Bose-Einstein Results from L3 and the Tau Model

W.J. Metzger

Radboud University Nijmegen

with T. Novák, T. Csörgő, W. Kittel

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BEC Introduction

\[ R_2 = \frac{\rho_2(p_1, p_2)}{\rho_1(p_1)\rho_1(p_2)} = \frac{\rho_2(Q)}{\rho_0(Q)} \]

Assuming particles produced incoherently with spatial source density \( S(x) \),

\[ R_2(Q) = 1 + \lambda |\tilde{S}(Q)|^2 \]

where \( \tilde{S}(Q) = \int dx \ e^{iQx} S(x) \) – Fourier transform of \( S(x) \)

\( \lambda = 1 \) — \( \lambda < 1 \) if production not completely incoherent

Assuming \( S(x) \) is a Gaussian with radius \( r \)

\[ R_2(Q) = 1 + \lambda e^{-Q^2r^2} \]
The L3 Data

- $e^+e^- \rightarrow \text{hadrons at } \sqrt{s} \approx M_Z$
- about $36 \cdot 10^6$ like-sign pairs of well measured charged tracks from about $0.8 \cdot 10^6$ events
- about $0.5 \cdot 10^6$ 2-jet events — Durham $y_{\text{cut}} = 0.006$
- about $0.3 \cdot 10^6 > 2$ jets, “3-jet events”
- use mixed events for reference sample, $\rho_0$
Previous Results: Elongation

Results in **LCMS frame**: Longitudinal = Thrust axis

\[ \frac{R_L}{R_{side}} \]

- L3: \( 1.25 \pm 0.03^{+0.36}_{-0.05} \)
- OPAL: \( 1.19 \pm 0.03^{+0.08}_{-0.01} \)

(ZEUS finds similar results in ep)

\( \sim 25\% \) elongation along thrust axis

**OPAL**: Elongation larger for narrower jets

- **Conclusion**: Elongation, but it is relatively small.
- **So**: Ignore it. — Assume spherical.
Transverse Mass dependence of $r$

$r$ decreases with $m_t$ (or $k_t$) for all directions

Smirnova & Lörstad, 7th Int. Workshop on Correlations and Fluctuations (1996)

Van Dalen, 8th Int. Workshop on Correlations and Fluctuations (1998)

Results on $Q$ from $L_3 Z$ decay

$$R_2 = \gamma \cdot [1 + \lambda G] \cdot (1 + \epsilon Q)$$

- **Gaussian**
  $$G = \exp \left( -(rQ)^2 \right)$$

- **Edgeworth expansion**
  $$G = \exp \left( -(rQ)^2 \right) \cdot [1 + \frac{\kappa}{3!} H_3(rQ)]$$
  Gaussian if $\kappa = 0$
  $$\kappa = 0.71 \pm 0.06$$

- **symmetric Lévy**
  $$G = \exp \left( -|rQ|^{\alpha} \right)$$
  $$0 < \alpha \leq 2$$
  $$\alpha = 1.34 \pm 0.04$$

Poor $\chi^2$. Edgeworth and Lévy better than Gaussian, but poor.
Problem is the dip of $R_2$ in the region $0.6 < Q < 1.5$ GeV
Summary

- BEC depend (approximately) only on $Q$, not its components.
- BEC depend on $m_t$.
- Gaussian and similar parametrizations do not fit.

Turn now to a model providing such dependence.
The $\tau$-model


- Assume avg. production point is related to momentum:
  \[ \bar{x}^\mu (p^\mu) = a \tau p^\mu \]
  where for 2-jet events, \( a = 1/m_t \)
  \[ \tau = \sqrt{t^2 - r_z^2} \]
  is the “longitudinal” proper time
  and \( m_t = \sqrt{E^2 - p_z^2} \) is the “transverse” mass

- Let \( \delta_\Delta (x^\mu - \bar{x}^\mu) \) be dist. of prod. points about their mean, and \( H(\tau) \) the dist. of \( \tau \). Then the emission function is
  \[ S(x, p) = \int_0^\infty d\tau H(\tau) \delta_\Delta (x - a \tau p) \rho_1(p) \]

- In the plane-wave approx.
  \[ \rho_2(p_1, p_2) = \int d^4x_1 d^4x_2 S(x_1, p_1) S(x_2, p_2) (1 + \cos ( [p_1 - p_2] [x_1 - x_2] ) ) \]

- Assume \( \delta_\Delta (x - a \tau p) \) is very narrow — a \( \delta \)-function. Then
  \[ R_2(p_1, p_2) = 1 + \lambda \text{Re} \tilde{H} \left( \frac{a_1 Q^2}{2} \right) \tilde{H} \left( \frac{a_2 Q^2}{2} \right) , \quad \tilde{H}(\omega) = \int d\tau H(\tau) \exp(i\omega \tau) \]
BEC in the $\tau$-model

- **Assume** a Lévy distribution for $H(\tau)$
  Since no particle production before the interaction, $H(\tau)$ is one-sided.
  Characteristic function is
  $$\tilde{H}(\omega) = \exp \left[ -\frac{1}{2} (\Delta \tau |\omega|)^{\alpha} \left( 1 - i \text{sign}(\omega) \tan \left( \frac{\alpha \pi}{2} \right) \right) + i \omega \tau_0 \right], \quad \alpha \neq 1$$
  where
  - $\alpha$ is the index of stability
  - $\tau_0$ is the proper time of the onset of particle production
  - $\Delta \tau$ is a measure of the width of the dist.

