Theoretical status of the hadronic contributions to $(g-2)_\mu$

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AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

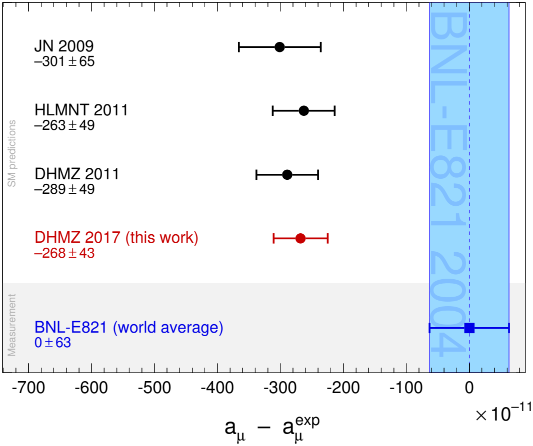
"Getting to Grips with QCD", Paris, 4-6 April 2018

Outline

Outline

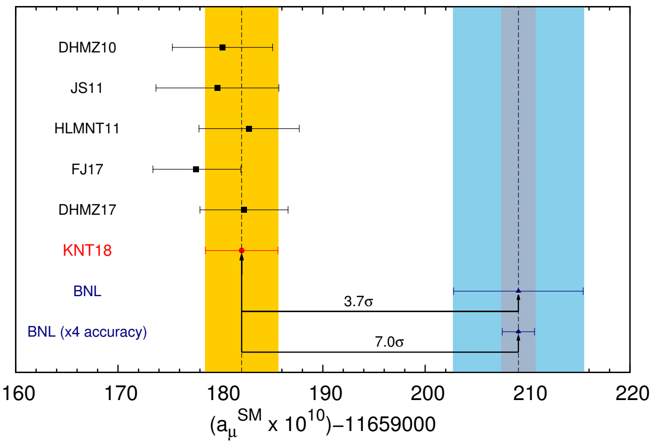
Status of $(g-2)_\mu$, experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2017



Status of $(g-2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)



Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon "$(g-2)$ Theory Initiative" lead by A. El-Khadra and C. Lehner

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.97	0.07
electroweak, total	153.6	1.0
HVP (LO) [KNT 18]	6 932.7	24.6
HVP (NLO) [KNT 18]	−98.2	0.4
HLbL [update of Glasgow consensus–KNT 18]	98.0	26.0
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.0	2.0
theory	116 591 820.5	35.6

Status of $(g-2)_\mu$, experiment vs SM

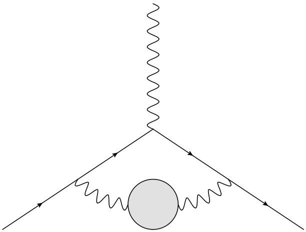
KNT 18

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 268.5 \pm 72.4 \qquad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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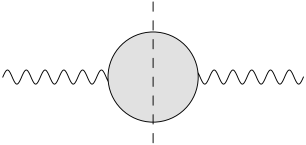
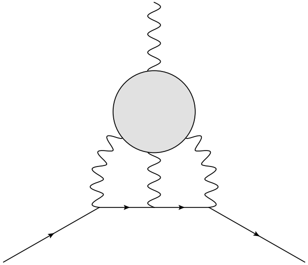


  - basic principles: unitarity and analyticity
  - direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
  - dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND → Z.P. Zhang's talk
  - alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD) → A. Gerardin's talk

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



- 4-point fct. of em currents in QCD
- "*it cannot be expressed in terms of measurable quantities*"
- until recently, only model calculations
- lattice QCD is making fast progress

→ A. Gerardin's talk

Muon $g-2$ Theory Initiative

Steering Committee:
GC
Michel Davier
Simon Eidelman
Aida El-Khadra (co-chair)
Christoph Lehner (co-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Andreas Nyffeler
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner

Workshops:

- First plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- Second plenary meeting, Mainz, 18-22 June 2018

Outline

Introduction

Hadronic Vacuum Polarization contribution to $(g-2)_\mu$
- Dispersive calculations

Hadronic light-by-light contribution to $(g-2)_\mu$
- Dispersive approach
- Master Formula
- A dispersion relation for HLbL
- Numerics

Conclusions

HVP, gauge invariance and analyticity

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle = \left(q_\mu q_\nu - g_{\mu\nu} q^2 \right) \Pi(q^2)$$

where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\hat{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\hat{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} dt \frac{\mathrm{Im}\hat{\Pi}(t)}{t(t-q^2)}$$

HVP, gauge invariance and analyticity

$$\Pi_{\mu\nu}(q) = i\int d^4x e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)$$

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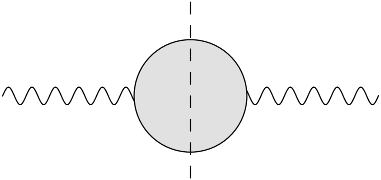
Easy!

