Extensive quantum entanglement and localization in quantum spin chains

Fumihiko Sugino

Center for Theoretical Physics of the Universe, Institute for Basic Science

ICHEP 2018 Seoul, July 6, 2018

Mainly based on

Bravyi et al, Phys. Rev. Lett. **118** (2012) 207202, arXiv: 1203.5801
R. Movassagh and P. Shor, Proc. Natl. Acad. Sci. **113** (2016) 13278, arXiv: 1408.1657
F.S. and P. Padmanabhan, J. Stat. Mech. **1801** (2018) 013101, arXiv: 1710.10426
P. Padmanabhan, F.S. and V. Korepin, arXiv: 1804.00978

Outline

Introduction 1

Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics

Introduction 1

Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics
- From pure state of the full system S: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem A: $\rho_A = \mathrm{Tr}_{S-A}\,\rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\mathrm{Tr}_A\left[\rho_A \ln \rho_A\right].$$

This is purely a quantum property.

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems (with local interactions) typically exhibit the area law behavior of the entanglement entropy: $S_A \propto (\text{area of } A)$
- Gapped systems in 1D are proven to obey the area law.
  [Hastings 2007]

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems (with local interactions) typically exhibit the area law behavior of the entanglement entropy: $S_A \propto$ (area of A)
- Gapped systems in 1D are proven to obey the area law. [Hastings 2007]
  (Area law violation) $\Rightarrow$ Gapless
- For gapless case, $(1+1)$-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln$ (volume of A). [Calabrese, Cardy 2009]

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems (with local interactions) typically exhibit the area law behavior of the entanglement entropy: $S_A \propto$ (area of A)
- Gapped systems in 1D are proven to obey the area law. [Hastings 2007]
  (Area law violation) $\Rightarrow$ Gapless
- For gapless case, $(1+1)$-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln$ (volume of A). [Calabrese, Cardy 2009]
- Recently, 1D solvable spin chain model which exhibit extensive entanglement entropy have been discussed.
  - Beyond logarithmic violation: $S_A \propto \sqrt{(\text{volume of } A)}$
    [Movassagh, Shor 2014], [Salberger, Korepin 2016]

Motzkin spin model 1

[Bravyi et al 2012]

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

$$|u\rangle \Leftrightarrow \nearrow, \qquad |d\rangle \Leftrightarrow \searrow, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

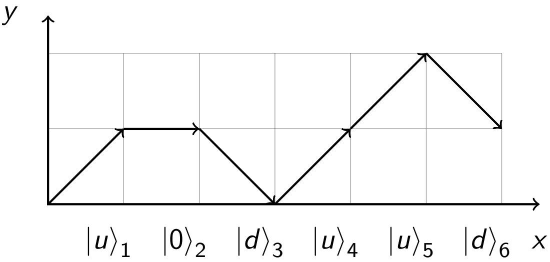
Motzkin spin model 1

[Bravyi et al 2012]

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

$$|u\rangle \Leftrightarrow \nearrow, \qquad |d\rangle \Leftrightarrow \searrow, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

- Each spin configuration $\Leftrightarrow$ length-$2n$ walk in (x, y) plane
  Example)



Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

$$|U\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, u\rangle - |u,\, 0\rangle\right),$$

$$|F\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, 0\rangle - |u,\, d\rangle\right).$$

Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

$\Leftrightarrow$

$$|U\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, u\rangle - |u,\, 0\rangle\right),$$

$$|F\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, 0\rangle - |u,\, d\rangle\right).$$

Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

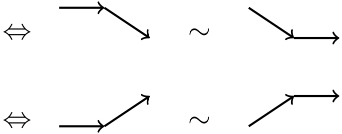
$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

$$|U\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, u\rangle - |u,\, 0\rangle\right),$$

$$|F\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, 0\rangle - |u,\, d\rangle\right).$$



Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

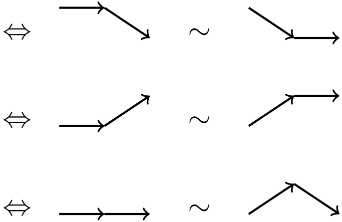
$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

$$|U\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, u\rangle - |u,\, 0\rangle\right),$$

$$|F\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, 0\rangle - |u,\, d\rangle\right).$$



"gauge equivalence".

