Power-law ensembles: fluctuations of volume or temperature?

Grzegorz WILK
The Andrzej Sołtan Institute for Nuclear Studies; Warsaw, Poland
Zbigniew WŁODARCZYK
Institute of Physics, Jan Kochanowski University, Kielce, Poland
Wojciech WOLAK
Kielce University of Technology, Kielce, Poland

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Content:

Introductory remarks – parametrizing intrinsic fluctuations by nonextensive parameter q

Intrinsic fluctuations:
- of temperature $T$ (?)
- of volume $V$ (?)

Comparison of nonextensivity parameters q evaluated from different characteristics of multiparticle production processes:
- $dN/dy$
- $dN/dp_T$
- $P(N)$

for pp and AA collisions
Whereas hard probes (described by the perturbative QCD (pQCD)) are customarily assumed to measure *dynamical* properties of the collision process, the soft interactions can be only modelled because pQCD is not applicable here.

The model of the first choice in this case is usually some variant of the statistical/thermodynamical model, either purely phenomenological or incorporating some features from the QCD (like quark-gluon structure of the hadronizing matter and its equation of state).

The border between these two schemes is not well defined, usually soft regime is supposed to be characterized by the exponential distribution in transverse momenta, whereas hard regime is connected with the power law behavior of the corresponding distributions and is therefore attributed to pQCD.

However, recent findings confirm that substantial part of the power-like region can be attributed not so much to pQCD but rather to some intrinsic fluctuations in the hadronizing system:

- either to temperature T fluctuations (when formulating statistical model description using the so called Tsallis nonextensive statistics instead of the usual Boltzman-Gibbs one [1])
- or to fluctuations of the volume (in purely phenomenological approach, apparently not referring to Tsallis statistics [2]).

The Gamma distribution is given by
\[ f(E) = \frac{1}{T} \exp \left( -\frac{E}{T} \right) \]

is derived in [1].

For plausible dynamical/stochastical justification of this formula see:

The Tsallis distribution is given by
\[ h_q(E) = \int_0^\infty f(E)g(1/T)d(1/T) = \frac{2-q}{T} \left[ 1 - (1-q)\frac{E}{T_0} \right]^{\frac{1}{1-q}}. \]

Gamma distribution is transformed to the Tsallis distribution with
\[ g(1/T) = \frac{1}{\Gamma \left( \frac{1}{q-1} - s \right)} \frac{T_0}{q-1} \left( \frac{1}{q-1} \frac{T_0}{T} \right)^{\frac{1}{q-1} - 1 - s} \cdot \exp \left( \frac{1}{q-1} \frac{T_0}{T} \right) \]

with
\[ q = 1 + \frac{\omega}{1 + s \cdot \omega}, \quad \text{where} \quad \omega = \frac{\text{Var}(T)}{\langle T \rangle^2}. \]

and \( T_0 \) denoting the values of \( T \) around which one has fluctuations.

1 ≤ q ≤ 2
From N-particle Tsallis distribution under condition that one gets the NBD form of $P(N)$ which for $q \to 1$ ($k \to \infty$) becomes Poisson distribution and for $q \to 2$ ($k \to 1$) geometrical distribution.

For large $N$ and $\langle N \rangle$ one gets the known KNO distribution with $z = N/\langle N \rangle$. 

$$h \left( \{E_i=1,...,N\} \right) = C_N \left[ 1 - (1 - q) \frac{\sum_{i=1}^{N} E_i}{\lambda} \right]^{\frac{1}{1-q} + 1-N}$$

$$\sum_{i=0}^{N} E_i \leq E \leq \sum_{i=0}^{N+1} E_i$$

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \left( \frac{\langle N \rangle}{k} \right)^N \left( 1 + \frac{\langle N \rangle}{k} \right)^{(N+k)}; \quad k = \frac{1}{q-1}.$$ 

$$P(N) = \frac{(\bar{N})^N}{N!} \exp(-\bar{N}) \quad \text{where} \quad \bar{N} = \frac{E}{\lambda}.$$ 

$$\langle N \rangle P(N) \approx \psi \left( z = \frac{N}{\langle N \rangle} \right) = \frac{k^k}{\Gamma(k)} z^{k-1} \exp(-kz),$$
\[ g(E_i) = C \exp \left( -\frac{E_i}{T} \right) \]

\[ h_q(E_i) = C_q \left[ 1 - (1 - q) \frac{E_i}{T_{\text{eff}}} \right]^{\frac{1}{1-q}} \]

where \( q = 1 + \frac{\text{Var}(T)}{\langle T \rangle^2} \)

and \( T_{\text{eff}} = T_0 + (q - 1)T_{\text{visc}} \)

\[ P(N) = \frac{\langle N \rangle^N}{N!} \exp \left( k \langle N \rangle \right) \exp \left( \frac{\langle N \rangle}{k} \right) \frac{1 + \langle N \rangle/k}{\Gamma(N + 1) \Gamma(k)} \]

where \( k = \frac{1}{q - 1} \) \( k \) and \( \text{Var}(N) = \langle N \rangle + (q - 1)\langle N \rangle^2 \)

Volume fluctuations have been introduced in the statistical model:

They were just assumed.
Suitable chosen form of fluctuations of the volume, $P(V)$, results in fluctuating $T$, in broadening of the $P(N)$ and in the power-like behavior of single particle spectra. all these apparently without resorting to Tsallis statistics.

