Extensive quantum entanglement and localization in quantum spin chains

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Mainly based on

Outline

Introduction

Motzkin spin model

Colored Motzkin model

SIS Motzkin model

Summary and discussion
Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics
Quantum entanglement

- Most surprising feature of quantum mechanics, No analog in classical mechanics
- From pure state of the full system $S$: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem $A$: $\rho_A = \text{Tr}_{S-A} \rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\text{Tr}_A [\rho_A \ln \rho_A].$$

This is purely a quantum property.
Area law of entanglement entropy

- Ground states of quantum many-body systems (with local interactions) typically exhibit the area law behavior of the entanglement entropy: \( S_A \propto \text{(area of } A) \)
- Gapped systems in 1D are proven to obey the area law. [Hastings 2007]
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  (Area law violation) $\Rightarrow$ Gapless
- For gapless case, $(1 + 1)$-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln (\text{volume of } A)$. [Calabrese, Cardy 2009]
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- Recently, 1D solvable spin chain model which exhibit extensive entanglement entropy have been discussed.
  - Beyond logarithmic violation: $S_A \propto \sqrt{\text{(volume of } A\text{)}}$
    [Movassagh, Shor 2014], [Salberger, Korepin 2016]
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Motzkin spin model 1

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

$$|u\rangle \leftrightarrow \uparrow, \quad |d\rangle \leftrightarrow \downarrow, \quad |0\rangle \leftrightarrow \longrightarrow$$
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\[
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\]

- Each spin configuration \( \Leftrightarrow \) length-2n walk in \((x, y)\) plane

Example)

\[
\begin{align*}
|u\rangle_1 & \quad |0\rangle_2 & \quad |d\rangle_3 & \quad |u\rangle_4 & \quad |u\rangle_5 & \quad |d\rangle_6 \\
\end{align*}
\]

\[
\begin{array}{c}
\text{y} \\
\end{array}
\begin{array}{c}
\text{x} \\
\end{array}
\]
Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: \( H_{\text{Motzkin}} = H_{\text{bulk}} + H_{\text{bdy}} \)

- Bulk part: \( H_{\text{bulk}} = \sum_{j=1}^{2n-1} \Pi_{j,j+1} \)

\[
\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|
\]

(local interactions) with

\[
|D\rangle \equiv \frac{1}{\sqrt{2}} (|0, d\rangle - |d, 0\rangle), \\
|U\rangle \equiv \frac{1}{\sqrt{2}} (|0, u\rangle - |u, 0\rangle), \\
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\]

\[
|U\rangle \equiv \frac{1}{\sqrt{2}} (|0, u\rangle - |u, 0\rangle),
\]

\[
|F\rangle \equiv \frac{1}{\sqrt{2}} (|0, 0\rangle - |u, d\rangle).
\]
Motzkin spin model 3

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Boundary part: $H_{bdy} = |d\rangle_1 \langle d| + |u\rangle_{2n} \langle u|$

⇒ Positive semi-definite spectrum

⇒ We find the unique zero-energy ground state.

⇒ Each projector in $H_{Motzkin}$ annihilates the zero-energy state.

⇒ Frustration free

⇒ The ground state corresponds to random walks starting at $(0, 0)$ and ending at $(2n, 0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).
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|$\Downarrow$|

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\( \Rightarrow \) Positive semi-definite spectrum

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Motzkin spin model 4

Example) $2n = 4$ case,

MWs:

\[
\begin{align*}
|P_4\rangle &= \frac{1}{\sqrt{9}} \left[ |0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle \\
&\quad + |u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\
&\quad + |uu0d\rangle + |u0ud\rangle + |0uud\rangle + |uudd\rangle \right].
\end{align*}
\]
Note
Forbidden paths for the ground state

1. Path entering \( y < 0 \) region

\[
\begin{array}{c}
\text{Forbidden by } H_{bdy}
\end{array}
\]

2. Path ending at nonzero height

\[
\begin{array}{c}
\text{Forbidden by } H_{bdy}
\end{array}
\]
Motzkin spin model 6

Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

$$C_k = \frac{1}{k+1} \binom{2k}{k}: \text{Catalan number}$$
Motzkin spin model 6

[Bravyi et al 2012]

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  $$C_k = \frac{1}{k+1} \binom{2k}{k}: \text{Catalan number}$$

- Consider to trace out the density matrix $\rho = |P_{2n}\rangle \langle P_{2n}|$ w.r.t. the subsystem $B = \{n+1, \cdots, 2n\}$.