Motzkin spin model 3

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Boundary part: $H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

$$\Downarrow$$

Motzkin spin model 3

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Boundary part: $H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

$$\Downarrow$$

- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum

- We find the unique zero-energy ground state.

Motzkin spin model 3

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Boundary part: $H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

$$\Downarrow$$

- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.
  - Each projector in $H_{Motzkin}$ annihilates the zero-energy state.

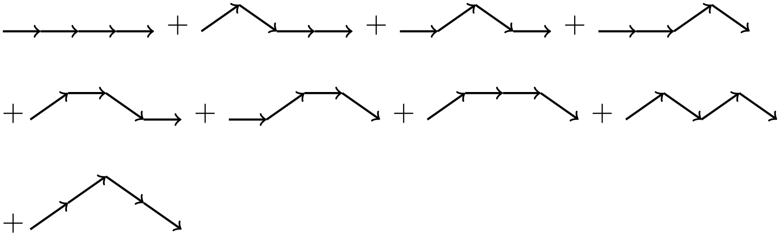
    $\Rightarrow$ Frustration free
- The ground state corresponds to randoms walks starting at $(0, 0)$ and ending at $(2n, 0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).

Motzkin spin model 4

[Bravyi et al 2012]

Example) $2n = 4$ case,

MWs:



$$\Updownarrow$$

Ground state:

$$
\begin{aligned}
|P_4\rangle \;\; = \;\; \frac{1}{\sqrt{9}} & \left[|0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle \right. \\
& + |u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\
& \left. + |uudd\rangle \right].
\end{aligned}
$$

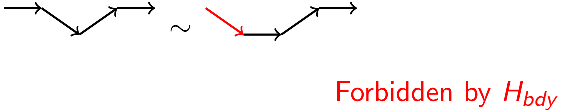
Motzkin spin model 5

[Bravyi et al 2012]

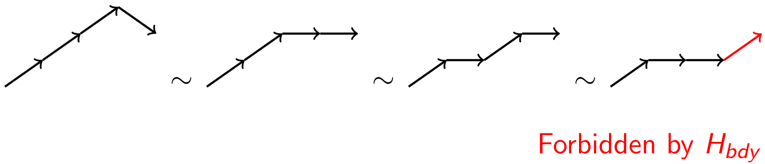
Note

Forbidden paths for the ground state

1. Path entering $y < 0$ region



2. Path ending at nonzero height



Motzkin spin model 6

[Bravyi et al 2012]

Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

$C_k = \frac{1}{k+1}\binom{2k}{k}$: Catalan number

Motzkin spin model 6

[Bravyi et al 2012]

Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

  $C_k = \frac{1}{k+1}\binom{2k}{k}$: Catalan number

- Consider to trace out the density matrix $\rho = |P_{2n}\rangle\langle P_{2n}|$ w.r.t. the subsystem $B = \{n+1, \cdots, 2n\}$.
  Schmidt decomposition:

$$|P_{2n}\rangle = \sum_{h\geq 0} \sqrt{p_{n,n}^{(h)}} \left| P_n^{(0\to h)} \right\rangle \otimes \left| P_n^{(h\to 0)} \right\rangle$$

with $p_{n,n}^{(h)} \equiv \frac{\left(M_n^{(h)}\right)^2}{M_{2n}}$.