Unitarity relation for HVP

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

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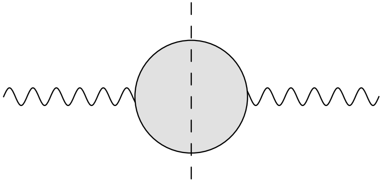


$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

$$a_\mu^{\mathrm{hvp}} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty}\frac{ds}{s}K(s)R(s)$$

Relevant issues in dispersive calculations of HVP

- Data on $\sigma(e^+e^- \to \text{hadrons})$ must be undressed of vacuum polarization effects (some experimental collaborations provide undressed data, some don't)
- FSR are usually included in the data: $\sigma(e^+e^- \to \text{hadrons}(\gamma))$, but fully inclusive measurements are impossible $\Rightarrow$ need theory input to complete the 4π photon emission cross sections
- other radiative corrections – energy-scan and radiative-return experiments have different issues
- combination of data (different data sets, clustering, χ^2 fitting, correlations, etc.)
- integration (trapezoidal rule or more sophisticated approaches?)

Updated dispersive calculation DHMZ17

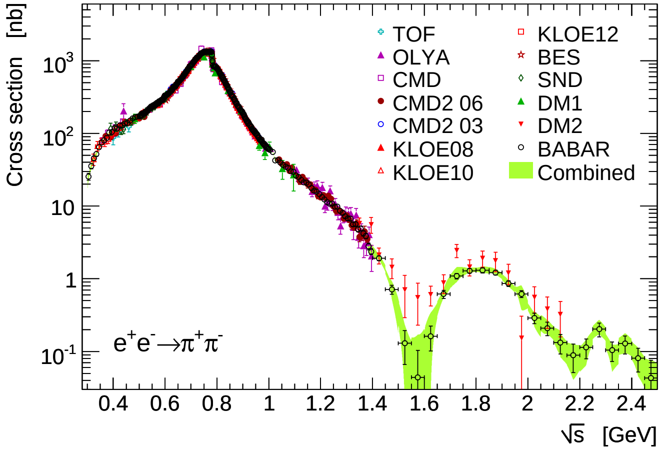
→ Z.P. Zhang's talk

Description of the method: Davier, Hoecker, Malaescu and Zhang 2017

"The integration of data points belonging to different experiments with their own data densities requires a careful treatment especially with respect to correlated systematic uncertainties within the same experiment and between different experiments. Quadratic interpolation of adjacent data points is performed for each experiment and a local combination between the interpolations is computed in bins of 1 MeV. Full covariance matrices are constructed between experiments and channels. Uncertainties are propagated using pseudo-experiment generation and closure tests with known distributions are performed to validate both the combination and integration. Where results from different experiments are locally inconsistent the combined uncertainty is rescaled according to the local χ^2 value following the well-known PDG approach. At present, for the dominant $\pi^+\pi^-$ channel such inconsistencies are limiting the precision of the combination. "

DHMZ17: 2π channel

→ Z.P. Zhang's talk



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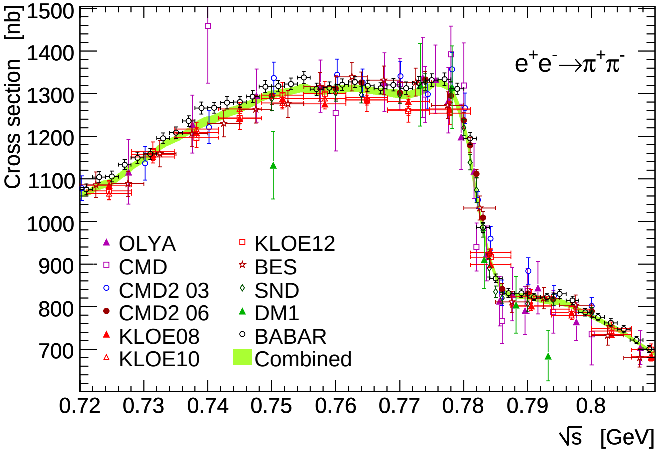

DHMZ17: 2π channel

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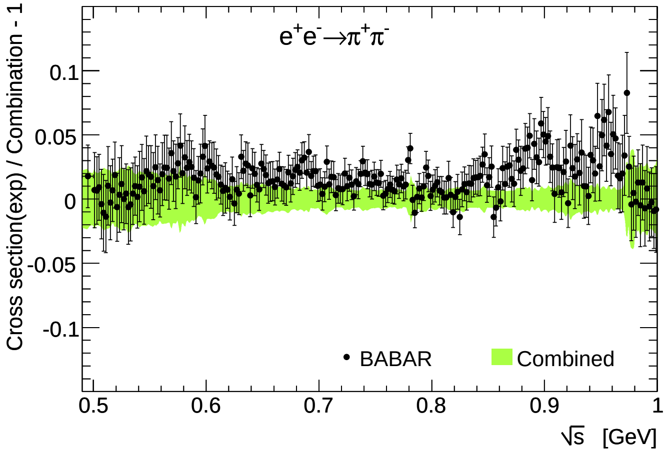
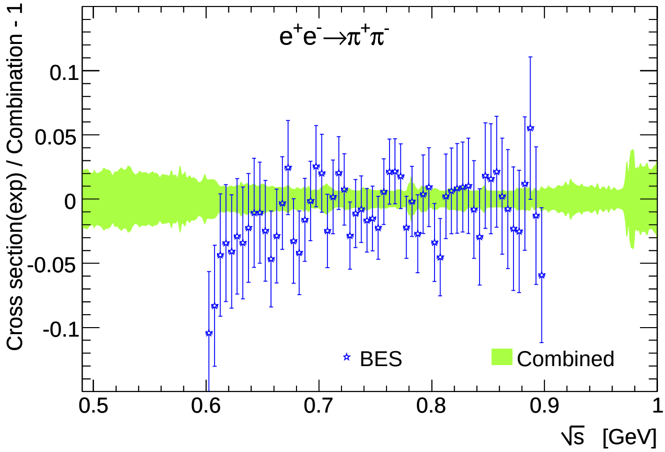


