Theory of the transverse momentum dependent (unintegrated) parton densities:
recent progress

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Based on collaboration with
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Transverse-momentum dependent (unintegrated) parton densities (TMD) accumulate information about intrinsic (longitudinal and transverse) motion of partons inside a hadron.

TMD can be defined as a natural generalization of the integrated collinear PDF:

\[ P_{\text{collinear}}(x, \mu) \rightarrow P_{\text{tmd}}(x, k, \mu, \zeta) \]

—depends on the longitudinal \( x = k^+/p^+ \) and transverse \( k_\perp \) components of parton’s momentum, as well as on the UV scale \( \mu \) and rapidity cutoff \( \zeta \).

Collinear PDFs (are expected to) restore after \( k_\perp \)-integration

\[ P_{\text{collinear}}(x, \mu) \sim \int d^2k_\perp P_{\text{tmd}}(x, k, \mu, \zeta) \]

Looks nice, but: the case is much more intricate than one could imagine 30 years ago!
WHY TMD?

- Nonperturbative domain of QCD: quark models, lattice, nonperturbative vacuum, instantons;
- Factorization and evolution;
- Gauge invariance;
- Structure of nucleon;
- Spin-related phenomena;
- Flavor structure
CURRENT and PLANNED “TMD" EXPERIMENTS

- **SIDIS process** $lH^\uparrow \rightarrow l'hX$: HERMES, COMPASS, JLab, EIC. To be studied: Sivers, Collins, transversity, Boer-Mulders, unpolarized X-sections, etc.

- **DY process** $H_1^{(\uparrow)} H_2^\uparrow \rightarrow l^+l^-X$: COMPASS, PAX, GSI, RHIC. To be studied: distribution functions, transversity.

- **Hadron collisions** $H_1^{(\uparrow)} H_2^\uparrow \rightarrow l^+l^-X$: RHIC

- $e^+e^- \rightarrow h_1 h_2 X$: BELLE, BaBar
• **INCLUSIVE PROCESSES (DIS)**

\[
W_{\mu\nu} = \frac{1}{2\pi} \Im m \left[ i \int d^4\xi \ e^{iq\xi} \langle P|T\{J_\mu(\xi)J_\nu(0)\}|P \rangle \right] = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x_B, Q^2) + \frac{1}{P \cdot q} \left( P_\mu - q_\mu \frac{P \cdot q}{q^2} \right) \left( P_\nu - q_\nu \frac{P \cdot q}{q^2} \right) F_2(x_B, Q^2)
\]

• **QCD FACTORIZATION in DIS**

\[
F(x_B, Q^2) = H(x_B, Q^2/\mu^2) \otimes F_D(\mu^2) = \sum_i \int_{x_B}^1 \frac{d\xi}{\xi} C_i \left( \frac{x}{\xi}, \frac{Q^2}{\mu^2} \right) F^i_D(\xi, \mu^2)
\]

• **RENORMALIZATION PROPERTIES: DGLAP**

\[
\mu \frac{d}{d\mu} Q_{i/h}(x, \mu) = \sum_j \int_x^1 \frac{dz}{z} P_{ij} \left( \frac{x}{z} \right) Q_{j/h}(x, \mu)
\]

• **MOMENTS** of collinear PDFs are related to matrix elements of the local twist-2 operators within the OPE
Completely gauge invariant (quark) density:

\[
Q_{i/h}(x, \mu) = \frac{1}{2} \int \frac{d\xi^-}{2\pi} \ e^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-, 0_\perp)[\xi^-, 0^-]\gamma^+\psi_i(0^-, 0_\perp)|h(P)\rangle
\]

Gauge invariance is saved by the insertion of the gauge link

\[
[y, x]_r = \mathcal{P} \ exp \left[-ig \int_{\tau_1}^{\tau_2} d\tau r^\mu A^a_\mu(r\tau)t^a \right] \quad r^\mu \tau_1 = x, \ r^\mu \tau_2 = y
\]

Note: distinguish between longitudinal \([, ]_{[n, v, v_0]}\) and transversal \([, ]_{[l]}\) gauge links!
QCD FACTORIZATION for TMD

→ Collins, Soper: NPB (1981);
→ Collins, Metz: PRL (2004);
→ Ji, Ma, Yuan: PRD (2005);

Standard factorization expected:

\[
F(x_B, z_h, P_{h\perp}, Q^2) = \sum_i e_i^2 \cdot H \otimes F_D \otimes F_F \otimes S
\]