↑ Paths from $(0,0)$ to (n,h)

Motzkin spin model 7

[Bravyi et al 2012]

- $M_n^{(h)}$ is the number of paths in $P_n^{(0\to h)}$.
  For $n \to \infty$, Gaussian distribution

$$p_{n,n}^{(h)} \sim \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{(h+1)^2}{n^{3/2}} e^{-\frac{3}{2}\frac{(h+1)^2}{n}} \times [1 + O(1/n)] \,.$$

- Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho = \sum_{h\geq 0} p_{n,n}^{(h)} \left| P_n^{(0\to h)} \right\rangle \left\langle P_n^{(0\to h)} \right|$$

- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h\geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= \frac{1}{2}\ln n + \frac{1}{2}\ln\frac{2\pi}{3} + \gamma - \frac{1}{2} \end{aligned}$$

(γ: Euler constant)

up to terms vanishing as $n \to \infty$.

Motzkin spin model 7

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT with $c = 3/2$.

Motzkin spin model 7

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT with $c = 3/2$.
- Gap scales as $O(1/n^z)$ with $z \geq 2$.
  The system cannot be described by relativistic CFT.
  Lifshitz type ? [Chen, Fradkin, Witczak-Krempa 2017]
- Excitations have not been much investigated.

Colored Motzkin spin model 1

[Movassagh, Shor 2014]

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$\left|u^k\right\rangle \Leftrightarrow \overset{k}{\nearrow}, \qquad \left|d^k\right\rangle \Leftrightarrow \overset{k}{\searrow}, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

Color d.o.f. decorated to Motzkin Walks

Colored Motzkin spin model 1

[Movassagh, Shor 2014]

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$\left|u^k\right\rangle \Leftrightarrow \overset{k}{\nearrow}, \qquad \left|d^k\right\rangle \Leftrightarrow \overset{k}{\searrow}, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  - Bulk part consisting of local interactions:

$$H_{bulk} = \sum_{j=1}^{2n-1} \left(\Pi_{j,j+1} + \Pi^{cross}_{j,j+1}\right),$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s} \left[\left|D^k\right\rangle_{j,j+1}\left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1}\left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1}\left\langle F^k\right|\right]$$

with

Colored Motzkin spin model 2

[Movassagh, Shor 2014]

$$\left|D^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, d^k\right\rangle - \left|d^k,\, 0\right\rangle\right),$$
$$\left|U^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, u^k\right\rangle - \left|u^k,\, 0\right\rangle\right),$$
$$\left|F^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, 0\rangle - \left|u^k,\, d^k\right\rangle\right),$$

and

$$\Pi^{cross}_{j,j+1} = \sum_{k \neq k'} \left|u^k,\, d^{k'}\right\rangle_{j,j+1}\left\langle u^k,\, d^{k'}\right|.$$

$\Rightarrow$ Colors should be matched in up and down pairs.

- Boundary part

$$H_{bdy} = \sum_{k=1}^{s}\left(\left|d^k\right\rangle_1\left\langle d^k\right| + \left|u^k\right\rangle_{2n}\left\langle u^k\right|\right).$$

Colored Motzkin spin model 3

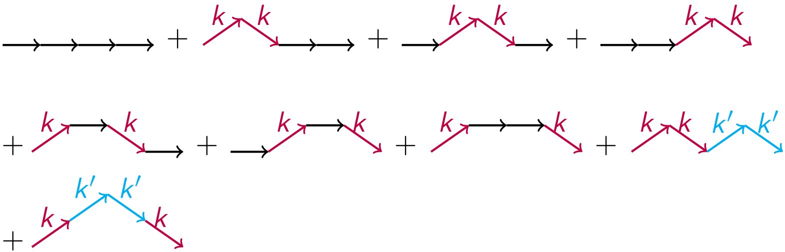
[Movassagh, Shor 2014]

- Still unique ground state with zero energy

Colored Motzkin spin model 3

[Movassagh, Shor 2014]

- Still unique ground state with zero energy
- Example) $2n = 4$ case,



$$|P_4\rangle = \frac{1}{\sqrt{1+6s+2s^2}}\left[|0000\rangle + \sum_{k=1}^{s}\left\{\left|u^k d^k 00\right\rangle + \cdots + \left|u^k 00 d^k\right\rangle\right\} + \sum_{k,k'=1}^{s}\left\{\left|u^k d^k u^{k'} d^{k'}\right\rangle + \left|u^k u^{k'} d^{k'} d^k\right\rangle\right\}\right].$$