DHMZ17: 2π channel

$\rightarrow$ Z.P. Zhang's talk



DHMZ17: 2π channel

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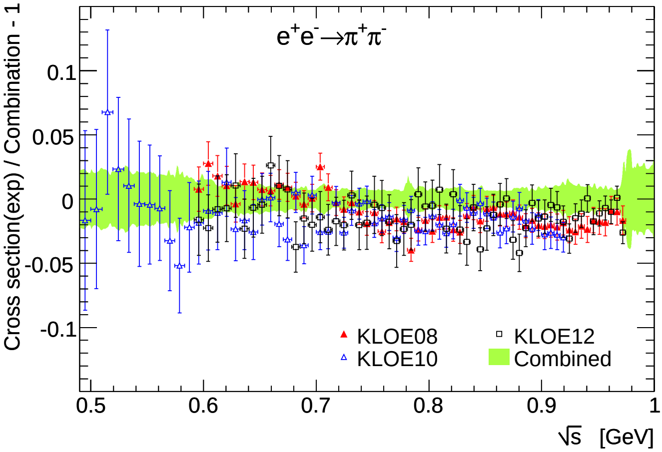
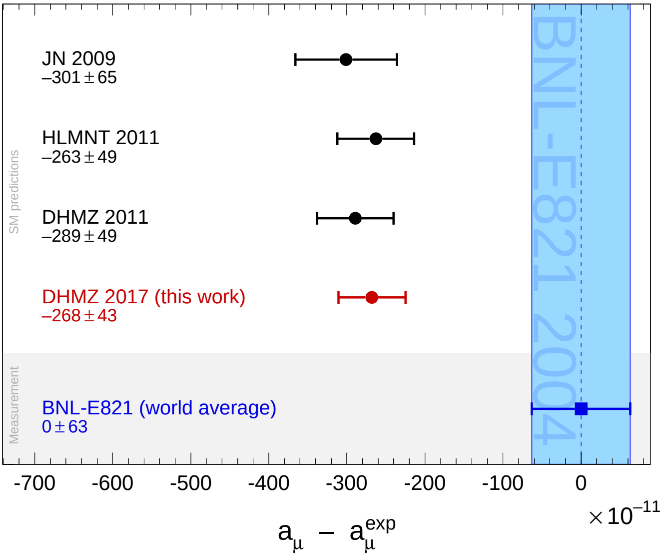


DHMZ17: final result

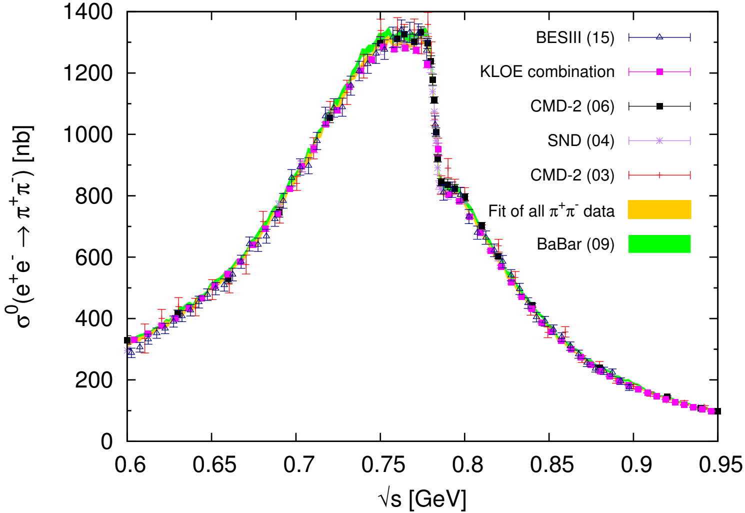
→ Z.P. Zhang's talk



KNT18: some remarks

- Completely new analysis: all data reanalyzed anew
- Thorough analysis of KLOE data in collaboration with the KLOE people to calculate and release a new correlation matrix for all KLOE data KLOE+KNT 17
- Integration by trapezoidal rule (as always)