- Then, $R_2$ depends on $Q, a_1, a_2$
  $$R_2(Q, a_1, a_2) = \gamma \left\{ 1 + \lambda \cos \left[ \frac{\tau_0 Q^2 (a_1 + a_2)}{2} + \tan \left( \frac{\alpha \pi}{2} \right) \left( \frac{\Delta \tau Q^2}{2} \right)^{\alpha} \frac{a_1^{\alpha} + a_2^{\alpha}}{2} \right] \right. \cdot \exp \left[ - \left( \frac{\Delta \tau Q^2}{2} \right)^{\alpha} \frac{a_1^{\alpha} + a_2^{\alpha}}{2} \right] \right\} \cdot (1 + \epsilon Q)$$
BEC in the $\tau$-model

$$R_2(Q, a_1, a_2) = \gamma \left\{ 1 + \lambda \cos \left[ \frac{\tau_0 Q^2(a_1+a_2)}{2} + \tan \left( \frac{\alpha \pi}{2} \right) \left( \frac{\Delta \tau Q^2}{2} \right)^\alpha \frac{a_1^\alpha + a_2^\alpha}{2} \right] \cdot \exp \left[ - \left( \frac{\Delta \tau Q^2}{2} \right)^\alpha \frac{a_1^\alpha + a_2^\alpha}{2} \right] \right\} \cdot (1 + \epsilon Q)$$

Simplification:

- Particle production begins immediately, $\tau_0 = 0$
- Effective radius, $R$, defined by $R^{2\alpha} = (\frac{\Delta \tau}{2})^\alpha \frac{a_1^\alpha + a_2^\alpha}{2}$
- Then
  $$R_2(Q) = \gamma \left[ 1 + \lambda \cos \left( (R_a Q)^{2\alpha} \right) \exp \left( - (R Q)^{2\alpha} \right) \right] \cdot (1 + \epsilon Q)$$

where $R_a^{2\alpha} = \tan \left( \frac{\alpha \pi}{2} \right) R^{2\alpha}$

Compare to sym. Lévy parametrization:

$$R_2(Q) = \gamma \left[ 1 + \lambda \exp \left[ - |r Q|^{\alpha} \right] \right] \cdot (1 + \epsilon Q)$$
2-jet Results on Simplified $\tau$-model from $L3$ $Z$ decay

$R_a$ free
$\chi^2$/dof = 91/94

$R_a^2 \alpha = \tan \left( \frac{\alpha \pi}{2} \right) R^{2\alpha}$
$\chi^2$/dof = 95/95
3-jet Results on Simplified $\tau$-model from L3 $Z$ decay

$R_a$ free
$\chi^2$/dof = 84/94

$R_a^{2\alpha} = \tan \left( \frac{\alpha \pi}{2} \right) R^{2\alpha}$
$\chi^2$/dof = 113/95
CL = 10%
## Summary of Simplified $\tau$-model

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$R$ (fm)</th>
<th>$R_a$ (fm)</th>
<th>CL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-jet</td>
<td>0.41 $\pm$ 0.02(^{+0.04}_{-0.06})</td>
<td>0.79 $\pm$ 0.04(^{+0.09}_{-0.19})</td>
<td>0.69 $\pm$ 0.04(^{+0.21}_{-0.09})</td>
<td>57%</td>
</tr>
<tr>
<td>3-jet</td>
<td>0.35 $\pm$ 0.01(^{+0.03}_{-0.04})</td>
<td>1.06 $\pm$ 0.05(^{+0.59}_{-0.31})</td>
<td>0.85 $\pm$ 0.04(^{+0.15}_{-0.05})</td>
<td>76%</td>
</tr>
<tr>
<td>3-jet</td>
<td>0.41 $\pm$ fixed</td>
<td>0.93 $\pm$ 0.03</td>
<td>0.76 $\pm$ 0.01</td>
<td>38%</td>
</tr>
<tr>
<td>2-jet</td>
<td>0.44 $\pm$ 0.01(^{+0.05}_{-0.02})</td>
<td>0.78 $\pm$ 0.04(^{+0.09}_{-0.16})</td>
<td>—</td>
<td>49%</td>
</tr>
<tr>
<td>3-jet</td>
<td>0.42 $\pm$ 0.01(^{+0.02}_{-0.04})</td>
<td>0.98 $\pm$ 0.04(^{+0.55}_{-0.14})</td>
<td>—</td>
<td>10%</td>
</tr>
<tr>
<td>3-jet</td>
<td>0.44 $\pm$ fixed</td>
<td>0.87 $\pm$ 0.01</td>
<td>—</td>
<td>3%</td>
</tr>
</tbody>
</table>