- Extra (rapidity) divergences;
- Complicated structure of gauge links: non-universality (generalized factorization):
  → Bacchetta, Bomhof, Mulders, Pijlman
- Generalized factorization may fail: several counter-examples exist
  → Collins, Qiu; Mulders, Rogers
PROBLEMS of OPERATOR DEFINITION of TMD

“Naive” definition:

\[
Q_i(x, k_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} e^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp}.
\]

\[
\langle P, S | \bar{\psi}_i(\xi^-, \xi_\perp) | \bar{\psi}_i(\xi^-, \xi_\perp; \infty^-, \xi_\perp) \rangle_{l} \langle \infty^-, \xi_\perp; \infty^-, \infty_\perp \rangle_{l} \cdot \gamma^+.
\]

Formally:

\[
\int d^2k_\perp Q_i(x, k_\perp) = Q_i(x)
\]
CLASSIFICATION of SINGULARITIES

1. $\sim \frac{1}{\varepsilon}$ poles, usual UV-singularities: removed by the standard $R-$operation and are controlled by renormalization-group evolution equations (DGLAP in integrated case)

2. pure rapidity divergences: give rise to logarithmic and double-logarithmic terms of the form $\sim \ln \eta$, $\ln^2 \eta$; have to be resumed

3. overlapping divergences: contain both UV and soft singularities simultaneously $\sim \frac{1}{\varepsilon} \ln \eta$; highly undesirable—depend on the parameters of the chosen gauge; prevents the removal of all UV-singularities by the standard $R-$procedure; a special generalized renormalization procedure is needed
EXAMPLE: three different definitions of a unintegrated quark distribution:

A. pure light-cone $\mathcal{F}_n$: $[n^2 = 0, \ n^+ = 0, \ n_\perp = 0]$

$$\mathcal{F}_n(x, k_\perp; \mu, \eta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \ e^{-ik\xi} \langle h | \bar{\psi}(\xi^-, \xi_\perp) [\xi^-, \xi_\perp; \infty^-, \infty_\perp]^\dagger [\infty^-, \xi_\perp; \infty^-, \infty_\perp]^\dagger l \gamma^+ [\infty^-, \infty_\perp; \infty^-, 0_\perp] l [\infty^-, 0_\perp; 0^-, 0_\perp] n \psi(0^-, 0_\perp) | h \rangle$$
EXAMPLE: three different definitions of a unintegrated quark distribution:

A. pure light-cone $F[n]: \left[ n^2 = 0 \; , \; n^+ = 0 \; , \; n_\perp = 0 \right]$ 

B. off-light-cone $F[v]: \left[ v^2 > 0 \; , \; v^- \gg v^+ \; , \; v_\perp = 0 \right] , \; \zeta = \frac{4(P \cdot v)^2}{v^2}$

$$F[v] (x, k_\perp; \mu, \zeta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi (2\pi)^2} \; e^{-ik\xi} \langle h | \bar{\psi}(\xi^- , \xi_\perp) [\xi^- , \xi_\perp; \infty^- , \xi_\perp]^\dagger_v [\infty^- , \xi_\perp; \infty^- , \infty_\perp]^\dagger_l \gamma^+$$

$$[\infty^- , \infty_\perp; \infty^- , 0_\perp]^\dagger_l [\infty^- , 0_\perp; 0^- , 0_\perp]^\dagger_v \; \psi(0^- , 0_\perp) | h \rangle$$
EXAMPLE: three different definitions of a unintegrated quark distribution:

A. pure light-cone $F_{[n]}$: \[ n^2 = 0 \, , \, n^+ = 0 \, , \, n_\perp = 0 \]

B. off-light-cone $F_{[v]}$: \[ v^2 > 0 \, , \, v^- \gg v^+ \, , \, v_\perp = 0 \] , \( \zeta = \frac{4(P \cdot v)^2}{v^2} \)

Γ. direct link $F_{[v_0]}$: \[ v_0^2 = v_0^2 < 0 \, , \, v^+ = 0 \, , \, \zeta = \frac{4(P \cdot v_0)^2}{v_0^2} \]

\[ F_{[v_0]} (x, k_\perp; \mu, \zeta_0) = \]

\[ = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \, e^{-ik\xi} \langle h | \bar{\psi}(\xi^-, \xi_\perp) \gamma^+ [\xi^-, \xi_\perp; 0^-, 0_\perp] v_0 \psi(0^-, 0_\perp) | h \rangle \]
**ONE – GLUON ORDER ANOMALOUS DIMENSIONS**