KNT18: 2π channel



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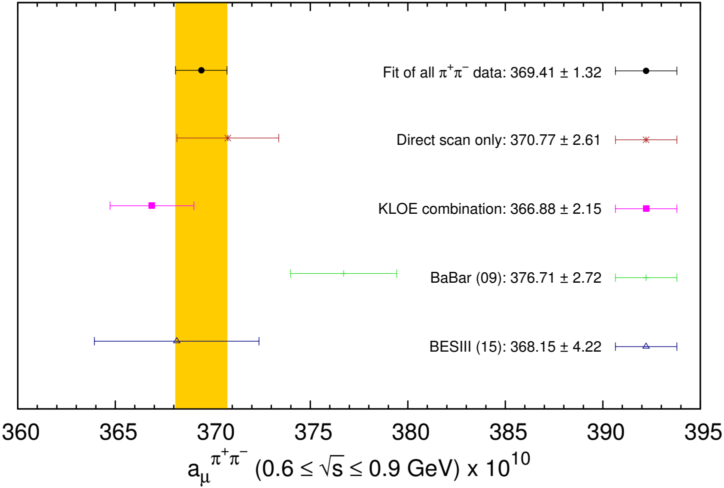
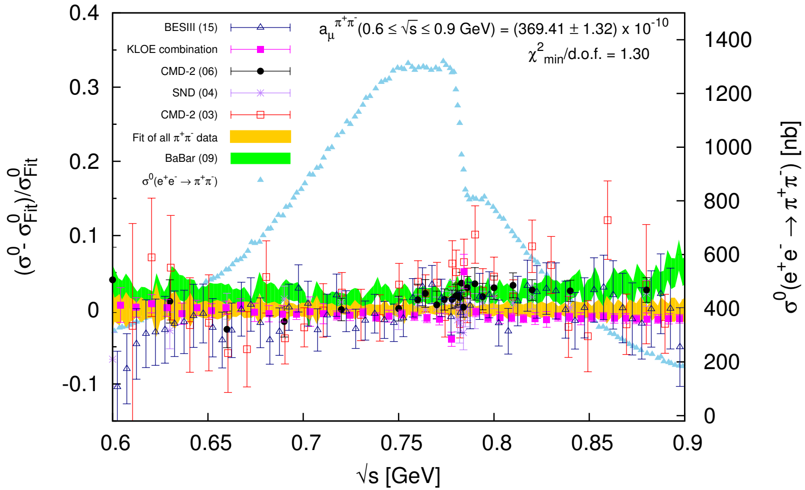
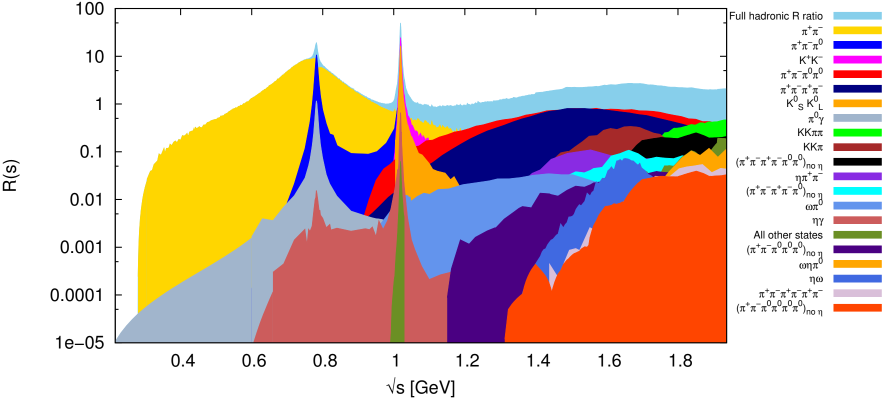