- Consistent values of $\alpha$
- $R_a^{2\alpha} = \tan\left(\frac{\alpha\pi}{2}\right)R^{2\alpha}$ to $0.5\sigma$ for 2-jet and to $1.5\sigma$ for 3-jet
- Simplified $\tau$-model works well
- $R$ seems to be larger for 3-jet than for 2-jet events
Full $\tau$-model for 2-jet events — $a = 1/m_t$

$$R_2(Q, m_{t1}, m_{t2}) = \gamma \left\{ 1 + \lambda \cos \left[ \frac{\tau_0 Q^2 (m_{t1} + m_{t2})}{2 (m_{t1} m_{t2})} + \tan \left( \frac{\alpha \pi}{2} \right) \left( \frac{\Delta \tau Q^2}{2} \right)^\alpha \frac{m_{t1}^\alpha + m_{t2}^\alpha}{2 (m_{t1} m_{t2})^\alpha} \right] \right. \right.$$ 

$$\times \exp \left[ - \left( \frac{\Delta \tau Q^2}{2} \right)^\alpha \frac{m_{t1}^\alpha + m_{t2}^\alpha}{2 (m_{t1} m_{t2})^\alpha} \right] \} \cdot (1 + \epsilon Q)$$

- Fit $R_2(Q)$ using avg $m_{t1}, m_{t2}$ in each $Q$ bin, $m_{t1} > m_{t2}$
- $\tau_0 = 0.00 \pm 0.02$
  so fix to 0
- $\chi^2$/dof = 90/95
Full $\tau$-model for 2-jet events

- $\tau$-model predicts dependence on $m_t$, $R_2(Q, m_{t1}, m_{t2})$
- Parameters $\alpha$, $\Delta \tau$, $\tau_0$ are independent of $m_t$
- $\lambda$ (strength of BEC) can depend on $m_t$

- divide $m_{t1}$-$m_{t2}$ plane in regions (equal statistics)
- in each region fit $R_2(Q)$ using avg $m_{t1}$, $m_{t2}$ in each $Q$ bin with $\alpha$, $\Delta \tau$, fixed to values found for entire plane and $\tau_0 = 0$ fits
Summary of $\tau$-model

- $\tau$-model with a one-sided Lévy proper-time distribution describes BEC well
  - in simplified form it provides a new parametrization of $R_2(Q)$ for both 2- and 3-jet events,
  - in full form for 2-jet events, $R_2(Q, m_{t1}, m_{t2})$
    - both $Q$- and $m_t$-dependence described correctly
    - Note: we found $\Delta \tau$ to be independent of $m_t$
      $\Delta \tau$ enters $R_2$ as $\Delta \tau Q^2 / m_t$
      In Gaussian parametrization, $r$ enters $R_2$ as $r^2 Q^2$
      Thus $\Delta \tau$ independent of $m_t$ corresponds to $r \propto 1/\sqrt{m_t}$

- BUT, what about elongation?
Elongation?

- Previous elongation results used fits of Gaussian or Edgeworth
- But we find that Gaussian and Edgeworth fit $R_2(Q)$ poorly
- $\tau$-model predicts no elongation and fits the data well
- Could the elongation results be an artifact of an incorrect fit function?
  or is the $\tau$-model in need of modification?
- So, we modify *ad hoc* the $\tau$-model description to allow elongation and see what we get
Elongation in the Simplified $\tau$-model?

LCMS: $Q^2 = Q^2_L + Q^2_{\text{side}} + Q^2_{\text{out}} - (\Delta E)^2$

$= Q^2_L + Q^2_{\text{side}} + Q^2_{\text{out}} (1 - \beta^2)$, $\beta = \frac{p_{1\text{out}} + p_{2\text{out}}}{E_1 + E_2}$

Replace $R^2 Q^2 \Longrightarrow A^2 = R^2 L Q^2_{\text{L}} + R^2_{\text{side}} Q^2_{\text{side}} + R^2_{\text{out}} Q^2_{\text{out}}$

Then in $\tau$-model,

$R_2(Q_L, Q_{\text{side}}, Q_{\text{out}}) = \gamma \left[ 1 + \lambda \cos \left( \tan \left( \frac{\alpha \pi}{2} \right) A^{2\alpha} \right) \exp \left( -A^{2\alpha} \right) \right]$

$\cdot (1 + \epsilon_L Q_L + \epsilon_{\text{side}} Q_{\text{side}} + \epsilon_{\text{out}} Q_{\text{out}})$

for 2-jet events:

<table>
<thead>
<tr>
<th></th>
<th>$R_{\text{side}}/R_L$ (fm)</th>
<th>$\chi^2$/dof</th>
<th>CL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$-model</td>
<td>0.61 $\pm$ 0.02</td>
<td>14847/14921</td>
<td>66%</td>
</tr>
<tr>
<td>Edgeworth</td>
<td>0.64 $\pm$ 0.02</td>
<td>14891/14919</td>
<td>56%</td>
</tr>
</tbody>
</table>

consistent

Elongation is real
Direct Test of $Q^2$-only Dependence

1. $Q^2 = Q^2_{\text{LE}} + Q^2_{\text{side}} + Q^2_{\text{out}}$
   where $Q^2_{\text{LE}} = Q^2_{\text{L}} - (\Delta E)^2$
   inv. boosts along thrust axis

2. $Q^2 = Q^2_{\text{L}} + Q^2_{\text{side}} + q^2_{\text{out}}$
   where $q_{\text{out}} = Q_{\text{out}}$ boosted ($\beta$) along out direction to rest frame of pair

