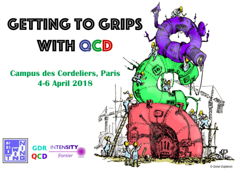
# Soft Photon Contributions to Hadronic Processes from Lattice QCD

Getting to Grips with QCD Workshop





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6 April 2018


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European Twisted Mass Collaboration

# Outline

## Introduction

- Motivation to include QED in QCD
- Why lattice QCD+QED
- If Lattice QCD is though, including QED is even harder!
- Include QED: the perturbative approach

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- Hadron Masses
- Decay rates
  1. In pure QCD (infrared finite)
  2. Ratio of decay rates (infrared finite)
  3. Single decay rate (infrared troubled)
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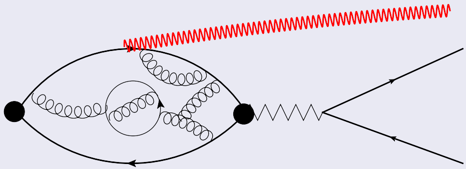
## Some final words

- Work in progress
- Future developments

# Dealing with photons

<u>Hard photons</u> - $E \sim$ many GeV



Perturbation theory

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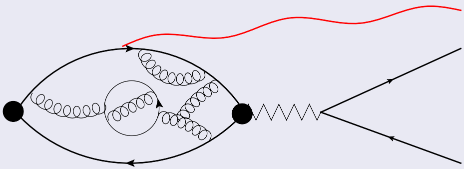

Point-like hadrons

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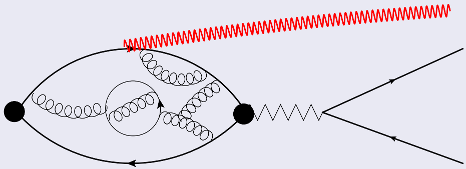

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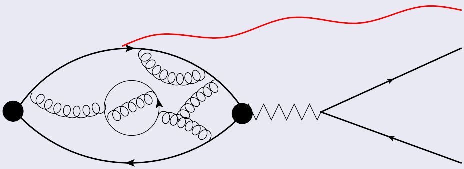

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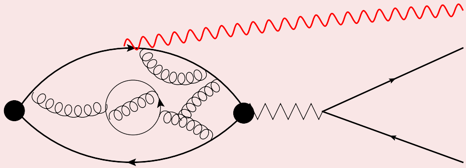
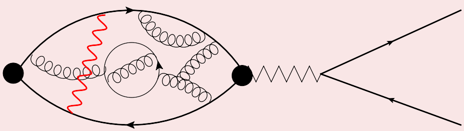
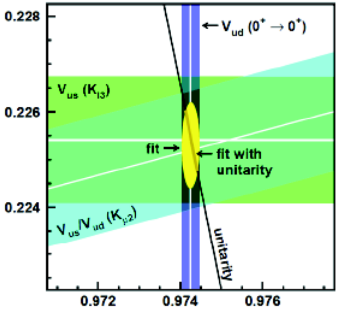
## What to do with soft photons?

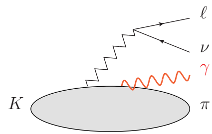




...Here we come to the rescue...

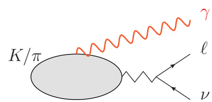
# Example: CKM matrix elements from semileptonic and leptonic $K$ and $\pi$ decays




## Semileptonic

$$\underbrace{\Gamma_{K\to\pi l\bar{\nu}(\gamma)}}_{experiments} \propto |V_{us}|^2 \underbrace{\left|f_+^{K\pi}(0)\right|^2}_{QCD}$$




## Leptonic

$$\underbrace{\frac{\Gamma_{K\to\ell\bar{\nu}(\gamma)}}{\Gamma_{\pi\to\ell\bar{\nu}(\gamma)}}}_{experiments} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{QCD}$$

### Hadronic matrix elements, lattice results

$$\begin{aligned} f_+^{K\pi}(0) &= 0.956\,(8) \\ f_K/f_\pi &= 1.193\,(5) \end{aligned}$$

in the **isospin symmetric** limit.

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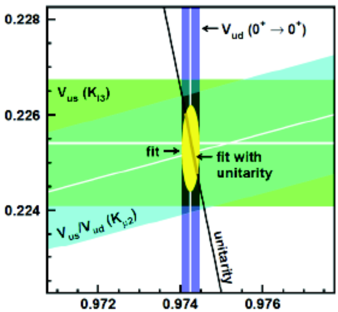


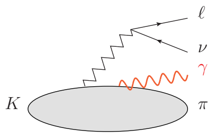


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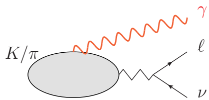


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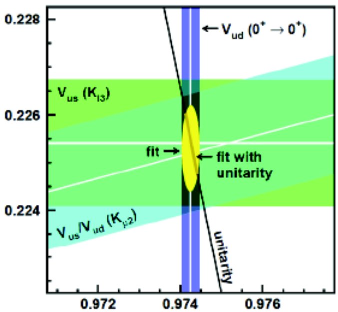


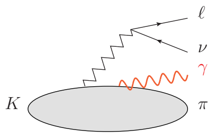


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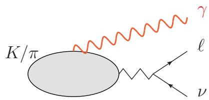