KNT18: 2π channel



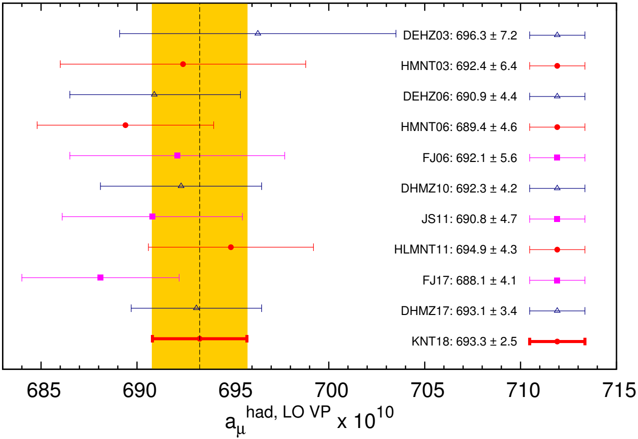
KNT18: final results



KNT18: final results

Channel	This work (KNT18)	HLMNT11 [9]	Difference
Chiral perturbation theory (ChPT) threshold contributions			
$\pi^0\gamma$	0.12 ± 0.01	0.12 ± 0.01	0.00 ± 0.01
2π	0.87 ± 0.02	0.87 ± 0.02	0.00 ± 0.03
3π	0.01 ± 0.00	0.01 ± 0.00	0.00 ± 0.00
$\eta\gamma$	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
Data based channels ($\sqrt{s} \leq 2$ GeV)			
$\pi^0\gamma$	4.46 ± 0.10	4.54 ± 0.14	-0.08 ± 0.17
$\pi^+\pi^-$	502.99 ± 1.97	505.77 ± 3.09	-2.78 ± 3.66
$\pi^+\pi^-\pi^0$	47.82 ± 0.89	47.51 ± 0.99	0.31 ± 1.33
$\pi^+\pi^-\pi^+\pi^-$	15.17 ± 0.21	14.65 ± 0.47	0.52 ± 0.51
$\pi^+\pi^-\pi^0\pi^0$	19.80 ± 0.79	20.37 ± 1.26	-0.57 ± 1.49
$(2\pi^+2\pi^-\pi^0)_{\text{no }\eta}$	1.08 ± 0.10	1.20 ± 0.10	-0.12 ± 0.14
$3\pi^+3\pi^-$	0.28 ± 0.02	0.28 ± 0.02	0.00 ± 0.03
$(2\pi^+2\pi^-2\pi^0)_{\text{no }\eta\omega}$	1.60 ± 0.20	1.80 ± 0.24	-0.20 ± 0.31
K^+K^-	23.05 ± 0.22	22.15 ± 0.46	0.90 ± 0.51
$K^0_SK^0_L$	13.05 ± 0.19	13.33 ± 0.16	-0.28 ± 0.25
$KK\pi$	2.80 ± 0.12	2.77 ± 0.15	0.03 ± 0.19
$KK2\pi$	2.42 ± 0.09	3.31 ± 0.58	-0.89 ± 0.59
$\eta\gamma$	0.70 ± 0.02	0.69 ± 0.02	0.01 ± 0.03
$\eta\pi^+\pi^-$	1.32 ± 0.06	0.98 ± 0.24	0.34 ± 0.25
$(\eta\pi^+\pi^-\pi^0)_{\text{no }\omega}$	0.63 ± 0.15	-	0.63 ± 0.15
$\eta 2\pi^+2\pi^-$	0.11 ± 0.02	0.11 ± 0.02	0.00 ± 0.03
$\eta\omega$	0.31 ± 0.03	0.43 ± 0.07	-0.12 ± 0.08
$\omega(\to \pi^0\gamma)\pi^0$	0.88 ± 0.02	0.77 ± 0.03	0.11 ± 0.04
$(\eta\phi)_{\text{no }\phi\to K^0\bar{K}^0}$	0.30 ± 0.02	0.46 ± 0.03*	-0.16 ± 0.04
$\phi \to$ unaccounted	0.04 ± 0.04	0.04 ± 0.04	0.00 ± 0.06
$\eta\omega\pi^0$	0.42 ± 0.10	-	0.42 ± 0.10
$K^0_SK^0_L\eta$	0.18 ± 0.03	-	0.18 ± 0.03
$p\bar{p}$	0.07 ± 0.00	0.06 ± 0.00	0.01 ± 0.00
$n\bar{n}$	0.06 ± 0.01	0.07 ± 0.02	-0.01 ± 0.02
Estimated contributions ($\sqrt{s} \leq 2$ GeV)			
$(\pi^+\pi^-3\pi^0)_{\text{no }\eta}$	0.53 ± 0.05	0.60 ± 0.05	-0.07 ± 0.07
$(\pi^+\pi^-4\pi^0)_{\text{no }\eta}$	0.25 ± 0.25	0.28 ± 0.28	-0.03 ± 0.38
$KK3\pi$	0.08 ± 0.03	0.08 ± 0.04	0.00 ± 0.05
$\omega(\to \text{npp})2\pi$	0.10 ± 0.02	0.11 ± 0.02	-0.01 ± 0.03
$\omega(\to \text{npp})3\pi$	0.20 ± 0.04	0.22 ± 0.04	-0.02 ± 0.06
$\omega(\to \text{npp})KK$	0.01 ± 0.00	0.01 ± 0.00	0.00 ± 0.00
$\eta\pi^+\pi^-2\pi^0$	0.11 ± 0.05	0.11 ± 0.06	0.00 ± 0.08
Other contributions ($\sqrt{s} > 2$ GeV)			
Inclusive channel	41.27 ± 0.62	41.40 ± 0.87	-0.13 ± 1.07
J/ψ	6.26 ± 0.19	6.24 ± 0.16	0.02 ± 0.25
ψ'	1.58 ± 0.04	1.56 ± 0.05	0.02 ± 0.06
$\Upsilon(1S-4S)$	0.09 ± 0.00	0.10 ± 0.00	-0.01 ± 0.00
pQCD	2.07 ± 0.00	2.06 ± 0.00	0.01 ± 0.00
Total	693.27 ± 2.46	694.91 ± 4.27	-1.64 ± 4.93