  Schmidt decomposition:

  $$|P_{2n}\rangle = \sum_{h \geq 0} \sqrt{p_{n,n}^{(h)}} \left| P_{n}^{(0 \rightarrow h)} \right\rangle \otimes \left| P_{n}^{(h \rightarrow 0)} \right\rangle$$

  with $p_{n,n}^{(h)} \equiv \left( \frac{M_{n}^{(h)}}{M_{2n}} \right)^2$.

Paths from $(0, 0)$ to $(n, h)$
Motzkin spin model 7

- $M_n^{(h)}$ is the number of paths in $P_n^{(0 \to h)}$.
  For $n \to \infty$, Gaussian distribution

\[
p_{n,n}^{(h)} \sim 3\sqrt{6} (h + 1)^2 \frac{e^{-\frac{3}{2} \frac{(h+1)^2}{n^2}}}{n^{3/2}} \times [1 + O(1/n)].
\]

- Reduced density matrix

\[
\rho_A = \text{Tr}_B \rho = \sum_{h \geq 0} p_{n,n}^{(h)} \left| P_n^{(0 \to h)} \right\rangle \left\langle P_n^{(0 \to h)} \right|
\]

- Entanglement entropy

\[
S_A = - \sum_{h \geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)}
\]

\[
= \frac{1}{2} \ln n + \frac{1}{2} \ln \left(\frac{2\pi}{3}\right) + \gamma - \frac{1}{2}
\]

($\gamma$: Euler constant)

up to terms vanishing as $n \to \infty$. 

[Bravyi et al 2012]
Notes

- The system is critical (gapless).
  $S_A$ is similar to the $(1 + 1)$-dimensional CFT with $c = 3/2$. 
Motzkin spin model 7  

Notes

- The system is critical (gapless).  
  \( S_A \) is similar to the \((1 + 1)\)-dimensional CFT with \( c = \frac{3}{2} \).

- Gap scales as \( O(1/n^z) \) with \( z \geq 2 \).
  
  The system cannot be described by relativistic CFT.

- Lifshitz type?  
  [Chen, Fradkin, Witczak-Krempa 2017]

- Excitations have not been much investigated.
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Summary and discussion
Colored Motzkin spin model 1

Introducing color d.o.f. \( k = 1, 2, \cdots, s \) to up and down spins as

\[
|u^k\rangle \iff \uparrow^k, \quad |d^k\rangle \iff \downarrow^k, \quad |0\rangle \iff \rightarrow
\]

Color d.o.f. decorated to Motzkin Walks
Colored Motzkin spin model 1

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

\[
\begin{align*}
|u^k\rangle & \leftrightarrow k, \\
|d^k\rangle & \leftrightarrow \overline{k}, \\
|0\rangle & \leftrightarrow \rightarrow
\end{align*}
\]

Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  
  - Bulk part consisting of local interactions:

\[
H_{bulk} = \sum_{j=1}^{2n-1} \left( \prod_{j,j+1} + \prod^{cross}_{j,j+1} \right),
\]

\[
\prod_{j,j+1} = \sum_{k=1}^{s} \left[ |D^k\rangle_{j,j+1} \langle D^k| + |U^k\rangle_{j,j+1} \langle U^k| + |F^k\rangle_{j,j+1} \langle F^k| \right]
\]

with
Colored Motzkin spin model 2

\[ |D^k\rangle \equiv \frac{1}{\sqrt{2}} \left( |0, d^k\rangle - |d^k, 0\rangle \right), \]

\[ |U^k\rangle \equiv \frac{1}{\sqrt{2}} \left( |0, u^k\rangle - |u^k, 0\rangle \right), \]

\[ |F^k\rangle \equiv \frac{1}{\sqrt{2}} \left( |0, 0\rangle - |u^k, d^k\rangle \right), \]

and

\[ \prod_{j, j+1}^{\text{cross}} = \sum_{k \neq k'} |u^k, d^{k'}\rangle_{j, j+1} \langle u^{k'}, d^{k'}| \]

⇒ Colors should be matched in up and down pairs.

