ANALYTICITY PROPERTIES OF SCATTERING AMPLITUDE IN HIGHER DIMENSIONAL FIELD THEORIES

Jnan Maharana

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INTRODUCTION

- It is recognized that theories which live in higher spacetime dimensions $D > 4$ play an important role in constructing unified theories of fundamental interactions. For example supersymmetric theories and supergravity theories are known to exist in higher dimensions. In the context of superstring theories, there have been a lot of activities not only to explore various types of string theories but also it is argued that experimental signatures of string theories might be observed at accelerator energies in the large compactification scale scenario. Thus a considerable attention has been focused to study particle productions and to explore what will be signatures of models with large extra dimensions.

INTRODUCTION

- It is recognized that theories which live in higher spacetime dimensions $D > 4$ play an important role in constructing unified theories of fundamental interactions. For example supersymmetric theories and supergravity theories are known to exist in higher dimensions. In the context of superstring theories, there have been a lot of activities not only to explore various types of string theories but also it is argued that experimental signatures of string theories might be observed at accelerator energies in the large compactification scale scenario. Thus a considerable attention has been focused to study particle productions and to explore what will be signatures of models with large extra dimensions.
- There were attempts to derive Froissart-like bounds for higher dimensional theories under two reasonable assumptions: (i) The amplitude is polynomially bounded in s (the c.m. energy squared) and (ii) the corresponding partial wave expansion of the amplitude converges inside a Lehmann-Martin ellipse. These assumptions were not proved.

- Moreover, the corresponding bound on the total cross section, σ_t will lead to only Greenberg-Low like bound which is weaker than Froissart bound. Thus further work is required.

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- Let us recapitulate the known rigorous results for $D = 4$ theories derived from axiomatic field theory. Froissart bound

$$\sigma_t \leq \frac{4\pi}{t_0} ln^2 \frac{s}{s_0}$$

t_0 is a parameter derived from first principle ($t_0 = 4m_\pi^2$) for most hadronic processes. s_0 is energy scale to make argument of *log* dimensionless and cannot be determined from axiomatic field theoretic frame work.

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- The bound is arrived at from the following ingredients which can be derived from axiomatic field theory.

- 1. Analyticity of scattering amplitude, $F(s,t)$, in the cut s-plane. $|F(s,t)| \leq |s|^N, N \in Z$, and it is polynomially bounded and satisfies dispersion relation.

  2. Crossing symmetry.

  3. Convergence of partial wave amplitude inside Lehmann-Martin ellipse.

  4. Unitarity. The partial wave amplitudes satisfy positivity condition

  $$0 \leq |f_l(s)|^2 \leq Im\ f_l(s)$$

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- What are our assumptions, results and how do we intend to accomplish them for $D > 4$ field theories? We adopt the axioms of LSZ such as existence of a massive scalar field theory of mass m. (i) Existence of a Hilbert space and observables are self adjoint operaors. (ii) Unitary representation of Poncaré group exists. (iii) The existence of unique Lorentz invariant vacuum. (iv) Local operators satisfy constraint of micro causality: $[O_1(x), O_2(y)] = 0$ for $(x-y)^2 < 0$. (v) Existence of $\phi(x)^{in,out}(x)$ and the condition of interpolation: $t \to \pm\infty,\ \phi(x) \to \phi(x)^{out,in}(x)$. (vi) We implement LSZ reduction technique to derive amplitudes.

HIGHER DIMENSIONAL FIELD THEORIES

- Results

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- The Froissart Bound for D-dimensional spacetime

$$\sigma_{total}(s) \leq B(\lambda)\left(\frac{1}{2\sqrt{4m^2-\epsilon}}\right)^{D-2}(ln\ s)^{D-2}$$

  where $B(\lambda)$ is a constant depending on $\lambda = \frac{1}{2}(D-3)$.

- (i) Existence of Lehmann-Martin ellipse. (ii)Derivation of generalized Jost-Lehmann-Dyson representation for the retarded function $F_R(q) = \int d^D z e^{iq.z}\theta(z_0) < Q_f|Rj(z/2)j(-z/2)|Q_i >$, Q_i, Q_f are arbitrary fixed states and R is the retarded product. $j(x)$ is the source current for the interacting field $\phi(x)$.

- (iii)Proof of the Martin's theorem: Inside the Lehmann-Martin ellipse there is a domain, $\mathcal{R}$, independent of s inside which the absorptive parts $A_s(s', t)$ and $A_u(u', t)$ satisfy certain positivity constraints. Moreover, the amplitude is analytic in the topological domain $D_s \otimes D_t$ The domain is defined by $|t| < \mathcal{R}$ and s outside the cut $s_{thr} + \eta = 4m^2 + \eta, \eta > 0$.

- In view of the results derived, as stated above, the analyticity properties are derived from LSZ axiomatic approach
- Consider following three mathematically defined objects where $|Q_f>$ *and* $|Q_i>$ are states carrying momenta Q_f, Q_i. We hold Q_i and Q_f fixed here in defining the matrix elements.

$$F_R(q) = \int d^4 z e^{iq.z} \theta(z_0) < Q_f|[j_l(z/2), j_m(-z/2)]|Q_i >$$

$$F_A(q) = -\int d^4 z e^{iq.z} \theta(-z_0) < Q_f|[j_l(z/2), j_m(-z/2)]|Q_i >$$

$$F_C(q) = \int d^4 z e^{iq.z} < Q_f|[j_l(z/2), j_m(-z/2)|Q_i >$$

It is easy to see that

$$F_C(q) = F_R(q) - F_A(q)$$

$F_R(q) = F_A(q)$ for $F_C(q) = 0$. The Fourier transform $\tilde{F}_C(z) = 0$ when $z^2 < 0$ due to microcausality.