Colored Motzkin spin model 4

[Movassagh, Shor 2014]

Entanglement entropy

- Paths from $(0,0)$ to (n,h), $P_n^{(0\to h)}$, have h unmatched up steps.
  Let $\tilde{P}_n^{(0\to h)}(\{\kappa_m\})$ be paths with the colors of unmatched up steps fixed.
  (unmatched up from height $(m-1)$ to m) $\to u^{\kappa_m}$
- Similarly,

  $$P_n^{(h\to 0)} \to \tilde{P}_n^{(h\to 0)}(\{\kappa_m\}),$$

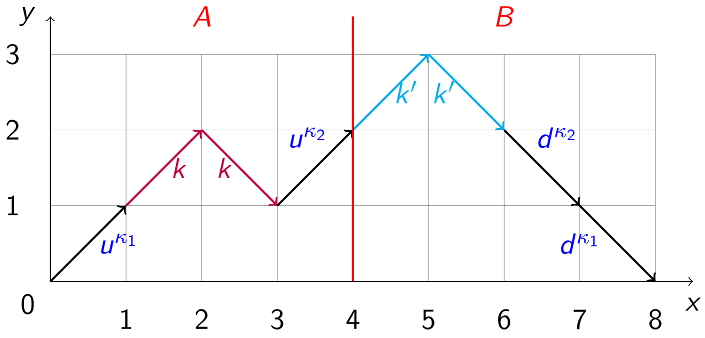
  (unmatched down from height m to $(m-1)$) $\to d^{\kappa_m}$.
- The numbers satisfy $M_n^{(h)} = s^h \, \tilde{M}_n^{(h)}$.

Colored Motzkin spin model 5

[Movassagh, Shor 2014]

Example

$2n = 8$ case, $h = 2$



Colored Motzkin spin model 6

[Movassagh, Shor 2014]

- Schmidt decomposition

$$|P_{2n}\rangle = \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s}\sqrt{p_{n,n}^{(h)}} \times \left|\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right\rangle \otimes \left|\tilde{P}_n^{(h\to 0)}(\{\kappa_m\})\right\rangle$$

with

$$p_{n,n}^{(h)} = \frac{\left(\tilde{M}_n^{(h)}\right)^2}{M_{2n}}.$$

- Reduced density matrix

$$\rho_A = \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s} p_{n,n}^{(h)} \times \left|\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right\rangle\left\langle\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right|.$$

Colored Motzkin spin model 7

[Movassagh, Shor 2014]

- For $n \to \infty$,

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma n)^{3/2}} (h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$.

Note: Effectively $h \lesssim O(\sqrt{n})$.

- Entanglement entropy

$$S_A = -\sum_{h \geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)}$$

Colored Motzkin spin model 7

[Movassagh, Shor 2014]

- For $n \to \infty$,

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma n)^{3/2}} (h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$.

Note: Effectively $h \lesssim O(\sqrt{n})$.

- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h \geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= (2 \ln s)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} - \ln s \end{aligned}$$

up to terms vanishing as $n \to \infty$.

Grows as $\sqrt{n}$.

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- Matching color $\Rightarrow$ s^{-h} factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in S_A

Colored Motzkin spin model 8 [Movassagh, Shor 2014]

Comments

- $$\begin{aligned} \text{Matching color} \quad &\Rightarrow \quad s^{-h} \text{ factor in } p_{n,n}^{(h)} \\ &\Rightarrow \quad \text{crucial to } O(\sqrt{n}) \text{ behavior in } S_A \end{aligned}$$
- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- Matching color $\Rightarrow$ s^{-h} factor in $p^{(h)}_{n,n}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in S_A
- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]
- Deformation of models to achieve the volume law behavior ($S_A \propto n$)
  Weighted Motzkin/Dyck walks [Zhang et al, Salberger et al 2016]