Boundary part

\[ H_{\text{bdy}} = \sum_{k=1}^{s} \left( |d^k\rangle_1 \langle d^k| + |u^k\rangle_{2n} \langle u^k| \right). \]
Colored Motzkin spin model 3

- Still unique ground state with zero energy

Example: \( n = 4 \) case,
\[
|P_4\rangle = \sqrt{1 + 6s + 2s^2} [|0000\rangle + s\sum_{k=1}^{n} \{|u_kd_k\rangle + \cdots + |u_kd_k\rangle\} + s\sum_{k, k'=1}^{n} \{|u_kd_ku_{k'}d_{k'}\rangle + |u_ku_{k'}d_kd_{k'}\rangle\}].
\]
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- Still unique ground state with zero energy
- Example) $2n = 4$ case,

\[ |P_4\rangle = \frac{1}{\sqrt{1 + 6s + 2s^2}} \left[ |0000\rangle + \sum_{k=1}^{s} \left\{ |u^k d^k 00\rangle + \cdots + |u^k 00 d^k\rangle \right\} \right. \]
\[ \left. + \sum_{k,k'=1}^{s} \left\{ |u^k d^k u^{k'} d^{k'}\rangle + |u^k u^{k'} d^{k'} d^k\rangle \right\} \right]. \]
Entanglement entropy

Paths from \((0, 0)\) to \((n, h)\), \(P_n^{(0 \rightarrow h)}\), have \(h\) unmatched up steps.

Let \(\tilde{P}_n^{(0 \rightarrow h)}(\{\kappa_m\})\) be paths with the colors of unmatched up steps fixed.

\[
\text{(unmatched up from height \((m - 1)\) to } m) \rightarrow u^{\kappa_m}
\]

Similarly,

\[
P_n^{(h \rightarrow 0)} \rightarrow \tilde{P}_n^{(h \rightarrow 0)}(\{\kappa_m\}),
\]

\[
\text{(unmatched down from height } m \text{ to } (m - 1)) \rightarrow d^{\kappa_m}.
\]

The numbers satisfy \(M_n^{(h)} = s^h \tilde{M}_n^{(h)}\).
Example

$2n = 8$ case, $h = 2$
Colored Motzkin spin model 6

- Schmidt decomposition

\[ |P_{2n} \rangle = \sum_{h \geq 0} \sum_{\kappa_1 = 1}^{s} \cdots \sum_{\kappa_h = 1}^{s} \sqrt{p_{n,n}^{(h)}} \]

\[ \times |\tilde{P}_{n}^{(0 \rightarrow h)} (\{\kappa_m\}) \rangle \otimes |\tilde{P}_{n}^{(h \rightarrow 0)} (\{\kappa_m\}) \rangle \]

with

\[ p_{n,n}^{(h)} = \left( \tilde{M}_{n}^{(h)} \right)^2 \]

\[ \frac{M_{2n}}{M_{2n}}. \]

- Reduced density matrix

\[ \rho_A = \sum_{h \geq 0} \sum_{\kappa_1 = 1}^{s} \cdots \sum_{\kappa_h = 1}^{s} p_{n,n}^{(h)} \]

\[ \times |\tilde{P}_{n}^{(0 \rightarrow h)} (\{\kappa_m\}) \rangle \langle \tilde{P}_{n}^{(0 \rightarrow h)} (\{\kappa_m\}) |. \]
For $n \to \infty$, 

$$p^{(h)}_{n,n} \sim \frac{\sqrt{2} s^{-h}}{\sqrt{\pi} (\sigma n)^{3/2}} (h + 1)^2 e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s+1}}$. Note: Effectively $h \lesssim O(\sqrt{n})$.

Entanglement entropy

$$S_A = -\sum_{h \geq 0} s^h p^{(h)}_{n,n} \ln p^{(h)}_{n,n}$$
For \( n \to \infty \),

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p_{n,n}^{(h)} \sim \frac{\sqrt{2} s^{-h}}{\sqrt{\pi} (\sigma n)^{3/2}} (h + 1)^2 e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]
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with \( \sigma \equiv \frac{\sqrt{s}}{2\sqrt{s+1}} \).

Note: Effectively \( h \lesssim O(\sqrt{n}) \).

