SSB in tensor theories and matrices

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Tensor models appear in the context of

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Tensor and matrix models interest

Matrix models
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- Quantum gravity in $d = 2$.
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Tensor models appear in the context of
- Entanglement.
- Quantum gravity description $d > 2$.
- SYK and holography. Recently SYK has been linked to tensor models [Witten’16]. Holography ($d > 2$)
Definition of color TM
Tensors with no additional symmetry assumed
Definition of color TM

Tensors with no additional symmetry assumed

\[ \Phi = \Phi_{i_1 i_2 \ldots i_d} e^{i_1} \otimes \cdots \otimes e^{i_d}, \quad e^{i_k} \in \mathbb{C}^{N_k}, \quad i_k = 1, \ldots, N_k. \]
Tensors with no additional symmetry assumed

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Under the action of \( G_d \equiv U(N_1) \otimes \cdots \otimes U(N_d) \)

\[ \Phi_{j_1 j_2 \ldots j_d} = \sum_{i_1, \ldots, i_d} (g_1)_{j_1}^{i_1} \cdots (g_d)_{j_d}^{i_d} \Phi_{i_1 \ldots i_d} \]

\[ \overline{\Phi}_{j_1 j_2 \ldots j_d} = \sum_{i_1, \ldots, i_d} (\overline{g}_1)_{j_1}^{i_1} \cdots (\overline{g}_d)_{j_d}^{i_d} \overline{\Phi}_{i_1 \ldots i_d}. \]
Definition of color TM

Tensors with no additional symmetry assumed

\[ \Phi = \Phi_{i_1i_2...i_d} e^{i_1} \otimes ... \otimes e^{i_d}, \quad e^{i_k} \in \mathbb{C}^{N_k}, \quad i_k = 1, \ldots, N_k. \]

Under the action of \( G_d \equiv U(N_1) \otimes \cdots \otimes U(N_d) \)

\[ \Phi_{j_1j_2...j_d} = \sum_{i_1,...,i_d} (g_1)_{j_1}^{i_1} \cdots (g_d)_{j_d}^{i_d} \Phi_{i_1...i_d} \]
\[ \overline{\Phi}_{j_1j_2...j_d} = \sum_{i_1,...,i_d} (\overline{g_1})_{j_1}^{i_1} \cdots (\overline{g_d})_{j_d}^{i_d} \overline{\Phi}_{i_1...i_d}. \]

The action of the free theory \( S = \Phi_{i_1i_2...i_d} \overline{\Phi}^{i_1i_2...i_d}. \)
Counting tensor invariants problem
Invariants are constructed by contracting indices of $\Phi$ and $\Phi$. 

Examples:

- $n = 1, d = 3$ $\rightarrow \Phi_{i_1} \Phi_{i_2} \Phi_{i_3}$
- $n = 2, d = 3$ $\rightarrow \{ \Phi_{i_1}^1 \Phi_{i_2}^1 \Phi_{i_3}^1, \Phi_{i_2}^1 \Phi_{i_3}^1 \Phi_{i_3}^2, \Phi_{i_1}^1 \Phi_{i_2}^2 \Phi_{i_3}^3, \Phi_{i_1}^2 \Phi_{i_2}^1 \Phi_{i_3}^3 \}$
- $n = 3$ $\rightarrow 11$ invariants.
- $n = 4$ $\rightarrow 43$ invariants.
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Invariants are constructed by contracting indices of $\Phi$ and $\Phi^\dagger$. Examples:

$\triangleright \ n = 1, \ d = 3 \rightarrow \Phi_{i_1i_2i_3} \Phi^{i_1i_2i_3}$
Invariants are constructed by contracting indices of $\Phi$ and $\Phi$. Examples:

- $n = 1, \ d = 3 \rightarrow \Phi_{i_1 i_2 i_3} \Phi_{i_1 i_2 i_3}^\dagger$
- $n = 2, \ d = 3$

$$\rightarrow \left\{ \Phi_{i_1 i_2 i_3}^{i_1} \Phi_{i_1 i_2 i_3}^{i_2}, \Phi_{i_1 i_2 i_3}^{i_3} \Phi_{i_1 i_2 i_3}^{i_2} \Phi_{i_1 i_2 i_3}^{i_3}, \Phi_{i_1 i_2 i_3}^{i_1} \Phi_{i_1 i_2 i_3}^{i_2} \Phi_{i_1 i_2 i_3}^{i_3} \Phi_{i_1 i_2 i_3}^{i_2} \Phi_{i_1 i_2 i_3}^{i_3}, \Phi_{i_1 i_2 i_3}^{i_1} \Phi_{i_1 i_2 i_3}^{i_2} \Phi_{i_1 i_2 i_3}^{i_3} \Phi_{i_1 i_2 i_3}^{i_2} \Phi_{i_1 i_2 i_3}^{i_3} \right\}$$
Counting tensor invariants problem