Colored Motzkin spin model 8

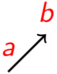
[Movassagh, Shor 2014]

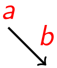
Comments

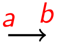
- $$\text{Matching color} \quad \Rightarrow \quad s^{-h} \text{ factor in } p_{n,n}^{(h)}$$
  $$\Rightarrow \quad \text{crucial to } O(\sqrt{n}) \text{ behavior in } S_A$$
- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]
- Deformation of models to achieve the volume law behavior ($S_A \propto n$)
  Weighted Motzkin/Dyck walks [Zhang et al, Salberger et al 2016]
- Next, we consider extension of the model from a different point of view.

SIS Motzkin model 1

[Sugino, Padmanabhan 2017]

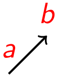
- Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.
- $a < b$ case: 'up' $\Leftrightarrow$ 

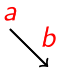
  $a > b$ case: 'down' $\Leftrightarrow$ 

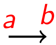
  $a = b$ case: 'flat' $\Leftrightarrow$ 

SIS Motzkin model 1

[Sugino, Padmanabhan 2017]

- Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.
- $a < b$ case: 'up' $\Leftrightarrow$ 

  $a > b$ case: 'down' $\Leftrightarrow$ 

  $a = b$ case: 'flat' $\Leftrightarrow$ 

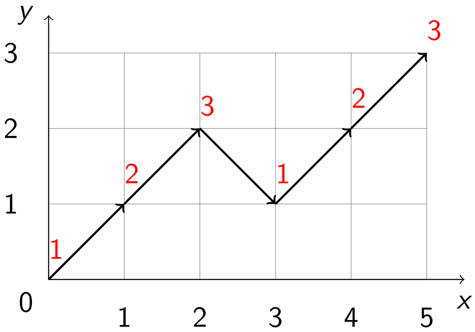
- We regard the configuration of adjacent sites $|(x_{a,b})_j\rangle\,|(x_{c,d})_{j+1}\rangle$ as a connected path for $b = c$.
  c.f.) Analogous to the product rule of Symmetric Inverse Semigroup ($\mathcal{S}_1^k$): $x_{a,b} * x_{c,d} = \delta_{b,c}\, x_{a,d}$

  a, b: semigroup indices
- Inner product: $\langle x_{a,b}|x_{c,d}\rangle = \delta_{a,c}\delta_{b,d}$
- Let us consider the $k = 3$ case.

SIS Motzkin model 2

[Sugino, Padmanabhan 2017]

- Maximum height is lower than the original Motzkin case.



SIS Motzkin model 3

[Sugino, Padmanabhan 2017]

Hamiltonian $H_{S31Motzkin} = H_{bulk} + H_{bulk,disc} + H_{bdy}$

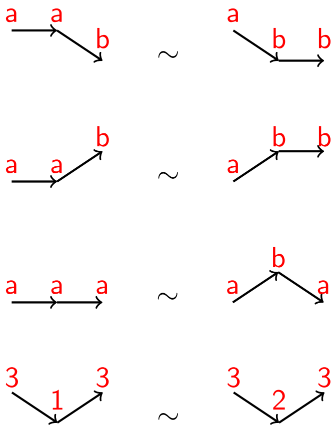
- H_{bulk}: local interactions corresponding to the following moves:

$(Down)$ $\sim$ $(a > b)$

(Up) $\sim$ $(a < b)$

$(Flat)$ $\sim$ $(a < b)$

$(Wedge)$ $\sim$



SIS Motzkin model 4

[Sugino, Padmanabhan 2017]

- $H_{bulk,disc}$ lifts disconnected paths to excited states.

$\Pi^{|\psi\rangle}$: projector to $|\psi\rangle$

$$H_{bulk,disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1;\, b\neq c}^{3} \Pi^{\left|(x_{a,b})_j,(x_{c,d})_{j+1}\right\rangle}$$

SIS Motzkin model 4

[Sugino, Padmanabhan 2017]

- $H_{bulk,disc}$ lifts disconnected paths to excited states.