In $\tau$-model, for case 1

$R_2(Q_{\text{LE}}, Q_{\text{side}}, Q_{\text{out}}) = \gamma \left[ 1 + \lambda \cos \left( \tan \left( \frac{\alpha \pi}{2} \right) B^{2\alpha} \right) \exp \left( -B^{2\alpha} \right) \right] b$

where $B^2 = R^2_{\text{LE}} Q^2_{\text{LE}} + R^2_{\text{side}} Q^2_{\text{side}} + R^2_{\text{out}} Q^2_{\text{out}}$

$b = 1 + \epsilon_{\text{LE}} Q_{\text{LE}} + \epsilon_{\text{side}} Q_{\text{side}} + \epsilon_{\text{out}} Q_{\text{out}}$

and comparable expression for case 2, $R_2(Q_{\text{L}}, Q_{\text{side}}, q_{\text{out}})$
**Direct Test of $Q^2$-only Dependence**

Compare fits with all ‘radii’ free to fits with all ‘radii’ constrained to be equal

<table>
<thead>
<tr>
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<th>case 2</th>
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<tr>
<td>$\alpha$</td>
<td>0.46 ± 0.01</td>
<td>0.44 ± 0.01</td>
</tr>
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<td>$R_{LE}$ (fm)</td>
<td>0.84 ± 0.04</td>
<td>0.82 ± 0.04</td>
</tr>
<tr>
<td>$R_{side}/R_{LE}$</td>
<td>0.60 ± 0.02</td>
<td>1</td>
</tr>
<tr>
<td>$R_{out}/R_{LE}$</td>
<td>0.986 ± 0.003</td>
<td>1</td>
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</tbody>
</table>

| $\chi^2$/DoF | 14590/14538 | 14886/14540 |
| CL | 38% | 2% |

$\Delta \chi^2 = 296/2 \approx 0$

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<td>0.82 ± 0.04</td>
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<tr>
<td>$R_{side}/R_L$</td>
<td>0.62 ± 0.02</td>
<td>1</td>
</tr>
<tr>
<td>$r_{out}/R_L$</td>
<td>1.23 ± 0.03</td>
<td>1</td>
</tr>
</tbody>
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| $\chi^2$/DoF | 10966/10647 | 11430/10649 |
| CL | 2% | $10^{-7}$ |

$\Delta \chi^2 = 464/2 \approx 0$

Dependence on components of $Q$ is strongly preferred.
**Q Dependence**

Case 2, $R_2(Q_L, Q_{side}, q_{out})$ vs.
- $Q_L$ for $Q_{side}, q_{out} < 0.08$ GeV
- $Q_{side}$ for $Q_L, q_{out} < 0.08$ GeV
- $q_{out}$ for $Q_L, Q_{side} < 0.08$ GeV

Dependence on components of $Q$ is preferred.
Summary

- $R_2$ depends, to some degree, separately on components of $Q$, i.e., on $\bar{Q}$
- contradicts $\tau$-model, where dependence is on $Q$, not on $\bar{Q}$
- Nevertheless, $\tau$-model with a one-sided Lévy proper-time distribution succeeds:
  - Simplified, provides a new parametrization of $R_2(Q)$ which works well
  - $R_2(Q, m_{t1}, m_{t2})$ successfully fits $R_2$ for 2-jet events both $Q$- and $m_t$-dependence described correctly
- But dependence of $R_2$ on components of $Q$ implies $\tau$-model is in need of modification
  Perhaps, $a$ should be different for transverse/longitudinal
  
\[ \bar{x}^\mu(p^\mu) = a \tau p^\mu, \quad a = 1/m_t \text{ for 2-jet} \]
Emission Function of 2-jet Events.

In the $\tau$-model, the emission function in configuration space is

$$S(\vec{x}, \tau) = \frac{1}{n} \frac{d^4 n}{d\tau d\vec{x}} = \frac{1}{n} \left( \frac{m_t}{\tau} \right)^3 H(\tau) \rho_1 \left( \vec{p} = \frac{m_t \vec{x}}{\tau} \right)$$

For simplicity, assume $\rho_1(\vec{p}) = \rho_y(y) \rho_{p_t}(p_t)/\bar{n}$ ($\rho_1$, $\rho_y$, $\rho_{p_t}$ are inclusive single-particle distributions)

Then $S(\vec{x}, \tau) = \frac{1}{n^2} H(\tau) G(\eta) I(r)$

Strongly correlated $x, p \implies$

$$\eta = y \quad r = p_t \tau / m_t$$

$$G(\eta) = \rho_y(\eta) \quad I(r) = \left( \frac{m_t}{\tau} \right)^3 \rho_{p_t}(r m_t / \tau)$$

So, using experimental $\rho_y(y)$, $\rho_{p_t}(p_t)$ dists. and $H(\tau)$ from BEC fits, we can reconstruct $S$. 

expt. – Factorization OK

$H(\tau)$

$\alpha = 0.47$

$\Delta \tau = 1.56 \text{ fm}$

$\tau_0 = 0$
Emission Function of 2-jet Events.

Integrating over $r$,

Integrating over $z$,

"Boomerang" shape

Particle production is close to the light-cone
Emission Function of 2-jet Events.