(*) Note that for constant total energy, $E=\text{const}$, both the volume $V$ and temperature $T$ are related: $E \sim V \ T^4$

(*) One can therefore write:

$$\langle T \rangle / T = \left( V / \langle V \rangle \right)^{(1/4)}$$
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(*) Because in [■] the \( <N>_{[\text{MCE}]} \approx <N>_{[\text{GCE}]} = \bar{N} \), therefore

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\bar{N} \sim VT^3 = \langle V \rangle \langle T \rangle^3 y
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\]

And because in [■] it was *assumed* that \(y\) fluctuates according to gamma distribution in the form of \(\Psi(z \to y)\), it means that \(\bar{N}\) fluctuates in the same way (\(\langle N \rangle\) in \(P(N)\) is regarded to be constant).
It should be stressed that:

- The scaling distributions $\psi(y) = \psi(z)$ assumed in [1] is the same as $g(1/T)$ describing $T$ fluctuations in [1], therefore, due to the equality $y = \langle T \rangle / T$, one can argue that volume fluctuations are equivalent to temperature fluctuations.

Because $P(T)$ derived in [1] has the form of gamma distribution (identical with our $g(1/T)$), therefore single particle spectra in [1] must have the form of Tsallis distributions. [ The small differences in the corresponding powers in Tsallis distribution arises from different normalization factor ($\sim y^4$ in [1] and $\sim y$ in our exponential distribution [1]).]

The scaling function $\psi(z)$ is only some approximation of the NBD (because it neglects the Poissonian distribution with which multiplicity fluctuates in GCE). However, if one takes

$$g(\bar{N}) = \psi(z = \frac{\bar{N}}{\langle N \rangle}) = \frac{\gamma^k \bar{N}^{k-1} \exp(-\gamma \bar{N})}{\Gamma(k)}$$

where $\gamma = k/\langle N \rangle$ and $\bar{N} = \langle N \rangle y$.

one gets the NBD as given before.

\[ P(N) = \int_0^\infty \frac{\bar{N}}{N!} \exp(-\bar{N}) g(\bar{N}) d\bar{N} = \frac{\Gamma(N + k)}{\Gamma(N + 1)\Gamma(k)} \frac{\gamma^k}{(\gamma + 1)^{N+k}}, \]
To summarize this part:

1. In PRC78, 024904 (2008) (BGG): $E=\text{const}$, $V$ fluctuates (nothing is said on $T$)


But: $V^{(1/4)} \sim 1/T$

what means that:

- in $\square$ if $V$ fluctuates then also $T$ fluctuates
- in $\square$ if $T$ fluctuates then also $V$ fluctuates

i.e., in this sense both approaches are equivalent:

$V$ fluctuates: $\sim$ T fluctuates:
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Comparison of nonextensivity parameters $q$ evaluated from different characteristics of multiparticle production processes:

- $dN/dy$
- $dN/dp_T$
- $P(N)$

for pp and AA collisions

(Wilk, Włodarczyk, Wolak, in preparation).

Notice that $q - 1 \approx 1/V$

therefore we expect that $q(\text{hadronic}) \gg q(\text{nuclear})$

because $V(\text{hadronic}) \ll V(\text{nuclear})$
The most recent examples of using Tsallis distribution [3]: fitting the transverse momentum spectra all the way up to ~12 GeV/C.

Notice: all curves are fitted by single, 2-parameter \((T,n)\) formula.
The most recent examples of using Tsallis distribution [3]: fitting the transverse momentum spectra all the way up to ~12 GeV/C.