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## Indeed ChPT estimates of these effects are:

| | |
|---|---|
| $\left(f_+^{K^+\pi^0}/f_+^{K^-\pi^+}-1\right)^{QCD}=2.9(4)\%$ <br> A. Kastner, H. Neufeld (EPJ C57, 2008) | $\left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}-1\right)^{QCD}=-0.22(6)\%$ <br> V. Cirigliano, H. Neufeld (Phys.Lett.B700, 2011) |

# Why lattice QCD

## Lattice QCD = QCD

- ~~Only~~ Powerful method to solve **non-perturbative QCD** from the first theory principles
- Only parameters present in QCD lagrangian
- Precision only limited by available **computational power** (in principle)
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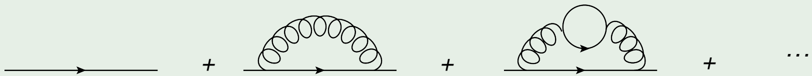
## Perturbative vs. nonperturbative

- Perturbative: computing order by order in the couplings

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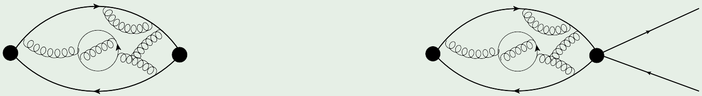

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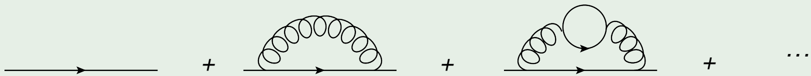


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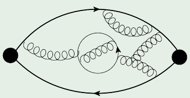
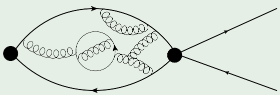
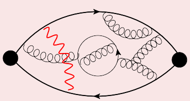
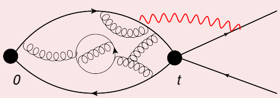




## Lattice QCD can incorporate non-factorizable QED contributions





# Lattice in a nutshell

## Discretize the theory

- Replace continuous space-time with $N_x \times N_y \times N_z \times N_t$ points with spacing $a$.
- Write a discretized action having QCD as limit when $a \to 0$.

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## Observables computed as ordinary multi-dimensional integrals

$$\langle O \rangle = Z^{-1} \int D\left[A, \psi\right] O e^{-S(\psi, U)}$$

1. sample the fields configuration space $[A, \psi]$ with weight $\propto Z^{-1} \exp\left(-S\right)$
   - EXTREMELY costly: years of continuous calculation on supercomputers
   - current dataset produced between 2007-2014 (ETM Collaboration)
2. measure observable of interests: $\langle O \rangle = \frac{1}{N} \sum_{i=1}^{N} O_{[A,\psi]_i}$: order of magnitudes cheaper.

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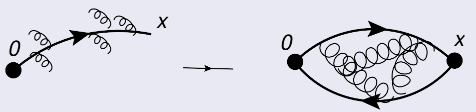
## Correlation functions

- Compute propagators by solving numerically the Dirac equation: $D_{x,y} S_{y,0} = \delta_{x,0}$ on a fixed gauge fields background,
- combine them building correlation functions:




At the end take the **continuum limit** $(a \to 0)$.

# Why Lattice QCD is so computationally demanding?

*"Give the energy of a nuclear pant to the latticists, and hurray!, they will be able to tell you the mass of the proton"*

*[quote from an AdS/CFT entusiast Nobel laureate]*

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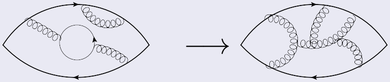
Simulation cost: rapidly grows as quark masses are lowered (zero mode builds up)

Early solution: quenching = drop virtual pair contributions from functional integral



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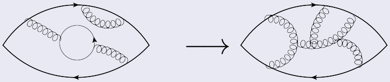

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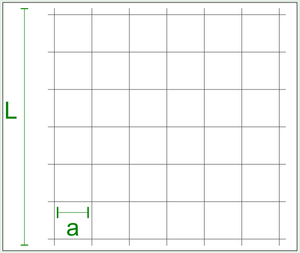
## Lattice size dependence




- **Simulation cost:** $[\#points]^{k>1} = \left[(L/a)^4\right]^k$ (scales: $a \ll 1/M_H$, $L \gg 1/M_\pi$)
- **Early solution:** $\#points = 4^4$
- **Nowadays:** $\#points = 48^3 \times 96 \div 64^3 \times 128$
  - D physics: $M_D/M_\pi \sim 15$, $M_{J/\psi}/M_\pi \sim 22$
  - B physics: $M_B/M_\pi \sim 40$, $M_\Upsilon/M_\pi \sim 70$

# State of the art

## Nowadays

1. Physical light quarks and large volumes ($\gtrsim (6\,\mathrm{fm})^3$ ),
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## What helped these improvements?

Increase in computing power


Conceptual developments

- Improved regularizations of LQCD
- Better understanding of behavior of Monte Carlo w.r.t $(m_0,\, g_0)$

Algorithm breakthroughs

- Multiple timescale Molecular Dynamic integrators
- Deflation, Multigrid, Domain Decomposition solvers, etc.

# More complications from QED

## The target: Fully unquenched QCD + QED

$$\mathcal{L} = \sum_i \bar{\psi}_i \left[ m_i - i\not{D}_i \right] \psi_i + \mathcal{L}_{gluons} + \mathcal{L}_{photon}, \quad D_{i,\mu} = \partial_\mu + igA^a_\mu T^a + ie_i A_\mu$$

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## Introducing photons

Power-like Finite Volume Effects due to long range interaction

Zero mode from photon propagator: $\int \frac{\delta_{\mu\nu}}{k^2} d^4k \;\to\; \sum_k \frac{\delta_{\mu\nu}}{k^2}$
massive photons, removal of zero mode, $C^*$ boundary conditions...