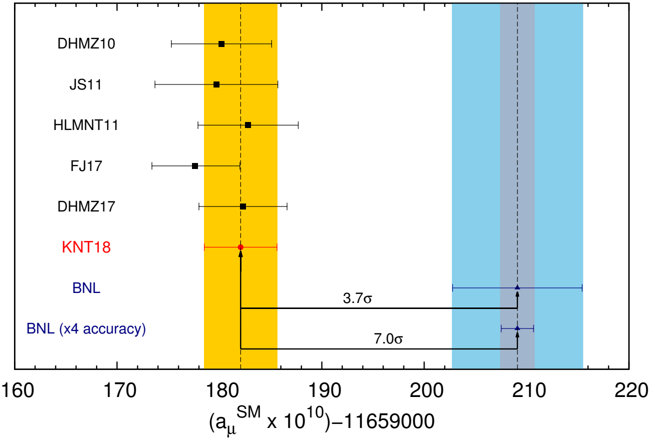
KNT18: final results



KNT18: final results

Channel	This work (KNT18)	DHMZ17 [77]	Difference
Data based channels ($\sqrt{s} \le 1.8$ GeV)			
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29 ± 0.14
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	-3.40 ± 3.24
$\pi^+\pi^-\pi^0$ (data + ChPT)	47.70 ± 0.89	46.20 ± 1.45	1.50 ± 1.70
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31 ± 0.36
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12 ± 0.92
$(2\pi^+2\pi^-\pi^0)_{\text{no }\eta}$	0.79 ± 0.08	0.69 ± 0.08	0.10 ± 0.11
$3\pi^+3\pi^-$	0.10 ± 0.01	0.11 ± 0.01	-0.01 ± 0.01
$(2\pi^+2\pi^-2\pi^0)_{\text{no }\eta\omega}$	0.77 ± 0.11	0.72 ± 0.17	0.05 ± 0.20
K^+K^-	23.00 ± 0.22	22.81 ± 0.41	0.19 ± 0.47
$K^0_S K^0_L$	13.04 ± 0.19	12.82 ± 0.24	0.22 ± 0.31
$KK\pi$	2.44 ± 0.11	2.45 ± 0.15	-0.01 ± 0.19
$KK2\pi$	0.86 ± 0.05	0.85 ± 0.05	0.01 ± 0.07
$\eta\gamma$ (data + ChPT)	0.70 ± 0.02	0.65 ± 0.02	0.05 ± 0.03
$\eta\pi^+\pi^-$	1.18 ± 0.05	1.18 ± 0.07	0.00 ± 0.09
$(\eta\pi^+\pi^-\pi^0)_{\text{no }\omega}$	0.48 ± 0.12	0.39 ± 0.12	0.09 ± 0.17
$\eta 2\pi^+2\pi^-$	0.03 ± 0.01	0.03 ± 0.01	0.00 ± 0.01
$\eta\omega$	0.29 ± 0.02	0.32 ± 0.03	-0.03 ± 0.04
$\omega(\to \pi^0\gamma)\pi^0$	0.87 ± 0.02	0.94 ± 0.03	-0.07 ± 0.04
$(\eta\phi)_{\text{no }\phi\to K^0K^0}$	0.22 ± 0.02	0.36 ± 0.03	-0.14 ± 0.04*
$\phi \to$ unaccounted	0.04 ± 0.04	0.05 ± 0.00	-0.01 ± 0.04
$\eta\omega\pi^0$	0.10 ± 0.05	0.06 ± 0.04	0.04 ± 0.06
$K^0_S K^0_L \eta$	0.12 ± 0.02	0.01 ± 0.01	0.11 ± 0.02*
Estimated contributions ($\sqrt{s} \le 1.8$ GeV)			
$(\pi^+\pi^-3\pi^0)_{\text{no }\eta}$	0.40 ± 0.04	0.35 ± 0.04	0.05 ± 0.06
$(\pi^+\pi^-4\pi^0)_{\text{no }\eta}$	0.12 ± 0.12	0.11 ± 0.11	0.01 ± 0.16
$KK3\pi$	-0.02 ± 0.01	-0.03 ± 0.02	0.01 ± 0.02
$\omega(\to \text{npp})2\pi$	0.08 ± 0.01	0.08 ± 0.01	0.00 ± 0.01
$\omega(\to \text{npp})3\pi$	0.10 ± 0.02	0.36 ± 0.01	-0.26 ± 0.02
$\omega(\to \text{npp})KK$	0.00 ± 0.00	0.01 ± 0.00	-0.01 ± 0.00
$\eta\pi^+\pi^-2\pi^0$	0.03 ± 0.01	0.03 ± 0.01	0.00 ± 0.01
Other contributions			
J/ψ	6.26 ± 0.19	6.28 ± 0.07	-0.02 ± 0.20
ψ'	1.58 ± 0.04	1.57 ± 0.03	0.01 ± 0.05
$\Upsilon(1S-4S)$	0.09 ± 0.00	-	0.09 ± 0.00**
Contributions by energy region			
$1.8 \le \sqrt{s} \le 3.7$ GeV	34.54 ± 0.56 (data)	33.45 ± 0.65 (pQCD)***	1.09 ± 0.86
$3.7 \le \sqrt{s} \le 5.0$ GeV	7.33 ± 0.11 (data)	7.29 ± 0.03 (data)	0.04 ± 0.11
$5.0 \le \sqrt{s} \le 9.3$ GeV	6.62 ± 0.10 (data)	6.86 ± 0.04 (pQCD)	-0.24 ± 0.11
$9.3 \le \sqrt{s} \le 12.0$ GeV	1.12 ± 0.01 (data+pQCD)	1.21 ± 0.01 (pQCD)	-0.09 ± 0.02**
$12.0 \le \sqrt{s} \le 40.0$ GeV	1.64 ± 0.00 (pQCD)	1.64 ± 0.00 (pQCD)	0.00 ± 0.00
> 40.0 GeV	0.16 ± 0.00 (pQCD)	0.16 ± 0.00 (pQCD)	0.00 ± 0.00
Total	693.3 ± 2.5	693.1 ± 3.4	0.2 ± 4.2

