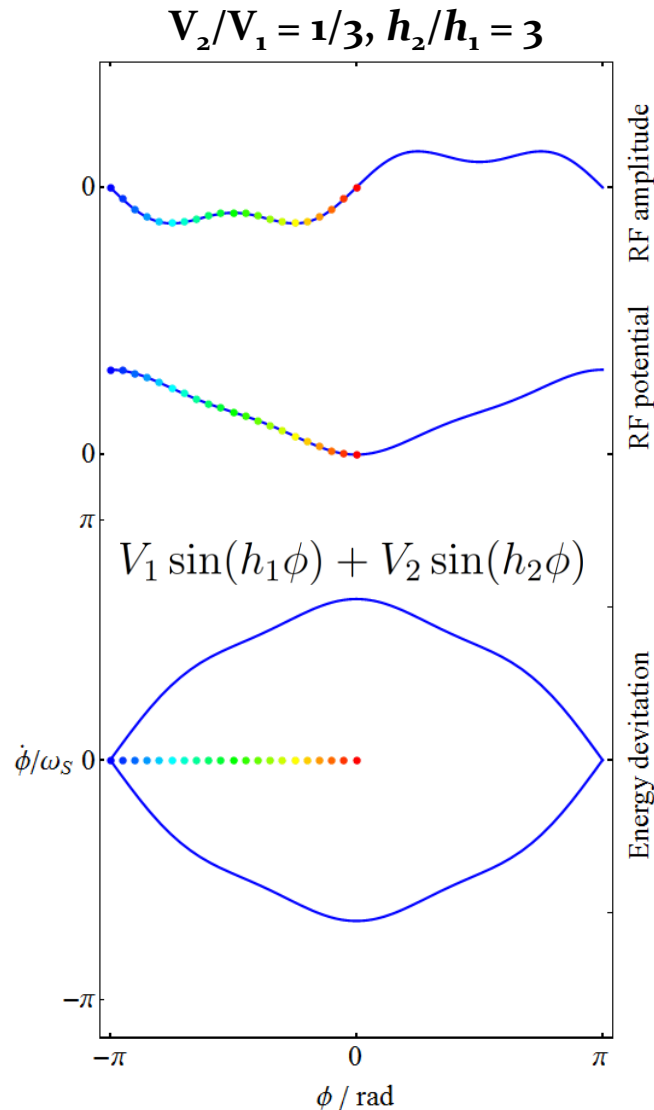
→ RF system at twice the main frequency and at half amplitude



- **Both RF systems in phase**
- **Important increase in synchrotron frequency spread**
- **Improves stability**