Renormalization pattern gets more complicated

Additional divergencies arises!

UV completeness: Nobody knows how to tame QED to all orders!

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## Practical problem

- Traditionally, gauge configuration datasets include only gluons
- Dedicated simulations with huge cost
- Even greater cost due to additional zero modes.

# The Roman approach - RM123 collaboration

## Pioneering papers

- "*Isospin breaking effects due to the up-down mass difference in Lattice QCD*", [JHEP 1204 (2012)]
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### 3) Roma Tre

D.Giusti, V.Lubicz, S.Romiti, F.S, S.Simula, C.Tarantino


### 1) La Sapienza

M.Di Carlo, G.Martinelli

### 2) Tor Vergata

G.deDivitiis, P.Dimopoulos, R.Frezzotti, N.Tantalo

⋆ **Guest Star** from Southampton University: C.T.Sachrajda

# The Roman approach - RM123 collaboration

## Perturbative expansion

Work on top of the isospin symmetric theory $\boldsymbol{\mathcal{L} = \mathcal{L}_{Iso\,symm} + \mathcal{L}_{Iso\,break}}$

$$\mathcal{L}_{Iso\,break} = e\mathcal{L}_{QED} + \delta m \mathcal{L}_{mass}, \quad e^2 = \frac{4\pi}{137.04}, \quad \delta m = (m_d - m_u)/2$$

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## Pros

Cleaner: Factorize small parameters $e$ and $\delta m$, introduce QED only when needed

Cheaper: No need to generate new QCD gauge field backgrounds.

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→ Only method to include QED in matrix elements (currently known)

# The perturbative expansion in $e^2$

## Keep QCD to all orders and QED to $\mathcal{O}\left(e^2\right)$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int D\left[A_\mu, U^{QCD}, \psi, \bar{\psi}\right] O \,\left(1 - e^2 S_1 + \mathcal{O}\left(e^4\right)\right) \exp\left[-S_0\right]$$

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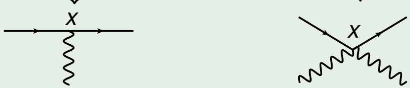
## Which on the lattice means...

$$S_1 = \underbrace{\left[\int dx\, V_\mu\left(x\right) A_\mu\left(x\right)\right]^2}_{} + \underbrace{\int dx\, T_\mu\left(x\right) A_\mu^2\left(x\right)}_{}$$




- $V^2$: Two photon-fermion-fermion vertex (as in the continuum)
- $T$: One photon-photon-fermion-fermion vertex (lattice special).

# The case of the pion

## Basic correlation function

$$C\left(t\right)=\sum_{\vec{x}}\left\langle P\left(\vec{x},t\right)P^{\dagger}\left(0\right)\right\rangle_{QCD+QED}, \qquad P=\bar{\psi}\gamma_5\psi$$

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## Functional integral

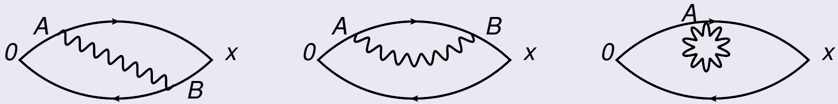
$$C(t) = C_0(t) + C_1(t) =$$

$$\left\langle P(\vec{x}, t)\, P^{\dagger}(0) \right\rangle_{QCD} - e^2 \left\langle P(\vec{x}, t) \sum_{y} S_1(y) P^{\dagger}(0) \right\rangle_{QCD}$$

Now take all Wick contractions...

# Diagrams

## Fermionically connected - easy part (so to say)




(gluons not drawn, connecting fermion lines in all possible ways)

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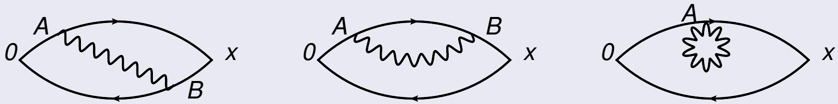


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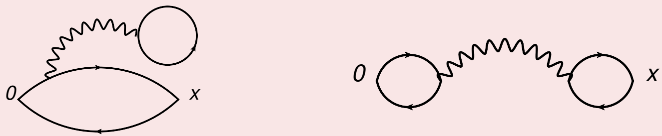
## Disconnected - various degree of nastiness





Subdominant (work is in progress to include)

# Hadron masses

## Infrared safe

- Hadron masses are finite quantities (after the action is properly renormalized)
- True without and with QED.

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- Large euclidean time behaviour (see next slide).