$\Pi^{|\psi\rangle}$: projector to $|\psi\rangle$

$$H_{bulk,disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1;\, b\neq c}^{3} \Pi^{\left|(x_{a,b})_j,(x_{c,d})_{j+1}\right\rangle}$$

-

$$H_{bdy} = \sum_{a>b} \Pi^{\left|(x_{a,b})_1\right\rangle} + \sum_{a<b} \Pi^{\left|(x_{a,b})_{2n}\right\rangle} + \Pi^{|(x_{1,3})_1,(x_{3,2})_2,(x_{2,1})_3\rangle} + \Pi^{|(x_{1,2})_{2n-2},(x_{2,3})_{2n-1},(x_{3,1})_{2n}\rangle}$$

The last 2 terms have no analog to the original Motzkin model.

SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- Ground states correspond to connected paths starting at $(0,0)$, ending at $(2n,0)$ and not entering $y<0$. $\mathcal{S}_1^3$ MWs

SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- Ground states correspond to connected paths starting at $(0, 0)$, ending at $(2n, 0)$ and not entering $y < 0$. $\mathcal{S}_1^3$ MWs
- The ground states have 5 fold degeneracy according to the initial and finial semigroup indices:
  $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ and $(3, 3)$ sectors
  The $(3, 3)$ sector is trivial, consisting of only one path:
  $$x_{3,3}x_{3,3}\cdots x_{3,3}.$$

SIS Motzkin model 5

[Sugino, Padmanabhan 2017]

- Ground states correspond to connected paths starting at $(0,0)$, ending at $(2n,0)$ and not entering $y<0$. $\mathcal{S}_1^3$ MWs
- The ground states have 5 fold degeneracy according to the initial and finial semigroup indices:
  $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ and $(3,3)$ sectors
  The $(3,3)$ sector is trivial, consisting of only one path:
  $x_{3,3}x_{3,3}\cdots x_{3,3}$.
- The number of paths can be obtained by recursion relations. For length-n paths from the semigroup index a to b ($P_{n,a\to b}$),

$$\begin{aligned} P_{n,1\to 1} &= x_{1,1}P_{n-1,1\to 1} + x_{1,2}\sum_{i=1}^{n-2} P_{i,2\to 2}\,x_{2,1}P_{n-2-i,1\to 1} \\ &\quad + x_{1,3}\sum_{i=1}^{n-2} P_{i,3\to 3}\,x_{3,1}P_{n-2-i,1\to 1} \\ &\quad + x_{1,3}\sum_{i=1}^{n-2} P_{i,3\to 3}\,x_{3,2}P_{n-2-i,2\to 1}, \qquad \text{etc.} \end{aligned}$$

SIS Motzkin model 6

[Sugino, Padmanabhan 2017]

Result

- The entanglement entropies $S_{A,1\to 1}$, $S_{A,1\to 2}$, $S_{A,2\to 1}$ and $S_{A,2\to 2}$ take the same form as in the case of the Motzkin model.

  Logarithmic violation of the area law

  - The form of $p_n^{(h)} \sim \frac{(h+1)^2}{n^{3/2}}\, e^{-(\mathrm{const.})\frac{(h+1)^2}{n}}$ is universal.
- $S_{A,3\to 3} = 0$.
- Colored version can also be constructed ($\mathcal{S}_2^3 \sim (\mathcal{S}_1^3)^2$):

$$S_{A,\,a\to b} = (2\ln 2)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln\frac{3}{2^{1/3}}$$

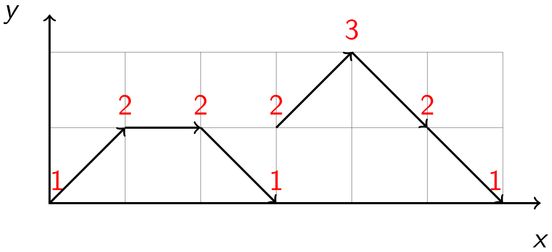
with $\sigma \equiv \frac{\sqrt{2}-1}{9\sqrt{2}}$ for $(a,b) = (1,1), (1,2), (2,1), (2,2)$.