Integrating over \( z \),

Integrating over \( r \),

“Boomerang” shape

Expanding ring

Particle production is close to the light-cone
\( \alpha_s \)

- LLA parton shower leads to a fractal in momentum space
  fractal dimension is related to \( \alpha_s \)
  Gustafson et al.
- Lévy dist. arises naturally from a fractal, or random walk, or anomalous diffusion
- strong momentum-space/configuration space correlation of \( \tau \)-model \( \implies \) fractal in configuration space with same \( \alpha \)
- generalized LPHD suggests particle dist. has same properties as gluon dist.
- Putting this all together leads to
  \[ \alpha_s = \frac{2\pi}{3} \alpha^2 \]
  Csőrgő et al.
- Using our value of \( \alpha = 0.47 \pm 0.04 \) yields \( \alpha_s = 0.46 \pm 0.04 \)
- This value is reasonable for a scale of 1–2 GeV, where production of hadrons takes place
  cf., from \( \tau \) decays \( \alpha_s(m_\tau \approx 1.8 \text{ GeV}) = 0.34 \pm 0.03 \) PDG
BEC Introduction

$q$-particle density

$$\rho_q(p_1, \ldots, p_q) = \frac{1}{\sigma_{\text{tot}}} \frac{d^q \sigma_q(p_1, \ldots, p_q)}{dp_1 \ldots dp_q}$$

2-particle correlation:

$$\frac{\rho_2(p_1, p_2)}{\rho_1(p_1)\rho_1(p_2)}$$

To study only BEC, not all correlations, let

$$\rho_0(p_1, p_2)$$

be the 2-particle density if no BEC ($= \rho_2$ of the ‘reference sample’) and define

$$R_2(p_1, p_2) = \frac{\rho_2(p_1, p_2)}{\rho_1(p_1)\rho_1(p_2)} \cdot \frac{\rho_1(p_1)\rho_1(p_2)}{\rho_0(p_1, p_2)} = \frac{\rho_2(p_1, p_2)}{\rho_0(p_1, p_2)}$$

Since 2-$\pi$ BEC only at small $Q$

$$Q = \sqrt{-(p_1 - p_2)^2} = \sqrt{M_{12}^2 - 4m_\pi^2}$$

integrate over other variables:

$$R_2(Q) = \frac{\rho(Q)}{\rho_0(Q)}$$
The usual parametrization assumes a symmetric Gaussian source. But, there is no reason to expect this symmetry in $e^+e^- \rightarrow q\bar{q}$. Therefore, do a 3-dim. analysis in the Longitudinal Center of Mass System (LCMS):

Boost each $\pi$-pair along event axis (thrust or sphericity) $\rho_{L1} = -\rho_{L2}$

$p_1 + p_2$ defines ‘out’ axis

$Q_{side} \perp (Q_L, Q_{out})$
Advantages of LCMS:

\[ Q^2 = Q_L^2 + Q_{\text{side}}^2 + Q_{\text{out}}^2 - (\Delta E)^2 \]
\[ = Q_L^2 + Q_{\text{side}}^2 + Q_{\text{out}}^2 (1 - \beta^2) \quad \text{where} \quad \beta \equiv \frac{p_{\text{out}1} + p_{\text{out}2}}{E_1 + E_2} \]

Thus, the energy difference, and therefore the difference in emission time of the pions couples only to the out-component, \( Q_{\text{out}} \).

Thus, \( Q_L \) and \( Q_{\text{side}} \) reflect only spatial dimensions of the source \( Q_{\text{out}} \) reflects a mixture of spatial and temporal dimensions.
Fit Results Simplified $\tau$-model