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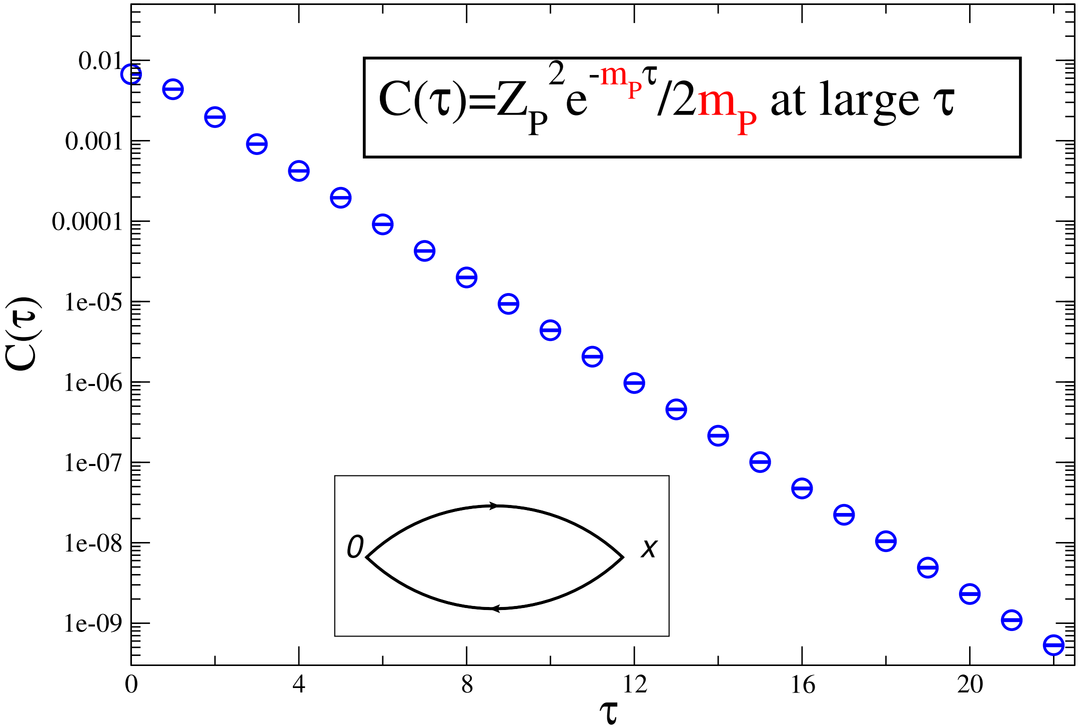
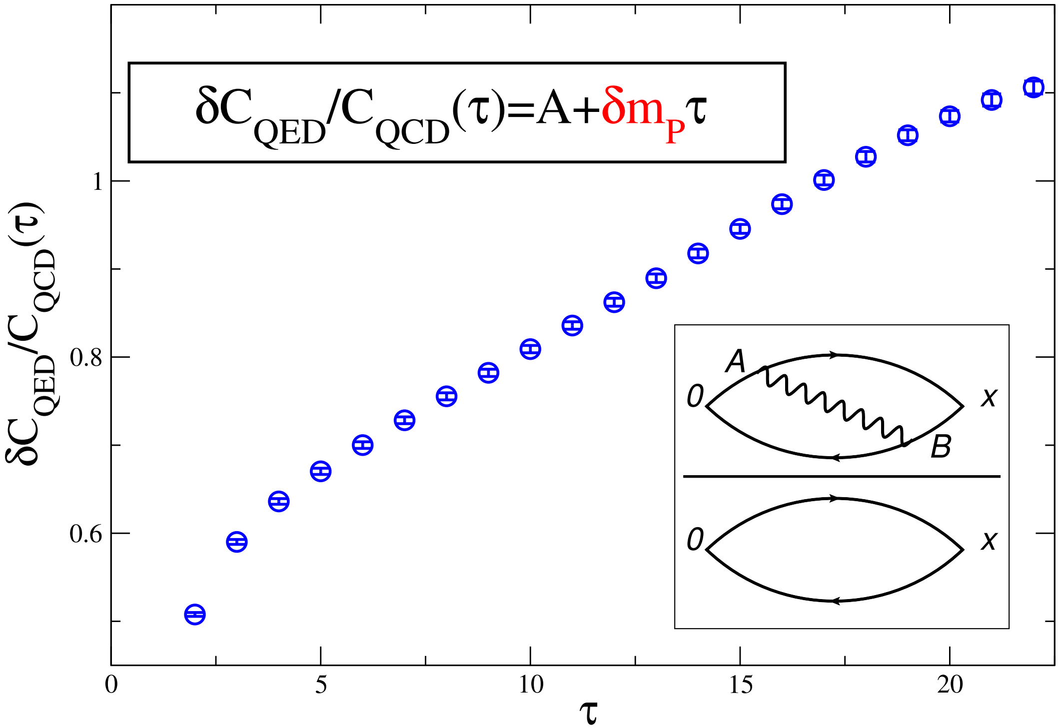
## Other collaborations, other approaches

- FNAL/MILC: valence quark contribution to all orders
- Budapest-Marseille-Wuppertal: fully dynamical simulation of QCD+QED
- QCDSF/UKQCD: fully dynamical simulation of QCD+QED
- RBC/UKQCD: comparison of perturbative and all-order approach.