SIS Motzkin model 7

Localizaion

[Padmanabhan, F.S., Korepin 2018]

- There are excited states corresponding to disconnected paths.
  Example) One such path in $2n = 6$ case,

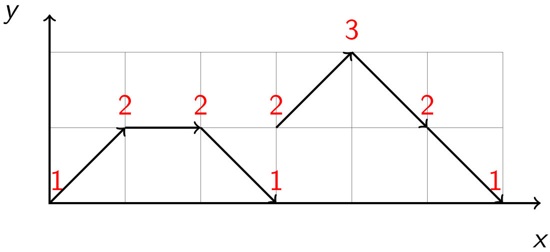


SIS Motzkin model 7

Localizaion

[Padmanabhan, F.S., Korepin 2018]

- There are excited states corresponding to disconnected paths.
  Example) One such path in $2n = 6$ case,



Corresponding excited state: $\left|P_{3,\,1\to1}\right\rangle \otimes \left|P^{(1\to0)}_{3,\,2\to1}\right\rangle$

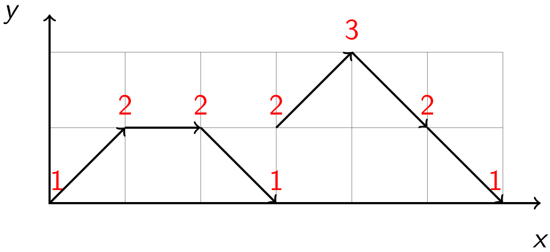
Each connected component has no entanglement with other components.

SIS Motzkin model 7

Localizaion

[Padmanabhan, F.S., Korepin 2018]

- There are excited states corresponding to disconnected paths.
  Example) One such path in $2n = 6$ case,



Corresponding excited state: $|P_{3,\,1\to1}\rangle \otimes \left|P^{(1\to0)}_{3,\,2\to1}\right\rangle$

Each connected component has no entanglement with other components.

$\Rightarrow$ 2pt connected correlation functions of local operators belonging to separate connected components vanish.

$\Rightarrow$ Localization!

Summary and discussion 1

- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.

Summary and discussion 1

- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
- We have extend the models by introducing additional d.o.f. based on Symmetric Inverse Semigroup.

Summary and discussion 1

- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
- We have extend the models by introducing additional d.o.f. based on Symmetric Inverse Semigroup.
- Extension of the Fredkin (spin 1/2) model

[Padmanabhan, F.S., Korepin 2018]

Summary and discussion 1

- We have reviewed the (colored) Motzkin spin models which yield large entanglement entropy proportional to the square root of the volume.
- We have extend the models by introducing additional d.o.f. based on Symmetric Inverse Semigroup.
- Extension of the Fredkin (spin 1/2) model

  [Padmanabhan, F.S., Korepin 2018]
- As a feature of the extended models,
  Anderson-like localization occurs in excited states corresponding to disconnected paths.
  - No need of noise.

Summary and discussion 2

- Continuum limit? (In particular, for colored case)

  [Chen, Fradkin, Witczak-Krempa 2017]

Summary and discussion 2

- Continuum limit? (In particular, for colored case)

  [Chen, Fradkin, Witczak-Krempa 2017]

- Holography? Application to quantum gravity or black holes?

  [Alexander, Klich 2018]

- Higher-dimensional models ($d = 2, 3, ...$)?

Summary and discussion 2

- Continuum limit? (In particular, for colored case)

  [Chen, Fradkin, Witczak-Krempa 2017]

- Holography? Application to quantum gravity or black holes?

  [Alexander, Klich 2018]

- Higher-dimensional models ($d = 2, 3, ...$)?

Thank you very much for your attention!