KNT18: final results



Other recent developments

- Get a_μ^{hvp} from $\Pi(t)$ in the spacelike region ($t < 0$):

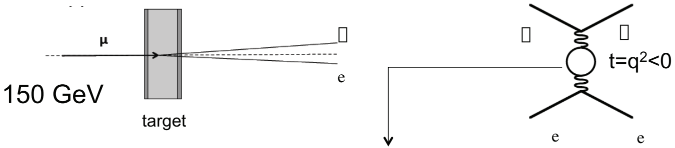
$$a_\mu^{\text{hvp}} = -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\Delta\alpha_{\text{had}}\left(-\frac{x^2}{1-x}m_\mu^2\right)$$

Carloni Calame, Passera, Trentadue, Venanzoni 2015

Other recent developments

Experimental approach:

Use of a 150 GeV μ beam on Be target at CERN (elastic scattering μe → μe)



$$\left|\frac{\alpha(t)}{\alpha_0}\right|^2 = \left|\frac{1}{1-\Delta\alpha(t)}\right|^2$$

Other recent developments

- Get $a_\mu^{\rm hvp}$ from $\Pi(t)$ in the spacelike region ($t<0$):

$$a_\mu^{\rm hvp} = -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\Delta\alpha_{\rm had}\left(-\frac{x^2}{1-x}m_\mu^2\right)$$

Carloni Calame, Passera, Trentadue, Venanzoni 2015

- Put on a more solid basis the calculation of a_μ from a few derivatives of $\Pi(q^2)$ at $q^2=0$ using the theory of Mellin-Barnes transforms Charles, Greynat, de Rafael 2017

Outline

Different model-based evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
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Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- heavier single-particle poles decreasingly important

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Dispersive approach to hadronic light-by-light

We address the calculation of the hadronic light-by-light tensor

GC, Hoferichter, Procura & Stoffer (14,...,17)

- model independent ⇒ rely on dispersion relations
- as data-driven as possible
- takes into account high-energy constraints [OPE, perturbative QCD] (work in progress, not discussed here)

Alternative dispersive approach for the μ-FF Pauk-Vanderhaeghen (14)

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

with Mandelstam variables

$$s = (q_1 + q_2)^2 \quad t = (q_1 + q_3)^2 \quad u = (q_2 + q_3)^2$$

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General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

The 54 scalar functions Π_i are free of kinematic singularities and zeros and as such are amenable to a dispersive treatment

HLbL contribution to a_μ

From gauge invariance:

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k-q_1-q_2) = -k^\rho \frac{\partial}{\partial k^\sigma} \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k-q_1-q_2).$$

Contribution to a_μ: $m := m_\mu$

$$a_\mu = \frac{-1}{48m} \mathrm{Tr}\Big\{ (\not{p}+m)[\gamma^\rho, \gamma^\sigma](\not{p}+m)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) \Big\}$$

$$\Gamma_{\rho\sigma} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{\gamma^\mu (\not{p}+\not{q}_1+m)\gamma^\lambda(\not{p}-\not{q}_2+m)\gamma^\nu}{((p+q_1)^2-m^2)((p-q_2)^2-m^2)} \times \\ \times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k-q_1-q_2)\Big|_{k=0}$$

BTT basis (no kin. singularities!) $\Rightarrow$ limit $k_\mu \to 0$ unproblematic

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- 5 integrals can be performed with Gegenbauer polynomial techniques

GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

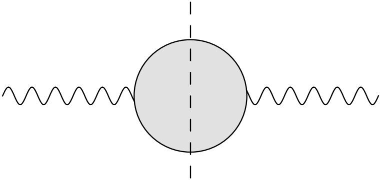
$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

Setting up the dispersive calculation for HLbL

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

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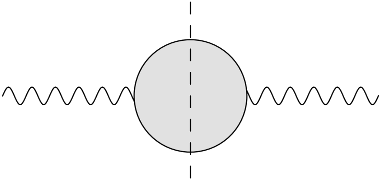


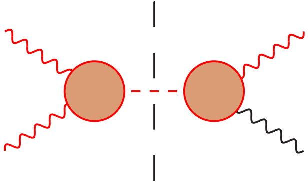
$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

For HLbL things are more complicated

Setting up the dispersive calculation for HLbL

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions

Projection on the BTT basis: easy ✓

Our master formula=explicit expressions in the literature ✓

Input: pion transition form factor Hoferichter, Kubis, Leupold, Niecknig, Schneider (14)

First results of direct lattice calculations Gerardin-Mayer-Nyffeler (16)

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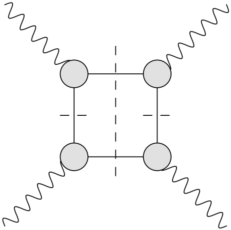
π-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole

– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

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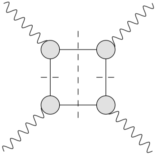
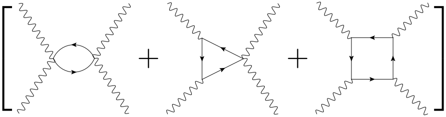