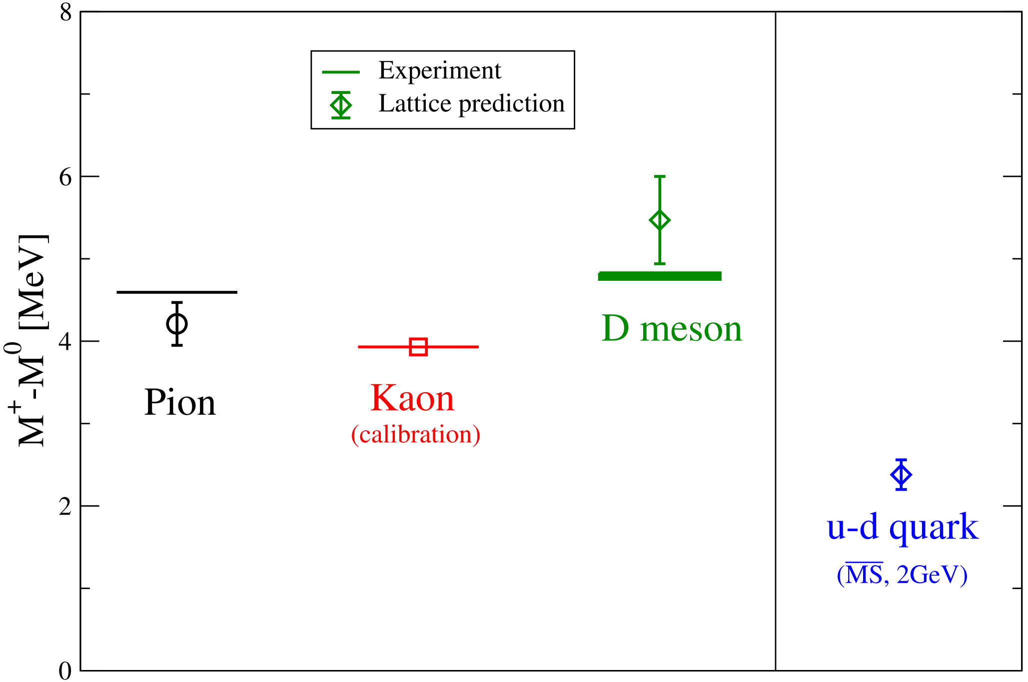
# Pseudoscalar meson 2pts. correlation function (no QED)



# Hadron masses at $\mathcal{O}\left(e^2\right)$

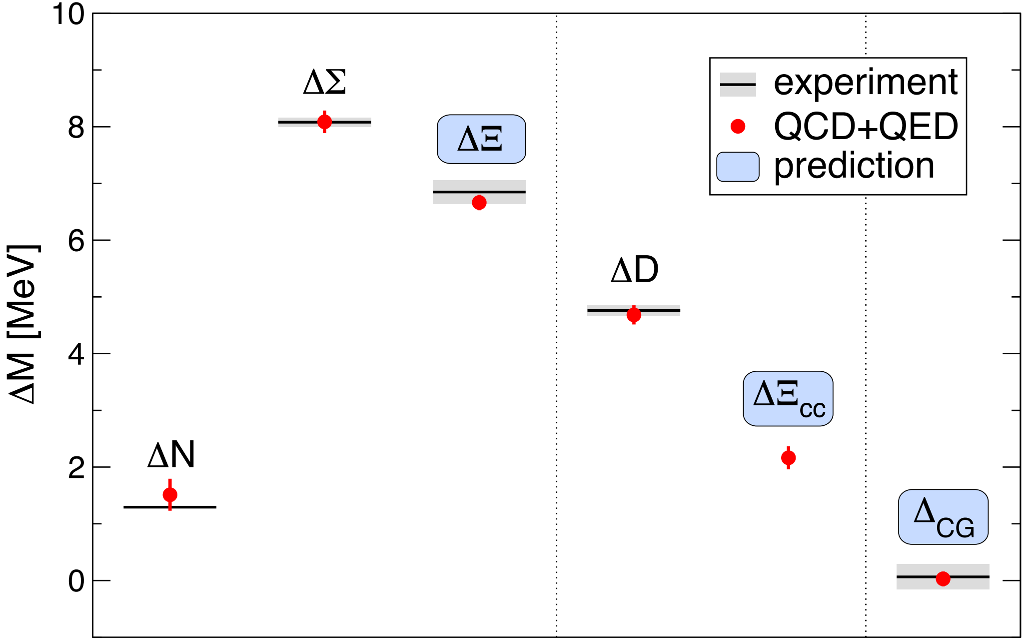


# Some results, meson mass (perturbative expansion)




RM123 coll., "*Leading isospin-breaking corrections to pion, K and D mesons*", PRD95 (2017)

# Some results, baryons (direct simulation)




BMW coll.: "*Ab initio calculation of the neutron-proton mass difference*", Science 347 (2015)

# Matrix elements

## More problems

- In general the amplitudes, are **infrared divergent**
- On the lattice, a natural infrared cutoff is provided by the **finite volume**
- But physically, only combinations of **Real** + **Virtual** contribution is finite

[cfr Einan Gardi talk on Wednseday]

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- We consider the leptonic decay of a **charged pion**
- The method is general (we are now applying it to semileptonic decays).