Additional non-linearity by double RF

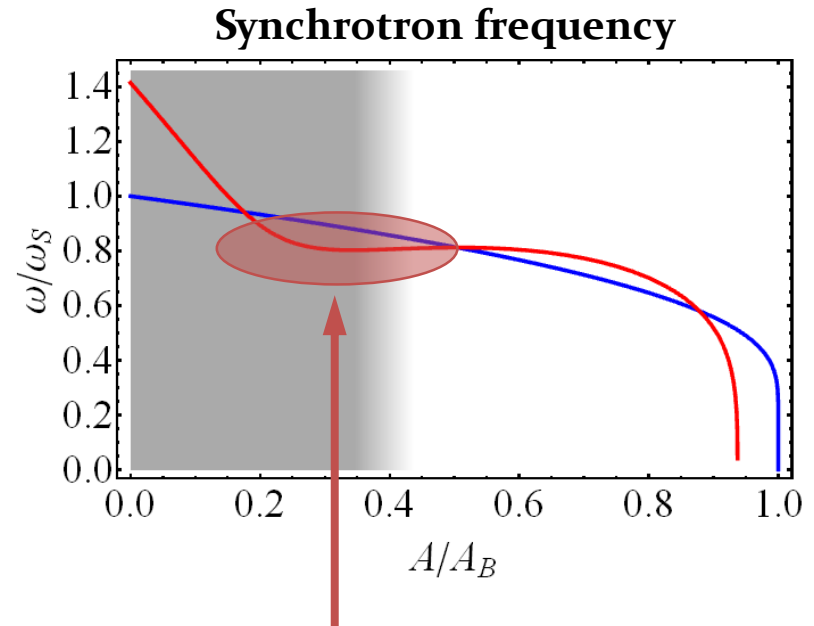
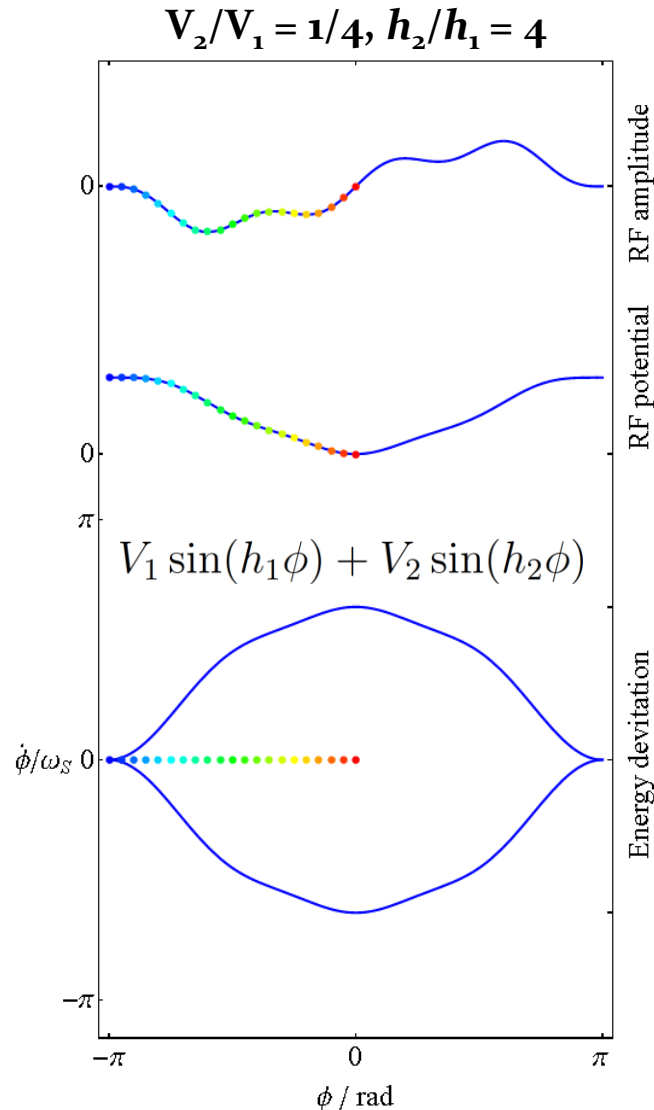
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- **Important increase in synchrotron frequency spread**
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Additional non-linearity by double RF

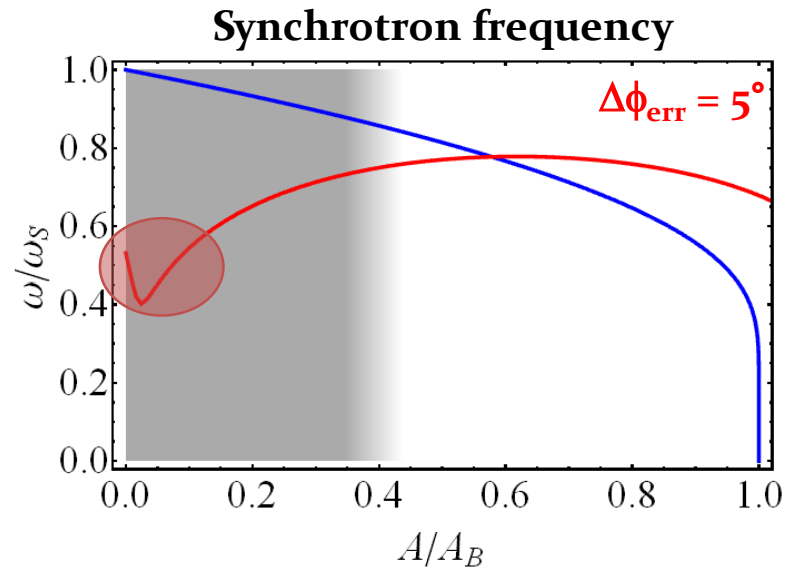
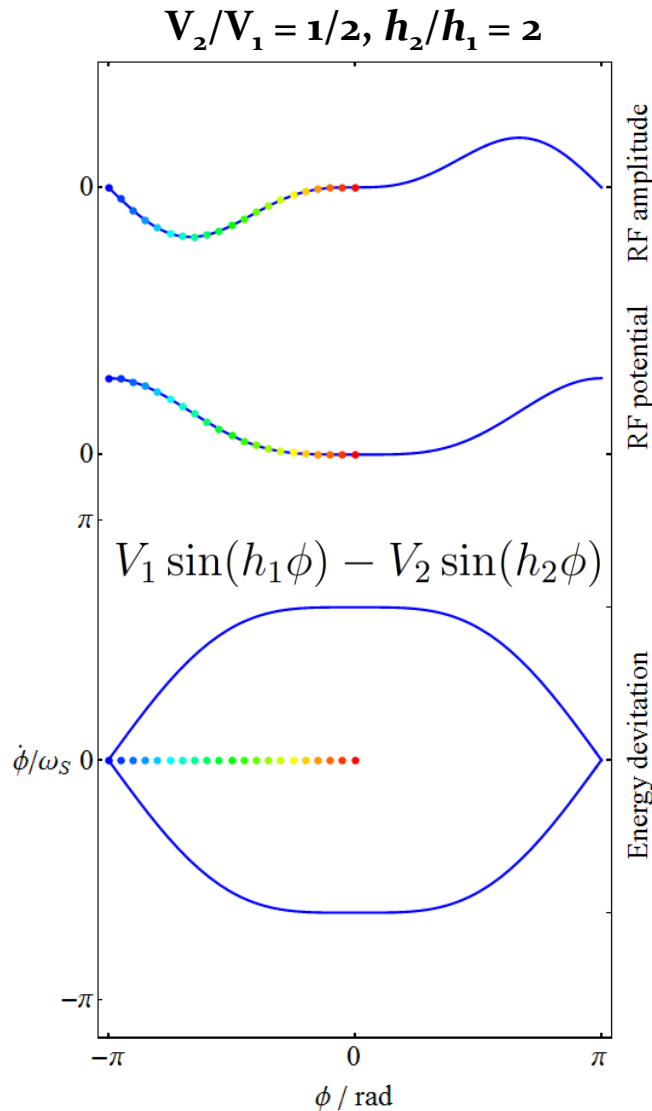
→ RF system at twice the main frequency and at half amplitude