<table>
<thead>
<tr>
<th>parameter</th>
<th>two-jet</th>
<th>three-jet</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$0.63 \pm 0.03^{+0.08}_{-0.35}$</td>
<td>$0.92 \pm 0.05^{+0.06}_{-0.48}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0.41 \pm 0.02^{+0.04}_{-0.06}$</td>
<td>$0.35 \pm 0.01^{+0.03}_{-0.04}$</td>
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<td>$R$ (fm)</td>
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</tr>
<tr>
<td>$\epsilon$ (GeV$^{-1}$)</td>
<td>$0.001 \pm 0.002^{+0.005}_{-0.008}$</td>
<td>$0.000 \pm 0.002^{+0.001}_{-0.007}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0.988 \pm 0.005^{+0.026}_{-0.012}$</td>
<td>$0.997 \pm 0.005^{+0.019}_{-0.002}$</td>
</tr>
<tr>
<td>$\chi^2$/DoF</td>
<td>$91/94$</td>
<td>$84/94$</td>
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<tr>
<td>confidence level</td>
<td>$57%$</td>
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<td>$\lambda$</td>
<td>$0.61 \pm 0.03^{+0.08}_{-0.26}$</td>
<td>$0.84 \pm 0.04^{+0.04}_{-0.37}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0.44 \pm 0.01^{+0.05}_{-0.02}$</td>
<td>$0.42 \pm 0.01^{+0.02}_{-0.04}$</td>
</tr>
<tr>
<td>$R$ (fm)</td>
<td>$0.78 \pm 0.04^{+0.09}_{-0.16}$</td>
<td>$0.98 \pm 0.04^{+0.55}_{-0.14}$</td>
</tr>
<tr>
<td>$\epsilon$ (GeV$^{-1}$)</td>
<td>$0.005 \pm 0.001 \pm 0.003$</td>
<td>$0.008 \pm 0.001 \pm 0.005$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$0.979 \pm 0.002^{+0.009}_{-0.003}$</td>
<td>$0.977 \pm 0.001^{+0.013}_{-0.008}$</td>
</tr>
<tr>
<td>$\chi^2$/DoF</td>
<td>95/95</td>
<td>113/95</td>
</tr>
<tr>
<td>confidence level</td>
<td>49%</td>
<td>10%</td>
</tr>
<tr>
<td>$m_t$ regions (GeV)</td>
<td>average $m_t$ (GeV)</td>
<td>confidence level (%)</td>
</tr>
<tr>
<td>---------------------</td>
<td>---------------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>$m_{t1}$ $m_{t2}$</td>
<td>$Q &lt; 0.4$</td>
<td>all</td>
</tr>
<tr>
<td>0.14 – 0.26</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>0.14 – 0.34</td>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
<td>0.14 – 0.46</td>
<td>0.37</td>
<td>0.37</td>
</tr>
<tr>
<td>0.14 – 0.66</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>0.26 – 0.42</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td>0.34 – 0.46</td>
<td>0.32</td>
<td>0.33</td>
</tr>
<tr>
<td>0.46 – 0.58</td>
<td>0.43</td>
<td>0.44</td>
</tr>
<tr>
<td>0.66 – 0.86</td>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>0.42 – 0.62</td>
<td>0.34</td>
<td>0.34</td>
</tr>
<tr>
<td>0.46 – 0.70</td>
<td>0.41</td>
<td>0.41</td>
</tr>
<tr>
<td>0.58 – 0.82</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>0.86 – 1.22</td>
<td>0.80</td>
<td>0.81</td>
</tr>
<tr>
<td>0.70 – 4.14</td>
<td>0.59</td>
<td>0.65</td>
</tr>
<tr>
<td>0.82 – 4.14</td>
<td>0.71</td>
<td>0.76</td>
</tr>
</tbody>
</table>
Fit Result \( R_2(Q, m_{t1}, m_{t2}) \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>0.58 ± 0.03^{+0.08}_{-0.24}</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.47 ± 0.01^{+0.04}_{-0.02}</td>
</tr>
<tr>
<td>( \Delta \tau ) (fm)</td>
<td>1.56 ± 0.12^{+0.32}_{-0.45}</td>
</tr>
<tr>
<td>( \epsilon ) (GeV(^{-1}))</td>
<td>0.001 ± 0.001 ± 0.003</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.988 ± 0.002^{+0.006}_{-0.002}</td>
</tr>
<tr>
<td>( \chi^2/\text{DoF} )</td>
<td>90/95</td>
</tr>
<tr>
<td>Confidence level</td>
<td>62%</td>
</tr>
</tbody>
</table>
Fit Results elongation in $\tau$-model for 2-jet events

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.49 ± 0.02</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.46 ± 0.01</td>
</tr>
<tr>
<td>$R_L$ (fm)</td>
<td>0.85 ± 0.04</td>
</tr>
<tr>
<td>$R_{\text{side}}/R_L$</td>
<td>0.61 ± 0.02</td>
</tr>
<tr>
<td>$R_{\text{out}}/R_L$</td>
<td>0.66 ± 0.02</td>
</tr>
<tr>
<td>$\epsilon_L$ (GeV$^{-1}$)</td>
<td>0.001 ± 0.001</td>
</tr>
<tr>
<td>$\epsilon_{\text{side}}$ (GeV$^{-1}$)</td>
<td>$-0.076 ± 0.003$</td>
</tr>
<tr>
<td>$\epsilon_{\text{out}}$ (GeV$^{-1}$)</td>
<td>$-0.029 ± 0.002$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.011 ± 0.002</td>
</tr>
</tbody>
</table>

$\chi^2$/DoF 14847/14921
CL 66%
Fit Results of direct tests for 2-jet events

<table>
<thead>
<tr>
<th>case 1</th>
<th>$\lambda$</th>
<th>$0.51 \pm 0.03$</th>
<th>$0.49 \pm 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$0.46 \pm 0.01$</td>
<td>$0.46 \pm 0.01$</td>
</tr>
<tr>
<td></td>
<td>$R_{LE} \text{ (fm)}$</td>
<td>$0.84 \pm 0.04$</td>
<td>$0.71 \pm 0.04$</td>
</tr>
<tr>
<td></td>
<td>$R_{side}/R_{LE}$</td>
<td>$0.60 \pm 0.02$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$R_{out}/R_{LE}$</td>
<td>$0.986 \pm 0.003$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\epsilon_{LE} \text{ (GeV}^{-1})$</td>
<td>$0.001 \pm 0.001$</td>
<td>$0.000 \pm 0.001$</td>
</tr>
<tr>
<td></td>
<td>$\epsilon_{side} \text{ (GeV}^{-1})$</td>
<td>$-0.069 \pm 0.003$</td>
<td>$-0.064 \pm 0.003$</td>
</tr>
<tr>
<td></td>
<td>$\epsilon_{out} \text{ (GeV}^{-1})$</td>
<td>$-0.032 \pm 0.002$</td>
<td>$-0.035 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>$1.010 \pm 0.002$</td>
<td>$1.012 \pm 0.002$</td>
</tr>
</tbody>
</table>

| | $\chi^2$/DoF | 14590/14538 | 14886/14540 |
| CL | 38% | 2% |
## Fit Results of direct tests for 2-jet events