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- We consider the leptonic decay of a **charged pion**
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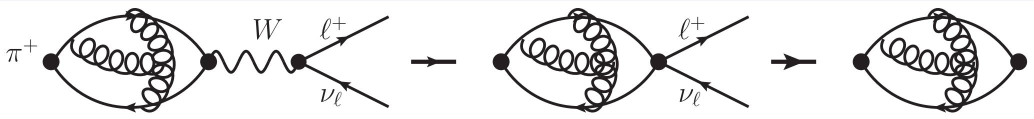
**Nobody has gone there before!**

# Leptonic decays of mesons (at tree level in QED: $e = 0$)

**Full process** | **Eff. weak hamiltonian** | **QCD side**



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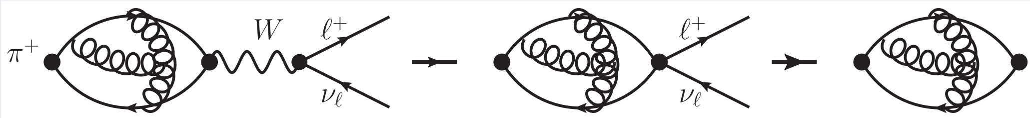
## Full process — Eff. weak hamiltonian — QCD side




## Two point correlation functions

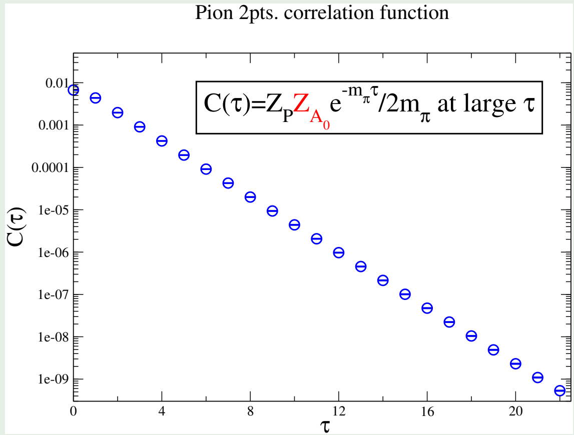
$$\Gamma_{\pi \to \ell \bar{\nu}} = \underbrace{|V_{xy}|^2}_{\text{CKM}} \underbrace{\mathcal{K}(m_\ell, m_M)}_{\text{kinematics}} \Big| \underbrace{f_\pi}_{\text{dec. constant}} \Big|^2$$

$$f_\pi = \frac{Z_A}{m_\pi} = \frac{\langle 0 | A_0 | \pi \rangle}{m_\pi}$$

### $Z$: coupling of current inducing decay

From lattice, 2 point correlation functions:

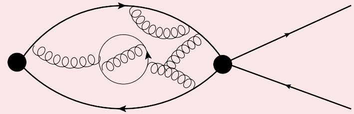
$$C(\tau) = \langle O^\dagger_{A_0}(\tau)\, O_P(0) \rangle, \; O = \bar{\psi} \Gamma \psi$$

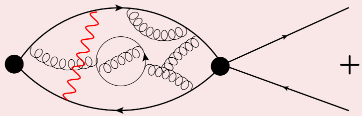
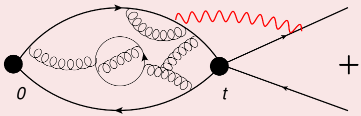


# Leptonic decays of mesons (with QED)

## Zero photons in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{0ph}_{\pi^+\to\ell^+\nu} = \left|A^0\right|^2 + 2e^2\left|A^0 A^1\right| + \mathcal{O}\left(e^4\right)$$

$A_0 =$ 

$A_1 =$  $+$ 

$+ \dots$

IR DIVERGENT 😱

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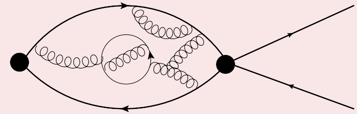
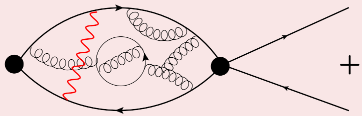
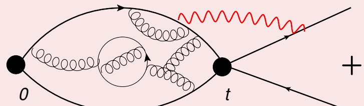
$A_0 =$ 

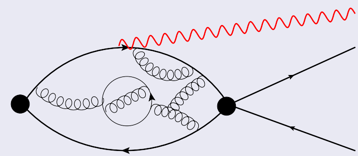
$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT

## One photon in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{1ph}_{\pi^+\to\ell^+\nu\gamma} = e^2\left|B^0\right|^2$$

$B_0 =$ 

Again, IR DIVERGENT

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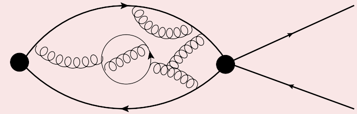
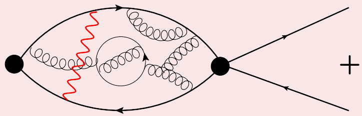
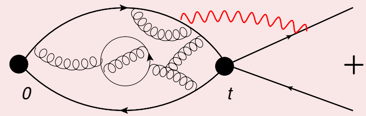
$A_0 =$ 

$A_1 =$  $+$ 

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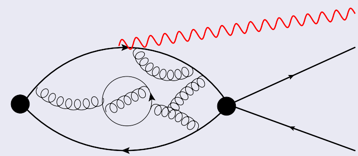
$B_0 =$ 

Again, IR DIVERGENT 🤔

## Solution

[Bloch and Nordsieck, PR52 (1937)]

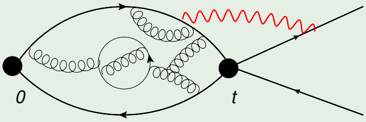
$$\Gamma = \Gamma^{0ph} + \Gamma^{1ph} \text{ is } \underline{\text{finite}}$$

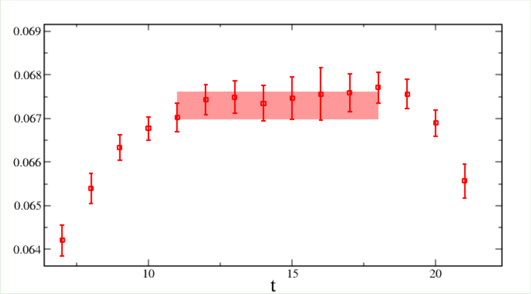
# The strategy

[N.Carrasco et al., PRD91 (2015)]

## Virtual photon

- Needs to implement leptons
- Not too demanding numerically.