<table>
<thead>
<tr>
<th>Particle</th>
<th>( d\sigma/dy ) (mb, ( \mu b ))</th>
<th>( T ) (MeV)</th>
<th>( n = -1/(1 - q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>( 43.5 \pm 2.0 \pm 1.9 )</td>
<td>( 112.7 \pm 2.9 \pm 1.1 )</td>
<td>( 9.57 \pm 0.11 \pm 0.03 )</td>
</tr>
<tr>
<td>( K )</td>
<td>( 4.0 \pm 0.1 \pm 0.5 )</td>
<td>( 132.7 \pm 3.8 \pm 7.2 )</td>
<td>( 10.04 \pm 0.16 \pm 0.27 )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( 5.1 \pm 1.1 \pm 3.9 )</td>
<td>( 119 \pm 10 \pm 30 )</td>
<td>( 9.68 \pm 0.18 \pm 0.49 )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( 4.3 \pm 0.3 \pm 0.4 )</td>
<td>( 109.7 \pm 6.9 \pm 6.7 )</td>
<td>( 9.78 \pm 0.24 \pm 0.18 )</td>
</tr>
<tr>
<td>( \eta' )</td>
<td>( 0.80 \pm 1.5 \pm 0.7 )</td>
<td>( 141 \pm 107 \pm 61 )</td>
<td>( 10.5 \pm 2.2 \pm 1.2 )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( 0.41 \pm 0.02 \pm 0.03 )</td>
<td>( 139 \pm 16 \pm 15 )</td>
<td>( 10.82 \pm 0.71 \pm 0.56 )</td>
</tr>
<tr>
<td>( J/\psi )</td>
<td>( 0.73 \pm 0.01 \pm 0.05 )</td>
<td>( 149 \pm 56 \pm 82 )</td>
<td>( 12.3 \pm 1.6 \pm 2.9 )</td>
</tr>
<tr>
<td>( \psi' )</td>
<td>( 0.13 \pm 0.03 \pm 0.02 )</td>
<td>( 164 \pm 10^3 \pm 10^2 )</td>
<td>( 14 \pm 12 \pm 6 )</td>
</tr>
<tr>
<td>( p )</td>
<td>( 1.63 \pm 0.05 \pm 0.11 )</td>
<td>( 107 \pm 13 \pm 12 )</td>
<td>( 12.2 \pm 1.0 \pm 0.7 )</td>
</tr>
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</table>

\( q \approx 1.07 \pm 1.1 \)
[3a] CMS Coll., Transverse-momentum and pseudorapidity distributions of charged hadrons in pp collisions at $\sqrt{s} = 0.9$ and 2.36 TeV, JHEP02(2010)041.

\[ \sqrt{s} = 900 \text{ GeV} \quad q = 1.11 \quad T = 140 \text{ MeV} \]

\[ \sqrt{s} = 2.3 \text{ TeV} \quad q = 1.13 \quad T = 130 \text{ MeV} \]

\[ \sqrt{s} = 7 \text{ TeV} \quad q = 1.15 \quad T = 145 \text{ MeV} \]
Parameter $q$ from $p_T$ distributions:

![Graph showing the energy dependence of the non-extensivity parameter $q$. The graph plots $q$ against $\sqrt{s}$ (GeV) for $T=130$ MeV. The line and data points show the variation of $q$ with energy, with $q \to 11/9 = 1.22$ for $\sqrt{s} \to \infty$ (?).]


$q \to 11/9 = 1.22$ for $\sqrt{s} \to \infty$ (?)
This is to be compared to $q_L$ from rapidity distributions:

The values of the nonextensivity parameter $q$ obtained in fits shown here and listed in Table I compared with the values of the parameter $k$ of Negative Binomial distribution fit to the corresponding multiplicity distributions (as given by C.Geich-Gimbel, Int. J. Mod. Phys. A4 (1989) 1527).

(*) From fits to rapidity distribution data one gets systematically $q>1$ with energy dependence the same as $1/k$ in NBD

(*) $y$-distributions $\Rightarrow$

'partition temperature'

$T \approx K \cdot \sqrt{s}/N$ (N=multiplicity)

(*) $q_L \approx q \Leftrightarrow$ fluctuating $T \Leftrightarrow$ fluctuating $N$

(*) Conjecture: $q-1$ measures the amount of fluctuations in $P(N)$
Different observables -> different fluctuations -> different parameters $q$

Summary on $q$ from pp collisions (or from peripheral AA collisions):

$q_L > q_T$

$q_L \approx q$ obtained from multiplicity distribution $P(N)$ (which has NBD form)

In general:

bigger $(q-1)$ means bigger fluctuations, $q=1$ means no fluctuations.

<table>
<thead>
<tr>
<th>$E_{cm}$ [GeV]</th>
<th>$q_L$</th>
<th>$T_L=1/\beta_L$</th>
</tr>
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<tbody>
<tr>
<td>200</td>
<td>1.203</td>
<td>12.12</td>
</tr>
<tr>
<td>546</td>
<td>1.262</td>
<td>22.38</td>
</tr>
<tr>
<td>900</td>
<td>1.291</td>
<td>29.47</td>
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<tr>
<td>900</td>
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Notice that
(*) values of q-1 from distributions of p_T are higher;
(*) there are substantial differences for central collisions (large number of participants N_p).
$q = q_T$ for different centralities from NA49 data on Pb+Pb collisions for negative pions (obtained from $p_T$ distributions)
Data are from:
C. Alt et al., PRC77(2008)024903 (*Pion and kaon production in central Pb+Pb collisions at 20A and 30A GeV: Evidence for the onset of deconfinement*)

\[
\frac{dN}{dy} = C\left(1 - (1 - q) \left(\frac{m_c \cosh y}{T}\right)\right)^{\frac{1}{1-q}}
\]
q = q_L from distributions in the longitudinal phase space

\[
\frac{dN}{dy} = C \left( 1 - (1 - q) \left( \frac{m_t \cosh y}{T} \right) \right)^{\frac{1}{1-q}}
\]