→ ICh, Stefanis: 2008, 2009

**Tree** approximation (distribution of *quark in a quark*):

\[
\mathcal{F}^{(0)}(x, k_\perp) = \\
= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \exp(-ik^+\xi^- + ik_\perp \cdot \xi_\perp) \langle p | \bar{\psi}(\xi^-, \xi_\perp) \gamma^+ \psi(0^-, 0_\perp) | p \rangle = \\
= \delta(1 - x) \delta^{(2)}(k_\perp)
\]

One-gluon exchanges, contributing to the UV-divergences:

\[+ (h.c.)\]
Source of the uncertainties and extra divergences: pole in the gluon propagator

\[ D_{LC}^{\mu\nu}(q) = \frac{1}{q^2} \left[ g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]^2} - \frac{q^\nu n^{-\mu}}{[q^+]^2} \right] \]

Possible pole prescriptions:

\[ d_{PV}^{\mu\nu}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu}) \frac{1}{2} \left( \frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta} \right) \]

\[ d_{Adv/Ret}^{\mu\nu}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu}) \frac{1}{q^+ \mp i\eta} \]

Mandelstam-Leibbrand prescription

\[ \frac{1}{[q^+]_{ML}} = \begin{cases} \\ \frac{1}{q^+ + i0q^-} \\ \frac{q^-}{q^+q^- + i0} \end{cases} \]
UV divergent part reads:

$$\Sigma_{\text{left}}(p, \alpha_s; \epsilon) =$$

$$= -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left[ -\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} + i\pi C_\infty \right] + \alpha_s C_F \frac{1}{\epsilon} [iC_\infty] =$$

$$= -\frac{\alpha_s}{\pi} C_F \frac{1}{\epsilon} \left[ -\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} \right]$$

prescription dependence is canceled

$$C_\infty = \begin{cases} 
0, \text{ Advanced } & \frac{1}{[q^+]} = \frac{1}{q^+ - i\eta} \\
-1, \text{ Retarded } & \frac{1}{[q^+]} = \frac{1}{q^+ + i\eta} \\
-\frac{1}{2}, \text{ PrincipalValue } & \frac{1}{[q^+]} = \frac{1}{2} \left( \frac{1}{q^+ - i\eta} + \frac{1}{q^+ + i\eta} \right) 
\end{cases}$$
Full UV divergent part:

\[ \Sigma_{\text{tot}}(p, \alpha_s(\mu); \epsilon) = \Sigma_{\text{left}} + \Sigma_{\text{right}} = \]

\[ = -\frac{\alpha_s}{4\pi} C_F \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\eta}{p^+} \right) \]

Dependence on \( \eta \) remains:

- gauge invariance is not complete
- AD doesn’t coincide with AD\(_{2q}\)
**LO anomalous dimension** is defined via the renormalization constant

\[
\gamma = \frac{1}{2} \frac{1}{Z^{(1)}} \frac{\partial \alpha_s(\mu)}{\partial \mu} \frac{\partial Z^{(1)}(\mu, \alpha_s(\mu); \epsilon)}{\partial \alpha_s}
\]

\[
\gamma_{LC} = \gamma_{\text{smooth}} - \delta \gamma, \quad \gamma_{\text{smooth}} = \frac{3 \alpha_s}{4 \pi} C_F + O(\alpha_s^2)
\]

**Defect of anomalous dimension**

\[
\delta \gamma = -\frac{\alpha_s}{\pi} C_F \ln \frac{\eta}{p^+}
\]

contains undesirable \( p^+ \)-dependent term: must be removed by a consistent procedure.
\( \delta \gamma \) is nothing else, but the **cusp anomalous dimension**:

\[
p^+ = (p \cdot n^-) \sim \cosh \chi
\]

defines an angle \( \chi \) between the direction of the quark momentum \( p_\mu \) and the light-like vector \( n^- \). In the large \( \chi \) limit:

\[
\ln p^+ \to \chi, \quad \chi \to \infty
\]
\( \delta \gamma \) is nothing else, but the **cusp anomalous dimension**:

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defines an angle \( \chi \) between the direction of the quark momentum \( p_{\mu} \)
and the light-like vector \( n^\mu \). In the large \( \chi \) limit:

\[
\ln p^+ \to \chi, \quad \chi \to \infty
\]