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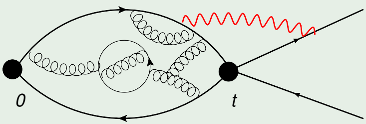
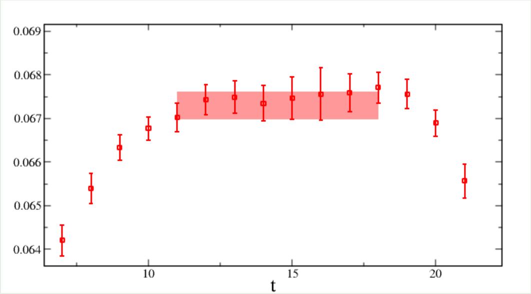
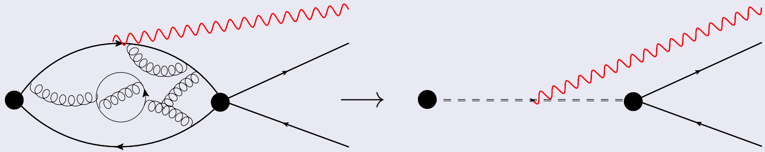



## Real photon

- More demanding numerically/different process
- For the time being, use point-like approximation and consider $E_\gamma < 20$ MeV

$B_0 =$ 

Cut-off appropriate experimentally ($\gamma$ detector sensitivity) and theoretically ($\pi$ inner structure)

(Work is in progress to compute on the Lattice)

# Intermediate step

$$\Gamma\left(\Delta E_\gamma\right) = \underbrace{\Gamma^{0ph}}_{\text{lattice in a box}} + \underbrace{\Gamma^{1ph}_{pt}\left(\Delta E_\gamma\right)}_{\text{perturbation theory, massive photon}}$$

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To ensure proper cancellation of IR divergence with different regulator, add and subtract $\Gamma^0_{pt}$

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## $\Gamma^0_{pt,V}$: V.Lubicz et al, Phys.Rev. D95 (2017)

- Perturbation theory with pointlike pion in finite volume ($V = L^4$)
- IR divergences $\propto \log L$ cancel in the difference $\Gamma^{0ph}(L) - \Gamma^0_{pt}(L)$ (fit the remainder)
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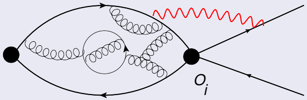
## $\lim_{m_\gamma\to 0}\left[\Gamma^{0}_{pt}(m_\gamma) + \Gamma^{1ph}_{pt}(m_\gamma, \Delta E_\gamma)\right]$: N.Carrasco et al. Phys.Rev. D91 (2015)

- Perturbation theory with pointlike pion with finite photon mass
- Neglected structure dependence: estimated to be small (V.Cirigliano, I.Rosell, JHEP'07)
  This might not be the case for $D$ or $B$ mesons ($m_{B^*} - m_B \sim 45$ MeV)
- Reproduce the the total rate (Berman, PRL 1958, and Kinoshita, PRL 1959).

# Matching to the "real world"

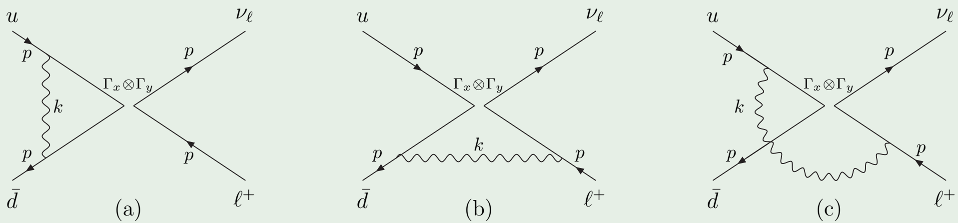
## Correlation functions computed with bare operators




Needs renormalization: $O_i^{ren} = Z_{ij} O_j^{bare}$

$$O_{1,2} = (V \mp A)_q \otimes (V - A)_\ell$$
$$O_{3,4} = (S \mp P)_q \otimes (S - P)_\ell$$
$$O_5 = \left(T + \tilde{T}\right)_q \otimes \left(T + \tilde{T}\right)_\ell$$

## Vertex




## Scheme

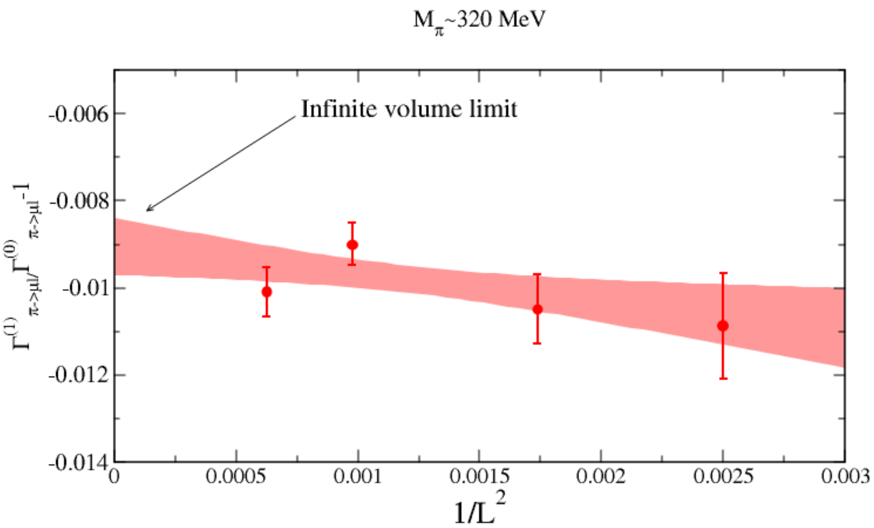
- As a first step: RI-MOM for QCD + perturbation theory for QED
- In the near future: RI-(S)MOM for QCD + QED