- Local regions of bunch with no f_s gradient
- Again prone to instability
- Reduce voltage of 2nd harmonic RF system
- Improving stability depends on appropriate voltage ratio

Two RF systems in counter-phase?

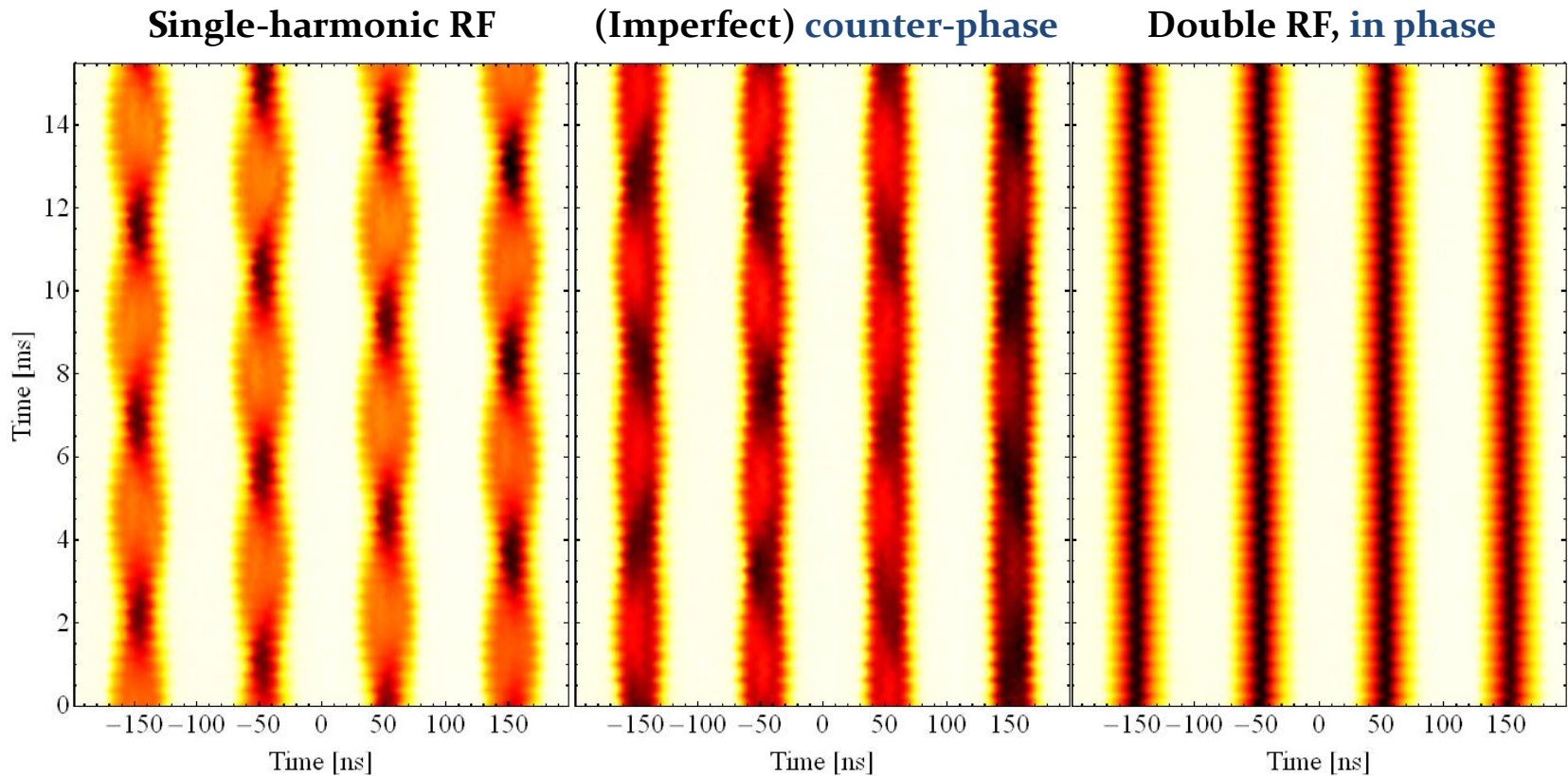
→ 2nd RF twice frequency, half amplitude in counter-phase