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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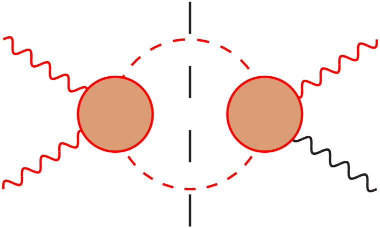


The "rest" with 2π intermediate states has cuts only in one channel and will be
calculated dispersively after partial-wave expansion

Setting up the dispersive calculation for HLbL

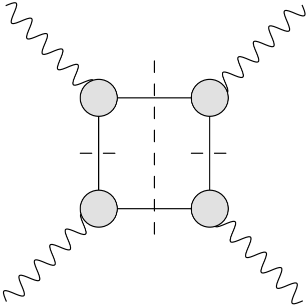
We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\text{-box}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states will not be discussed here

Pion box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

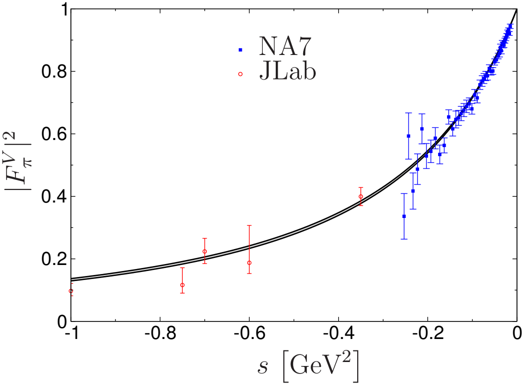
where

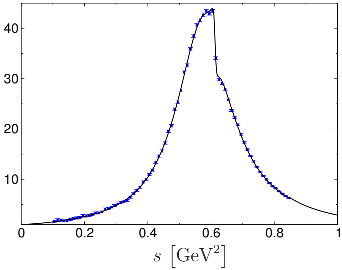
$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion box contribution





Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

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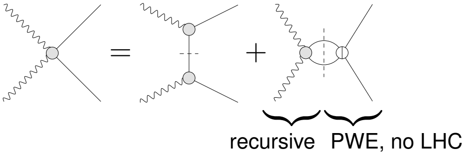
First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



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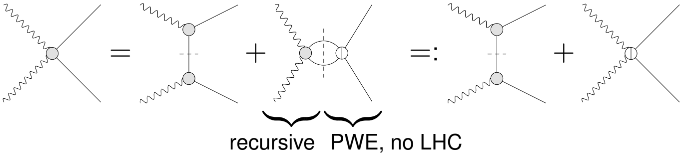

First evaluation of S- wave 2π-rescattering

Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method

  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

S-wave contributions:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

a_μ^{HLbL} in 10^{-11} units

cutoff	1 GeV	1.5 GeV	2 GeV	∞
$I=0$	-9.2	-9.5	-9.3	-8.8
$I=2$	2.0	1.3	1.1	0.9
sum	-7.3	-8.3	-8.3	-7.9

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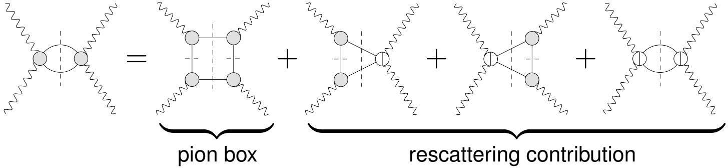
Our first numerical result

Two-pion contributions to HLbL:



$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1)\cdot 10^{-11}$$

Outline

Conclusions

- The $\sim$3.5 σ discrepancy between SM and measurement for $(g-2)_\mu$ continues to stay
- The Fermilab Muon $g-2$ experiment aims to reduce the experimental uncertainty by a factor four, potentially leading to a 7σ discrepancy
- I have reviewed current theoretical efforts based on dispersion relations to improve the calculations of the hadronic contributions, HVP and HLbL, in a model-independent way
- The prospects for matching the experimental reduction of the uncertainty with an analogous reduction on the theory side are becoming very concrete

Back-up slides

Gravitational effects in $(g-2)_\mu$? Morishima, Futamase, Shimizu, 1801.10246

Abstract: "... It implies that $(g-2)_\mu$ measured on the Earth contains the gravitational correction of $|a_\mu| = 2.1 \cdot 10^{-9}$ ..."

But in the text one finds Eq. (43):

$$a_\mu^{\mathrm{eff}} = \frac{\Omega_a(1+3\epsilon^2\phi)}{\Omega_L(1+3\epsilon^2\phi) - \Omega_a(1+3\epsilon^2\phi)} = \frac{\Omega_a}{\Omega_L - \Omega_a} = a_\mu$$

which means: no effect!

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which means: no effect! But below:

$$\vec{\Omega}_a^{\mathrm{eff}} = -\frac{e}{m}\left[a_\mu(1+3\epsilon^2\phi)\mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2-1} - \epsilon^2\phi K\right)\vec{\beta}\times\mathbf{E} + \ldots\right]$$

"If the γ was experimentally tuned to minimize the electric field contribution, the quantity, defined as

$$a_\mu^{\mathrm{mod}} = a_\mu - \epsilon^2\phi K$$

might have been measured."

Pion-box saturation with photon virtualities