<table>
<thead>
<tr>
<th></th>
<th>case 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$0.65 \pm 0.03$</td>
<td>$0.57 \pm 0.03$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0.41 \pm 0.01$</td>
<td>$0.44 \pm 0.01$</td>
</tr>
<tr>
<td>$R_L$ (fm)</td>
<td>$0.96 \pm 0.05$</td>
<td>$0.82 \pm 0.04$</td>
</tr>
<tr>
<td>$R_{\text{side}}/R_L$</td>
<td>$0.62 \pm 0.02$</td>
<td>$1$</td>
</tr>
<tr>
<td>$r_{\text{out}}/R_L$</td>
<td>$1.23 \pm 0.03$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\epsilon_L$ (GeV$^{-1}$)</td>
<td>$0.004 \pm 0.001$</td>
<td>$0.003 \pm 0.001$</td>
</tr>
<tr>
<td>$\epsilon_{\text{side}}$ (GeV$^{-1}$)</td>
<td>$-0.067 \pm 0.003$</td>
<td>$-0.059 \pm 0.003$</td>
</tr>
<tr>
<td>$\epsilon_{\text{out}}$ (GeV$^{-1}$)</td>
<td>$-0.022 \pm 0.003$</td>
<td>$-0.029 \pm 0.002$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$1.000 \pm 0.002$</td>
<td>$1.003 \pm 0.002$</td>
</tr>
<tr>
<td>$\chi^2$/DoF</td>
<td>10966/10647</td>
<td>11430/10649</td>
</tr>
<tr>
<td>CL</td>
<td>2%</td>
<td>$10^{-7}$</td>
</tr>
</tbody>
</table>
Transverse Mass dependence of $r$

$r$ decreases with $m_t$ (or $k_t$) for all directions.
\[ R_2 = \gamma \cdot \left[ 1 + \lambda G \right] \cdot \left( 1 + \epsilon Q \right) \]

- Gaussian
- Edgeworth expansion
- Symmetric Lévy

\[ G = \exp \left( - \frac{(rQ)^2}{2} \right) \cdot \left[ 1 + \kappa 3! H_3 (rQ) \right] \]

- Gaussian if \( \kappa = 0 \)
- \( 0.71 \pm 0.06 \) symmetric Lévy

- \( 0 < \alpha \leq 2 \)
- \( \alpha = 1.34 \pm 0.04 \)

**CL:**

- Poor \( \chi^2 \)
- Edgeworth and Lévy better than Gaussian, but poor.

**Problem:**

- Dip of \( R_2 \) in the region \( 0.6 < Q < 1.5 \) GeV
$R^2 = \gamma \cdot \left[ 1 + \lambda G \right] \cdot (1 + \epsilon Q)$

- Gaussian
  - $G = \exp\left(- \left(\frac{r Q}{\alpha}\right)^2\right)$
  - $\alpha = 1.34 \pm 0.04$

- Edgeworth expansion

- Lévy
  - $G = \exp\left(-|r Q|^{\alpha}\right)$
  - $0 < \alpha \leq 2$
  - $\alpha = 1.34 \pm 0.04$

Poor $\chi^2$.

Gauss, Edgew, Lévy better than Gaussian, but poor.

Problem is the dip of $R^2$ in the region $0.6 < Q < 1.5$ GeV.
$R^a_{\text{free}}$

$\chi^2/\text{dof} = 91/94$
\[ R_a = \tan \left( \frac{\alpha \pi}{2} \right) \]
$$R_{a}^{2\alpha} = \tan \left( \frac{\alpha \pi}{2} \right) R_{a}^{2\alpha}$$

$$\chi^2/\text{dof} = 95/95$$
\[2 \alpha = \tan \left( \frac{\pi}{2} \right)\]
$R_{a, \text{ free}}$

$\chi^2 / \text{dof} = 84/94$

CL = 10%

p. 44
\[ R^2 = \tan(\frac{\pi}{2}) \]

\[ R^2 \chi^2 / \text{dof} = 113/95 \]

\[ CL = 10\% \]

p. 44
\[ R_{a}^{2\alpha} = \tan \left( \frac{\alpha \pi}{2} \right) R_{a}^{2\alpha} \]

\[ \chi^2 / \text{dof} = 113/95 \]

CL = 10%
$$\chi^2/\text{dof} = 84/94$$

$$R^2 = \tan(\alpha \pi/2)$$

$$\chi^2/\text{dof} = 113/95$$

CL = 10%
$$R_2 = \gamma \left\{ 1 + \lambda \cos \left( \tau_0 Q^2 (m_{t1} + m_{t2})^2 (m_{t1} m_{t2}) \alpha \right) \exp \left( -\Delta \tau Q^2 2 \right) \alpha \cdot (1 + \epsilon Q) \right\} \cdot (1 + \epsilon Q)$$

Fit $R_2$ using avg $m_{t1}$, $m_{t2}$ in each $Q$ bin, $m_{t1} > m_{t2}$

$\tau_0 = 0.00 \pm 0.02$ so fix to 0

$\chi^2 / \text{dof} = 90 / 95$ fit
Dependence on components of $Q$ is preferred.