Entanglement entropy

\[
S_A = - \sum_{h \geq 0} s^h p_{n,n}^{(h)} \ln p_{n,n}^{(h)}
\]

\[
= (2 \ln s) \sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi \sigma) + \gamma - \frac{1}{2} - \ln s
\]

up to terms vanishing as \( n \to \infty \). Grows as \( \sqrt{n} \).
Matching color \Rightarrow s^{-h} \text{ factor in } p_{n,n}^{(h)} \Rightarrow \text{crucial to } O(\sqrt{n}) \text{ behavior in } S_A
Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- Matching color $\Rightarrow s^{-h}$ factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in $S_A$

- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model)  
  [Salberger, Korepin 2016]
Comments

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- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. *(Fredkin model)* [Salberger, Korepin 2016]

- Deformation of models to achieve the volume law behavior ($S_A \propto n$)

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

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- Matching color \( \Rightarrow s^{-h} \) factor in \( p_{n,n}^{(h)} \)
  \( \Rightarrow \) crucial to \( O(\sqrt{n}) \) behavior in \( S_A \)

- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]

- Deformation of models to achieve the volume law behavior \( (S_A \propto n) \)


- Next, we consider extension of the model from a different point of view.
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SIS Motzkin model

Summary and discussion
Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots , k\}$.

- $a < b$ case: ‘up’ $\Leftrightarrow$

- $a > b$ case: ‘down’ $\Leftrightarrow$

- $a = b$ case: ‘flat’ $\Leftrightarrow$
SIS Motzkin model 1

- Change the spin d.o.f. as $|x_{a,b}\rangle$ with $a, b \in \{1, 2, \cdots, k\}$.

- $a < b$ case: ‘up’ $\iff$ \vdash \begin{array}{c} \scriptstyle b \\ \downarrow \end{array} \begin{array}{c} \scriptstyle a \\ \downarrow \end{array}$

- $a > b$ case: ‘down’ $\iff$ \vdash \begin{array}{c} \scriptstyle a \\ \downarrow \end{array} \begin{array}{c} \scriptstyle b \\ \downarrow \end{array}$

- $a = b$ case: ‘flat’ $\iff$ \vdash \begin{array}{c} \scriptstyle a \\ \downarrow \end{array} \begin{array}{c} \scriptstyle b \\ \downarrow \end{array}$

- We regard the configuration of adjacent sites $|(x_{a,b})_j\rangle |(x_{c,d})_{j+1}\rangle$ as a connected path for $b = c$.

- Analogous to the product rule of Symmetric Inverse Semigroup ($S_1^k$):

  \[ x_{a,b} \ast x_{c,d} = \delta_{b,c} x_{a,d} \]

  $a, b$: semigroup indices

- Inner product:

  \[ \langle x_{a,b} | x_{c,d} \rangle = \delta_{a,c} \delta_{b,d} \]

- Let us consider the $k = 3$ case.
Maximum height is lower than the original Motzkin case.
Hamiltonian $H_{S31Motzkin} = H_{bulk} + H_{bulk, disc} + H_{bdy}$

- $H_{bulk}$: local interactions corresponding to the following moves:

  (Down) \[ \begin{array}{c}
    a \rightarrow b \\
  \end{array} \sim \begin{array}{c}
    a \\ b \\
  \end{array} \] \quad (a > b)

  (Up) \[ \begin{array}{c}
    a \rightarrow b \\
  \end{array} \sim \begin{array}{c}
    a \\
    b \\
  \end{array} \] \quad (a < b)

  (Flat) \[ \begin{array}{c}
    a \rightarrow a \rightarrow a \\
  \end{array} \sim \begin{array}{c}
    a \\
    b \\
    a \\
  \end{array} \] \quad (a < b)

  (Wedge) \[ \begin{array}{c}
    3 \\
    1 \\
    3 \\
  \end{array} \sim \begin{array}{c}
    3 \\
    2 \\
    3 \\
  \end{array} \]
$H_{bulk, disc}$ lifts disconnected paths to excited states.

\[ H_{bulk, disc} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1; b\neq c}^{3} \Pi |(x_{a,b})_j, (x_{c,d})_{j+1} \rangle \]

$\Pi |\psi \rangle$: projector to $|\psi \rangle$
**SIS Motzkin model 4**

- $H_{bulk, \text{disc}}$ lifts disconnected paths to excited states.
  - $\Pi|\psi\rangle$: projector to $|\psi\rangle$

\[
H_{bulk, \text{disc}} = \sum_{j=1}^{2n-1} \sum_{a,b,c,d=1; b \neq c}^{3} \Pi|(x_{a,b})_{j}(x_{c,d})_{j+1}\rangle
\]

- $H_{bdy}$

\[
H_{bdy} = \sum_{a>b} \Pi|(x_{a,b})_{1}\rangle + \sum_{a<b} \Pi|(x_{a,b})_{2n}\rangle + \Pi|(x_{1,3})_{1}(x_{3,2})_{2}(x_{2,1})_{3}\rangle + \Pi|(x_{1,2})_{2n-2}(x_{2,3})_{2n-1}(x_{3,1})_{2n}\rangle
\]

The last 2 terms have no analog to the original Motzkin model.
Ground states correspond to connected paths starting at $(0, 0)$, ending at $(2n, 0)$ and not entering $y < 0$. 

\[ S^3 \] MWs
Ground states correspond to connected paths starting at (0, 0), ending at (2n, 0) and not entering \( y < 0 \).