Renormalization of the Wilson operators with obstructions requires **extra renormalization factor**

\( \rightarrow \) Korchemsky, Radyushkin: 1987

\[
Z_\chi = \left[ \langle 0 | \mathcal{P} \exp \left[ ig \int_\chi^{\infty} d\zeta^\mu A_{\mu}^a(\zeta) \right] |0 \rangle \right]^{-1}
\]

**Generalized renormalization**

\[
\mathcal{O}_{\text{ren}}(\chi, \ldots) = Z_\chi Z_R \mathcal{O}(\chi, \ldots)
\]
One-gluon exchanges for the generalized multiplicative renormalization factor

Figure 1: One-gluon exchanges for the generalized multiplicative renormalization factor
One-gluon exchanges for the generalized multiplicative renormalization factor

Generalized renormalization constant reads

\[
\hat{Z}_{\text{mod}} = 1 + \frac{\alpha_s}{4\pi} C_F \frac{2}{\epsilon} \left( -3 - 4 \ln \frac{\eta}{p^+} + 4 \ln \frac{\eta}{p^+} \right) = 1 - \frac{3\alpha_s}{4\pi} C_F \frac{2}{\epsilon}
\]

so that

\[
\frac{1}{2} \mu \frac{d}{d\mu} \ln \hat{Z}_{\text{mod}}(\mu, \alpha_s, p^+) = \frac{3\alpha_s}{4 \pi} C_F + O(\alpha_s^2)
\]
Renormalization group equations

\( B, \Gamma. \) *off-light-cone* and *direct link*

\[
\mu \frac{d}{d\mu} \mathcal{F}[v, v_0] = \gamma_0 \mathcal{F}[v, v_0], \quad \gamma_0 = \frac{3}{4} \frac{\alpha_s C_F}{\pi} + O(\alpha_s^2)
\]

extraction of the *soft factor*

\[
\mathcal{F}[v] \rightarrow \mathcal{F}[v] \cdot R_v^{-1}
\]

\[
\mu \frac{d}{d\mu} \left[ \mathcal{F}[v] \cdot R_v^{-1} \right] = (\gamma_0 - \gamma_R) \left[ \mathcal{F}[v] \cdot R_v^{-1} \right]
\]

\( \gamma_R \)—anomalous dimension of the soft factor
A. pure light-cone

\[ \mu \frac{d}{d\mu} F_n = (\gamma_0 - \gamma_{\text{cusp}}) F_n \]

generalized renormalization “restores” the anomalous dimension

\[ F_n(\eta) \rightarrow F_n(\eta) \cdot R^{-1}(\eta) \]

\[ \mu \frac{d}{d\mu} [F_n \cdot R_n^{-1}] = \gamma_0 (F_n \cdot R_n^{-1}) \]
A. pure light-cone with Mandelstam-Leibbrandt prescription

\[ \mu \frac{d}{d\mu} \left[ F_{ML}^{[n]} \cdot R_n^{-1} \right] = \mu \frac{d}{d\mu} F_{ML}^{[n]} = \]

\[ \gamma_0 \left[ F_{ML}^{[n]} \cdot R_n^{-1} \right] = \gamma_0 F_{ML}^{[n]} \]

anomalous dimension without light-cone artifacts from the very beginning!
Generalized definition of TMD:

\[ \mathcal{F}_n(x, k⊥) \cdot R_n^{-1} = \]

\[ \frac{1}{2} \int \frac{dξ^- d^2ξ⊥}{2\pi(2\pi)^2} e^{-ik^+ξ^- + ik⊥⋅ξ⊥} \langle h | \bar{ψ}(ξ^-, ξ⊥) | [ξ^-, ξ⊥; ∞^-, ξ⊥]_n \rangle \]

\[ \times [∞^-, ξ⊥; ∞^-, ξ⊥]_l^\dagger \gamma^+ [∞^-, ξ⊥; ∞^-, 0⊥]_l [∞^-, 0⊥; 0^-, 0⊥]_n |ψ(0^-, 0⊥) | h \rangle \times \]

\[ \times [\text{SOFT FACTOR}]^{-1} \]

SOFT FACTOR =

\[ \langle 0 | \mathcal{P} \exp \left[ ig \int_{C_{\text{cusp}}} dζ^μ t^a A^a_μ(ζ) \right] \cdot \mathcal{P}^{-1} \exp \left[ - ig \int_{C'_{\text{cusp}}} dζ^μ t^a A^a_μ(ξ + ζ) \right] |0 \rangle \]
Generalized-to-generalized definition of TMD: *Pauli Spin Interactions*