$R^2$ vs. $Q_L$ for $Q_{side}$, $q_{out} < 0.08$ GeV

$R^2$ vs. $Q_{L}$, $Q_{side}$ < 0.08 GeV

$R^2$ vs. $q_{out}$ for $Q_{L}$

$Q$ Dependence
Dependence on components of $Q$ is preferred.
Integrating over $r$,

Integrating over $z$,

“Boomerang” shape

Particle production is close to the light-cone

$S(r,\tau)$ from L3 2–jet events, $\tau = 0.01, 0.02, \ldots, 0.1$ fm
Integrating over $r$, 

```
<table>
<thead>
<tr>
<th>t (fm)</th>
<th>0</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(z,t) (fm$^{-2}$)</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>
```

Integrating over $z$, 

```
<table>
<thead>
<tr>
<th>$r_t$ (fm)</th>
<th>0.00</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(r_t,t)$ (fm$^{-3}$)</td>
<td>50</td>
<td>10</td>
</tr>
</tbody>
</table>
```

“Boomerang” shape

Particle production is close to the light-cone
Integrating over $r$,

Integrating over $z$,

“Boomerang” shape

Particle production is close to the light-cone
Integrating over $r$, "Boomerang" shape
Particle production is close to the light-cone

Integrating over $z$, $S(r_z, \tau)$ from L3 2–jet events, $\tau = 0.01, 0.02, ..., 0.1$ fm
Integrating over $r$, Integrating over $z$,

"Boomerang" shape

Particle production is close to the light-cone
Integrating over $r$,  

“Boomerang” shape

Integrating over $z$,  

Particle production is close to the light-cone
Integrating over $r$, 

Integrating over $z$, 

“Boomerang” shape

Particle production is close to the light-cone

$S(r, t)$ from L3 2–jet events, $t = 0.01, 0.02, ..., 0.1$ fm

Particle production is close to the light-cone
Integrating over $r$,

“Boomerang” shape
Particle production is close to the light-cone

Integrating over $z$, 

$S(r, \tau)$ from L3 2–jet events, $\tau = 0.01, 0.02, ..., 0.1$ fm
Integrating over $r$, "Boomerang" shape

Particle production is close to the light-cone

Integrating over $z$, $S(r,t)$ from L3 2–jet events, $t = 0.01, 0.02, ..., 0.1$ fm
Particle production is close to the light-cone
Integrating over \( r \), "Boomerang" shape

Integrating over \( z \),

\[
S(r_t, \tau)\text{ from L3 2–jet events, } \tau = 0.01, 0.02, ..., 0.1 \text{ fm}
\]
(Loading movie...)

Particle production is close to the light-cone p. 51
Integrating over $r$, “Boomerang” shape

Integrating over $z$, $\tau = 0.01$ fm

$\tau = 0.02$ fm

$\tau = 0.04$ fm

$\tau = 0.07$ fm

$\tau = 0.15$ fm

Expanding ring

Particle production is close to the light-cone

$p. 63$
Integrating over \( r \), "Boomerang" shape

\[ \tau = 0.01 \text{ fm} \]

Expanding ring

Particle production is close to the light-cone
Integrating over $r$, "Boomerang" shape

Integrating over $z$, $\tau = 0.01 \text{ fm}$, $\tau = 0.02 \text{ fm}$, $\tau = 0.04 \text{ fm}$, $\tau = 0.07 \text{ fm}$, $\tau = 0.15 \text{ fm}$.

Particle production is close to the light-cone p. 63
Integrating over $r$, "Boomerang" shape

$\tau = 0.04 \text{ fm}$

Expanding ring

Particle production is close to the light-cone
Integrating over \( r \), "Boomerang" shape

\[ \tau = 0.01 \text{ fm} \]

\[ \tau = 0.02 \text{ fm} \]

\[ \tau = 0.04 \text{ fm} \]

\[ \tau = 0.07 \text{ fm} \]

\[ \tau = 0.15 \text{ fm} \]

Particle production is close to the light-cone
Integrating over $r$, "Boomerang" shape

Integrating over $z$, $\tau = 0.01$ fm

$\tau = 0.02$ fm

$\tau = 0.04$ fm

$\tau = 0.07$ fm

$\tau = 0.15$ fm

Particle production is close to the light-cone

p. 63
Integrating over $r$, “Boomerang” shape

Integrating over $z$,

Expanding ring

Particle production is close to the light-cone
Integrating over $z$, "Boomerang" shape

Integrating over $r$, Expanding ring

Particle production is close to the light-cone

\[ \tau = 0.01 \text{ fm} \]

\[ \tau = 0.02 \text{ fm} \]

\[ \tau = 0.04 \text{ fm} \]

\[ \tau = 0.07 \text{ fm} \]

\[ \tau = 0.15 \text{ fm} \]
Integrating over $z$, 

"Boomerang" shape

Integrating over $r$, 

Expanding ring

Particle production is close to the light-cone
Integrating over $r$, "Boomerang" shape

Integrating over $z$, $\tau = 0.01 \text{ fm}$

$\tau = 0.02 \text{ fm}$

$\tau = 0.04 \text{ fm}$

$\tau = 0.07 \text{ fm}$

$\tau = 0.15 \text{ fm}$

Expanding ring

Particle production is close to the light-cone