(not to mention the matching to $W$-reg where $G_F$ is defined)

# Infinite volume extrapolation

## Volume dependence

- IR divergences $\propto \log L$ cancel in the difference $\Gamma^{0ph}(L) - \Gamma^{0}_{pt}(L)$
- $1/L$ cancel as well (Ward identity)
- Best fit with $1/L^2$ (and $1/L^3$) and extrapolate to $L \to \infty$



# Let's start from a slightly simpler quantity

## QED contribution to ratio of decay width of Kaon and Pion

$$\frac{\Gamma_{K^+\to\ell^+\nu(\gamma)}}{\Gamma_{\pi^+\to\ell^+\nu(\gamma)}} = \frac{\Gamma_{K^+\to\ell^+\nu}}{\Gamma_{\pi^+\to\ell^+\nu}}\left(1+\delta R_{K\pi}\right), \qquad \delta R_{K\pi} = \delta R_K - \delta R_\pi$$

- Reduction of noise
- Large cancellation of renormalization correction
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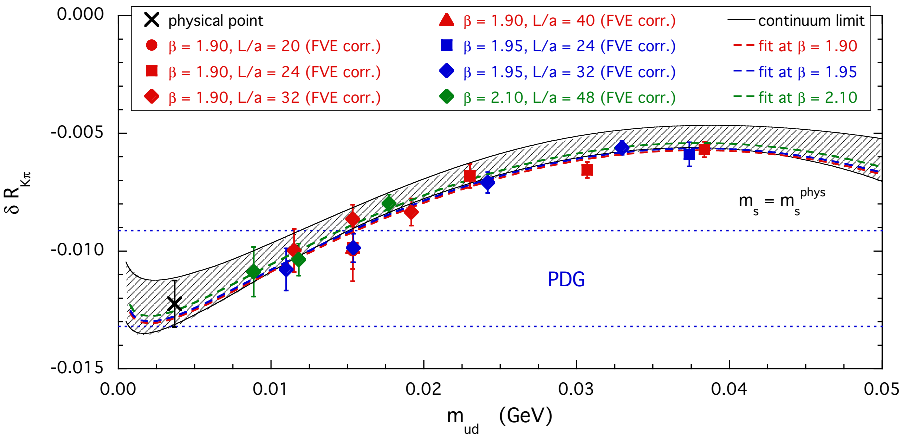
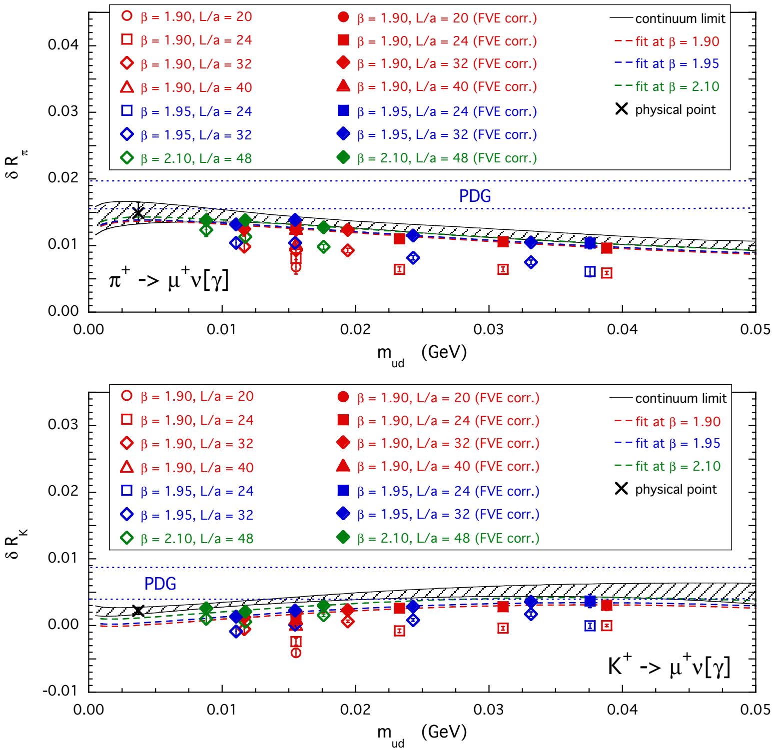



[D.Giusti et al., Phys. Rev. Lett. 120, 072001 (2018)]

# Separate Pion and Kaon corrections [PRELIMINARY]



# Development

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- Heavy mesons
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## Maybe one day

- Develop a strategy for semileptonic decays (analytic continuation to Minkowsky)
- Corrections to $K \to \pi\pi$
- ...

# THANK YOU!