- Large frequency spread at bunch centre **with perfectly adjusted phases**
 - Minor phase offset causes **locally unstable regions**
 - Works only for very short **bunches**
 - **Electron accelerators**

Example: damping observations in the PS

- Quadrupolar coupled-bunch oscillations at flat-top
- Main RF system: $h_1 = 21$, 10 MHz, 4 out of 18 bunches
- Higher-harmonic RF system: $h_2 = 84$, 40 MHz



Both RF systems in phase:

→ Highest peak current, but most stable

Summary

- **Longitudinal beam dynamics**
 - **Everything non-linear**
- **Longitudinal manipulations**
 - **Tricks to adjust length and distance of bunches**
 - **Do more with less RF**
- **Synchrotron frequency spread**
 - **More RF voltage may be less stability**
 - **Higher peak density may be more stable**
 - **Improve stability and control emittance**

A big Thank You

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Danilo Quartullo, Markus Ries, Elena Shaposhnikova,
Frank Tecker**

**Thank you very much
for your attention!**

References

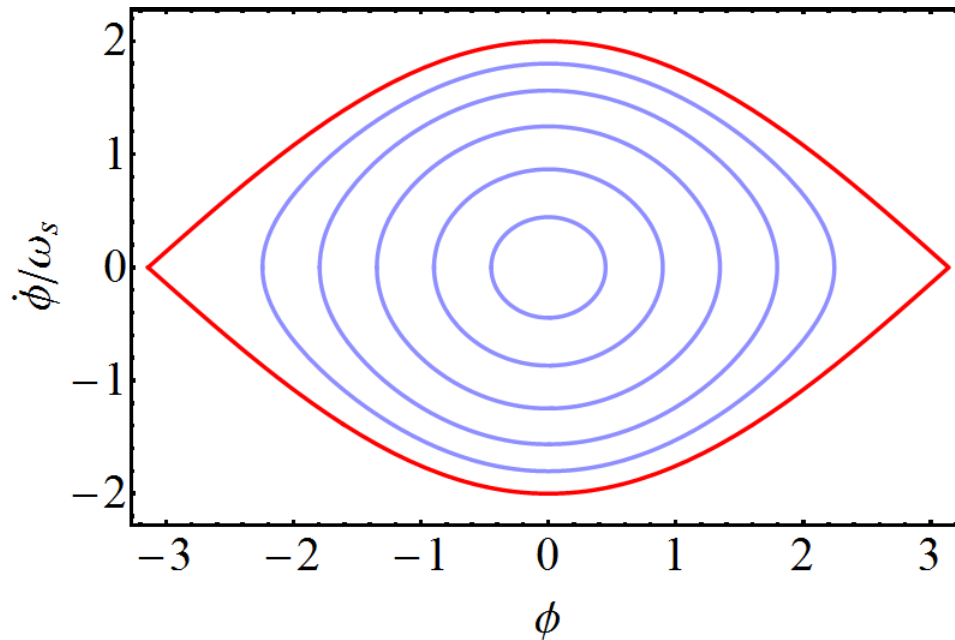
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Spare slides

Stationary bucket in normalized coordinates ⁵⁹

- RF bucket properties become independent from accelerator parameters
- Significant simplification of equations, **easy to use**

Example of stationary bucket



- **Bucket height**

$$\frac{\dot{\phi}_B}{\omega_S} = 2 \text{ rad}$$

- **Bucket area**

$$\frac{A_B}{\omega_S} = 16 \text{ rad}^2$$

- **Exception: conservation of longitudinal phase space**