ICh, A.I. Karanikas, N.G. Stefanis: 2010

Gauge potential $A^a_\mu$ is *spin-blind* → generalized conception of gauge invariance has to include coupling to spinning particles in terms of the Pauli term $\sim F^{\mu\nu} S_{\mu\nu}$, with the spin operator $S_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu]$:

- **Enhanced lightlike gauge link with Pauli term:**
  \[
  [[0^-, 0_\perp; 0^-, 0_\perp]] = \mathcal{P} \exp \left[ -ig \int_0^\infty d\sigma \, u_\mu A^\mu_a (u\sigma) t^a - ig \int_0^\infty d\sigma \, S_{\mu\nu} F^{\mu\nu}_{a} (u\sigma) t^a \right]
  \]

- **Enhanced transverse gauge link with Pauli term:**
  \[
  [[0^-, 0\perp; 0^-, 0\perp]] = \mathcal{P} \exp \left[ -i g \int_0^\infty d\tau \, l_\perp \cdot A^a_\perp (l\tau) t^a - i g \int_0^\infty d\tau \, S_{\mu\nu} F_{a}^{\mu\nu} (l\tau) t^a \right]
  \]
Collinear PDF from TMDs:

Definition A reproduces the DGLAP evolution after integration:

\[
\tilde{\int} d^2 k_\perp \, F_n(x, k_\perp, \mu) = F_n(x, \mu)
\]

\[
\mu \frac{d}{d\mu} F_n = K_{\text{DGLAP}} \otimes F_n
\]

Definition B fails to reproduce the DGLAP evolution after integration:

\[
\tilde{\int} d^2 k_\perp \, F_v(x, k_\perp, \mu) = F_v(x, \mu)
\]

\[
\mu \frac{d}{d\mu} F_v = \mathcal{K}_v \otimes F_v \neq \mathcal{K}_{\text{DGLAP}}
\]
FULL (?) SET of the EVOLUTION EQUATIONS for TMD

• UV-evolution (in the integrated case—DGLAP)

\[ \mu \frac{d}{d\mu} P(x, k_\perp, \mu, \zeta) = \mathcal{K}_{UV} \otimes P(x, k_\perp, \mu, \zeta) \]

• rapidity evolution (Collins-Soper)

\[ \zeta \frac{d}{d\zeta} P(x, k_\perp, \mu, \zeta) = \mathcal{K}_{CS} \otimes P(x, k_\perp, \mu, \zeta) \]

• BFKL evolution \rightarrow Collins-Soper?

\[ x \frac{d}{dx} P(x, k_\perp, \mu, \zeta) = \mathcal{K}_{BFKL} \otimes P(x, k_\perp, \mu, \zeta) \]

• ... ?
Current problems:

- **QCD factorization** within the **TMD** approach:
  
  \[ F_{\text{structure function}} = H_{\text{hard}} \otimes F_{\text{distribution}} \otimes F_{\text{fragmentation}} \otimes S_{\text{soft}} \]
  
  —expected, but not proved so far; (some counter-examples are known)

- Operator definition of **TMDs**: extra [overlapping = UV \( \otimes \) rapidity] divergences → satisfactory in the one-loop order (generalized renormalization); higher orders to be studied

- Very complicated structure of **gauge links** → non-universality of **TMDs**

- **Evolution equations** for **TMD**: (i) UV evolution, (ii) rapidity Collins-Soper evolution, (iii) small-\(X\) BFKL [?] evolution—full set of the **evolution equations**: [not known so far]

- Relationship between (unintegrated) **TMDs** and collinear (integrated) **PDFs**: [not known so far]

- Role of the **soft factor**: different within different frameworks

- **Lattice simulations**: problems with light-like gauge links

- ...A LOT OF WORK IN FRONT OF US!
Field-theoretic analysis of T M D is in progress!
ICh., A. Karanikas, N. Stefanis:


ICh., N. Stefanis:

- **AIP Conf. Proc.** 1105 (2009) 327
- **Mod. Phys. Lett A** 24 (2009) 2913
- **Phys. Rev. D** 80 (2009) 054008
- arXiv: 1008.0725 [hep-ph]
- arXiv: 0811.4357 [hep-ph]