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<th>80 GeV</th>
<th>40 GeV</th>
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<td>(q)</td>
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<td>(\frac{T}{m_t})</td>
<td>2.471 ± 0.092</td>
<td>1.948 ± 0.017</td>
<td>1.360 ± 0.022</td>
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Results for $q=q_L$ from $y$-distributions:

$$\frac{dN}{dy} = C(1 - (1 - q) \left(\frac{m_t \cosh y}{T}\right))^\frac{1}{1-q}$$

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These results must be confronted with results obtained from the multiplicity distributions $P(N)$:

$$q - 1 = \frac{1}{aN_p} \left(1 - \frac{N_p}{A}\right)$$

Notice that now $q_L - 1 >> q - 1$ (!)

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<tr>
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<td>179</td>
<td>174</td>
<td>169</td>
<td>160</td>
</tr>
<tr>
<td>$q$</td>
<td>1.000739</td>
<td>1.000771</td>
<td>1.000935</td>
<td>1.001108</td>
<td>1.001448</td>
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For central AA collisions:

- from rapidity: $dN/dy$ ($q_L$)
- from multiplicity: $P(N)$ ($q$)

Now $q_L \neq q$!
For central AA collisions:

- from rapidity: \( dN/dy \)
- from multiplicity: \( P(N) \)

\[ q_L \neq q \]
However:

When extracting values of parameter $q$ from the rapidity distributions tacit assumption was made that $m_t$ in $E=m_t \cosh(y)$ remains constant (i.e., it does not fluctuate). Perhaps this assumption is not true and should be lifted?

If so, one can take $m_t$ from PRC77(2008)024903 and PRC66(2002)054902 and calculate:

For $Z = \frac{m_t}{T}$ we have:

$$E/T = z \cosh(y)$$

It means that:

Therefore, if in rapidity distributions, dN/dy, $z=m_t/T$ fluctuates, then we can calculate fluctuations of $T$ which correspond to $q$ obtained from $P(N)$:

$$q - 1 = \frac{\text{Var}(T)}{\langle T \rangle^2} = \frac{\text{Var}(z)}{\langle z \rangle^2} - \frac{\text{Var}(m_t)}{\langle m_t \rangle^2}$$
from rapidity
from multiplicity distribution
from rapidity after accounting for fluctuations in $m_T$
Finally: Values of q obtained from P(N) (red stars) compared with q obtained from rapidity distributions (blues squares) corrected for fluctuations of $m_t$. Total error bars for q are indicated.
Different observables -> different fluctuations -> different parameters $q$

Summary on $q$ from AA collisions:

- For larger centralities, $q_T > q_L$, and $q_L$ differs substantially from $q$ obtained from $P(N)$.
- It becomes similar to it only after accounting for fluctuations in $m_T$ when calculating $E = m_T \cosh(y)$.

Different observables - different fluctuations - different parameters $q$.
Summary:

Fluctuations of different kinds are nowadays accessible and bear important information. They can be described in the usual statistical models by resorting to the so called $q$-statistics. In this language the best known so far are $T$ fluctuations (scale parameter fluctuations). However, it seems that

$$< V \text{ fluctuations } > \sim < T \text{ fluctuations } > .$$

The systematics of the parameter $q$ describing these fluctuations is still not fully understood and deserves further systematic phenomenological studies (which parallels the corresponding studies on interelation of fluctuations of different variables, both for pp and AA collisions). To this end one needs data from the same experiment on $dN/dy, dN/dp_T, P(N)$ (at least...).
Historical note: Since long time ago such power-like distributions were known but treated as simple parametrization interpolating between recognized exponential ("soft") physics equally recognized power-like ("hard") physics

First attempts to fit the whole range of $p_T$ are from 1977 (C.Michael) (*):

$$f(p_T) = C \left(1 + \frac{p_T}{p_0}\right)^{-n}$$

$$\exp\left(-\frac{n}{p_0} p_T\right)$$

for $p_T \rightarrow 0$  

$$\left(\frac{p_0}{p_T}\right)^n$$

for $p_T \rightarrow \infty$. 

"soft" (nonperturbative) physics

"hard" (perturbative) physics

- no special meaning of parameters $p_0$ and $n$ is offered ....