The ground states have 5 fold degeneracy according to the initial and final semigroup indices:

- \((1, 1)\), \((1, 2)\), \((2, 1)\), \((2, 2)\) and \((3, 3)\) sectors

The \((3, 3)\) sector is trivial, consisting of only one path:

\[ x_{3,3} x_{3,3} \cdots x_{3,3} \]
SIS Motzkin model 5

- Ground states correspond to connected paths starting at (0, 0), ending at (2n, 0) and not entering y < 0.
- The ground states have 5 fold degeneracy according to the initial and final semigroup indices: (1, 1), (1, 2), (2, 1), (2, 2) and (3, 3) sectors.
  The (3, 3) sector is trivial, consisting of only one path: $x_{3,3}x_{3,3} \cdots x_{3,3}$.
- The number of paths can be obtained by recursion relations. For length-n paths from the semigroup index $a$ to $b$ ($P_{n,a \rightarrow b}$),

$$P_{n,1 \rightarrow 1} = x_{1,1}P_{n-1,1 \rightarrow 1} + x_{1,2} \sum_{i=1}^{n-2} P_{i,2 \rightarrow 2} x_{2,1} P_{n-2-i,1 \rightarrow 1}$$

$$+ x_{1,3} \sum_{i=1}^{n-2} P_{i,3 \rightarrow 3} x_{3,1} P_{n-2-i,1 \rightarrow 1}$$

$$+ x_{1,3} \sum_{i=1}^{n-2} P_{i,3 \rightarrow 3} x_{3,2} P_{n-2-i,2 \rightarrow 1}, \text{ etc.}$$
Result

- The entanglement entropies $S_{A,1\rightarrow 1}$, $S_{A,1\rightarrow 2}$, $S_{A,2\rightarrow 1}$ and $S_{A,2\rightarrow 2}$ take the same form as in the case of the Motzkin model.

Logarithmic violation of the area law

- The form of $p_n^{(h)} \sim \frac{(h+1)^2}{n^{3/2}} e^{-\text{const.} \frac{(h+1)^2}{n}}$ is universal.

- $S_{A,3\rightarrow 3} = 0$.

- Colored version can also be constructed ($S_2^3 \sim (S_1^3)^2$):

$$S_{A, a\rightarrow b} = (2 \ln 2) \sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln \frac{3}{2^{1/3}}$$

with $\sigma \equiv \frac{\sqrt{2}-1}{9\sqrt{2}}$ for $(a, b) = (1, 1), (1, 2), (2, 1), (2, 2)$. 
There are excited states corresponding to disconnected paths.

Example) One such path in $2n = 6$ case,

.png
There are excited states corresponding to disconnected paths. Example) One such path in $2n = 6$ case,

$$|P_{3,1\rightarrow1}\rangle \otimes |P_{3,2\rightarrow1}^{(1\rightarrow0)}\rangle$$

Each connected component has no entanglement with other components.
There are excited states corresponding to disconnected paths. Example: One such path in $2n = 6$ case,

$|\psi_{3,1\rightarrow 1}\rangle \otimes |P_{3,2\rightarrow 1}\rangle$

Each connected component has no entanglement with other components.

$\Rightarrow$ 2pt connected correlation functions of local operators belonging to separate connected components vanish.

$\Rightarrow$ Localization!
Introduction

Motzkin spin model

Colored Motzkin model

SIS Motzkin model

Summary and discussion
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We have extended the models by introducing additional d.o.f. based on Symmetric Inverse Semigroup.

Extension of the Fredkin (spin $1/2$) model [Padmanabhan, F.S., Korepin 2018]

As a feature of the extended models, Anderson-like localization occurs in excited states corresponding to disconnected paths.

No need of noise.
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Summary and discussion 2

- Continuum limit? (In particular, for colored case)
  
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Summary and discussion 2

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Thank you very much for your attention!