- We open up the commutators and get difference of two matrix elements which are product of the two currents where the orders are $j(z/2)j(-z/2)$ and $j(-z/2)j(z/2)$ and define

$$F_C(q) = A_s(q) - A_u(q)$$

where

$$A_s(q) = \int d^4z e^{iq.z} \left[< Q_f | j_l(z/2) j_m(-z/2) | Q_i > \right]$$

and

$$A_u(q) = \int d^4z e^{iq.z} \left[< Q_f | j_m(-z/2) j_l(z/2) | Q_i > \right]$$

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Now introduce a complete set of states between the product of currents and get a spectral representation.

- $$A_s = \int d^4z e^{iq.z} \sum < Q_f | j_l(z/2) | p_n >< p_n | j_l(-z/2) | Q_i >$$

- The spectral representation is constrained since $|p_n>$ are physical states. $F_C(q) = 0$ only when A_s and A_u vanish simultaneously for some values of q.

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- A_s and A_u vanish simultaneously for real q which is in the unphysical region. This has very important implication. It is argued that A_s and A_u are analytic continuation of a function of several complex variables such that for certain real values they coincide - this is the *coinsidence region*. It is a formidable task to prove this *edge of the wedge theorem* result which is essentially a proof of crossing symmetry - that is the s-channel and u-channel amplitudes are really analyitically continued from the *coincidence* region. In case of $D = 4$ this result can be proved very rigorously for n-point function. However, for our case $D > 4$, the analog of this theorem (and hence crossing) can be only be proved for the 4-point amplitude.

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- Thus we can write a dispersion relation (up to subtractions) for the 4-point amplitude in the case of *Higher Dimensional Field Theories*.

- How the problem is different for $D > 4$ in contrast to $D = 4$?

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- The partial wave expansion is on a basis for $SO(D-1)$ symmetry and the amplitude is expanded in terms of Gegenbauer polynimials (instead of Legendre polynomial as in $D = 4$ case). Thus

$$F^{\lambda} = A_1 s^{-\lambda+1/2} \sum_{l=0}^{\infty} (l+\lambda) f_l^{\lambda} C_l^{\lambda}(cos\theta)$$

where $f_l^{\lambda}(s)$ is the partial wave amplitude, $C_l^{\lambda}(cos\theta)$ are the Gegebauer polynomials as a function of *cosine* of the c.m. scattering angle θ and $\lambda = \frac{1}{2}(D-3)$. The s-dependent prefactor is introduced so that the partial wave amplitudes satisfy unitarity and positivity condition

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- $$0 \leq |f_l^{\lambda}|^2 \leq \mathrm{Im} f_l^{\lambda} \leq 1$$

- We have proved the Jost-Lehmann-Dyson theorem for the absorptive part of 4-point function. Subsequently, the existence of small Lehmann Ellipse and Large Lehmann Ellipse was proved. This is not adequate. The proof of Martin's theorem (for $D = 4$) requires that the absorptive part and its t-derivatives satisfy certain positivity properties. Such inequalities were derived for $D > 4$ case by exploiting certain recursion relations of Gegenbauer polynomials and their derivatives.

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- We proved that the amplitude $F^{\lambda}(s,t)$ in analytic in the cut s-plane and within the Lehmann ellipse for $D > 4$. Finally, the analog of Jin-Martin bound that $F^{\lambda}(s,t)$ needs *only two subtractions* was proved. Thus it paved the way to prove the general Froissart bound .

  $\sigma_{total}(s) \leq B(\lambda)\left(\frac{1}{2\sqrt{4m^2-\epsilon}}\right)^{D-2}(ln\ s)^{D-2}$

  It is derived from rigorous Axiomatic field theory unlike earlier results.

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- Scattering of neutral massive scalar was considered in the LSZ formalism. Due to the temperedness of the amplitude it is polynomially bounded in s. We have proved the generalized J-L-D theorem. Next we have showed the existence of Lehmann ellipses for $D > 4$ theories.

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- Scattering of neutral massive scalar was considered in the LSZ formalism. Due to the temperedness of the amplitude it is polynomially bounded in s. We have proved the generalized J-L-D theorem. Next we have showed the existence of Lehmann ellipses for $D > 4$ theories.
- A partial wave expansion exists with Gegenbauer polynomial as basis functions. The positivity properties of absorptive amplitude and its t-derivatived have been proved. The analog of Jin-Martin bound was proved and the amplitude needs no more than two subtractions in s. Finally, analog of Froissart bound for $D > 4$ is proved rigorously from the Axioms of LSZ.

- If indeed there are extra dimensions with large compactification radius (Expt limit is $R > 600\ GeV$) then there are implications. If we find violation of Froissart bound experimentally (say at LHC energies) it might be due to decompactification of extra dimensions. We should explore this avenue before jumping into conclusion that either locality or causality or unitarity (*or all of them*) are violated.

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- **THANK YOU**