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$$
\rightarrow \left\{ \Phi_{i_1^1 i_2^1 i_3^1} \Phi_{i_1^2 i_2^2 i_3^2} \Phi^{i_1^1 i_2^1 i_3^1} \Phi^{i_1^2 i_2^2 i_3^2}, \Phi_{i_1^1 i_2^1 i_3^1} \Phi_{i_1^2 i_2^2 i_3^2} \Phi^{i_1^1 i_2^1 i_3^1} \Phi^{i_1^2 i_2^2 i_3^1}, \\
\Phi_{i_1^1 i_2^1 i_3^1} \Phi_{i_1^2 i_2^2 i_3^2} \Phi^{i_1^1 i_2^1 i_3^1} \Phi^{i_1^2 i_2^2 i_3^2}, \Phi_{i_1^1 i_2^1 i_3^1} \Phi_{i_1^2 i_2^2 i_3^2} \Phi^{i_1^1 i_2^2 i_3^1} \Phi^{i_1^2 i_2^1 i_3^1} \right\}
$$

- $n = 3 \rightarrow 11$ invariants.
Counting tensor invariants problem

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$$\rightarrow \left\{ \Phi_{i_1 i_2 i_3} \Phi_{i_1 i_2 i_3}^{i_1 i_2 i_3}, \Phi_{i_1 i_2 i_3} \Phi_{i_1 i_2 i_3}^{i_1 i_2 i_3}, \Phi_{i_1 i_2 i_3} \Phi_{i_1 i_2 i_3}^{i_1 i_2 i_3}, \Phi_{i_1 i_2 i_3} \Phi_{i_1 i_2 i_3}^{i_1 i_2 i_3} \right\}$$

- $n = 3 \rightarrow 11$ invariants.
- $n = 4 \rightarrow 43$ invariants.
Tensor and matrix counting of invariants

By means of representation theory (and some work):

\[ \text{dim} \{ \text{O} \}_{\text{G}^{d-\text{Inv}}_n} = \sum |\mu_1|, \ldots, |\mu_d| = n \prod (\mu_k) \leq N_k \]

\( g_{\mu_1, \ldots, \mu_d} \) are the Kronecker coefficients. Branching coefficients in the restriction \( U(N_1) \times \cdots \times U(N_d) \to U(N_1) \times \cdots \times U(N_d) \).

For multimatrix models we have

\[ \text{dim} \{ \text{O} \}_{\text{U}(N) - \text{Inv}_n} = \sum \mu \vdash n \nu_1 \vdash n_i (c_{\mu \nu_1}, \ldots, \nu_d) \]

\( n = n_1 + \cdots + n_d \)

\( c_{\mu \nu_1}, \ldots, \nu_d \) are LR numbers. Branching coefficients in the restriction \( S_n \to S_{n_1} \times \cdots \times S_{n_d} \).
Tensor and matrix counting of invariants

By means of representation theory (and some work):

\[ \dim \{ \mathcal{O}_{n}^{G_d - \text{Inv}} \} = \sum_{|\mu_1|, \ldots, |\mu_d| = n, l(\mu_k) \leq N_k} g_{\mu_1, \ldots, \mu_d}^2. \]
Tensor and matrix counting of invariants

By means of representation theory (and some work):

\[ \dim \{ \mathcal{O}^{G_d-\text{Inv}}_n \} = \sum_{|\mu_1|, \ldots, |\mu_d|=n, l(\mu_k) \leq N_k} g^2_{\mu_1, \ldots, \mu_d}. \]

\( g_{\mu_1, \ldots, \mu_d} \) are the **Kronecker coefficients**. Branching coefficients in the restriction \( U(N_1 \cdots N_d) \to U(N_1) \times \cdots \times U(N_d) \).
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\[ \dim \{ \mathcal{O}_{n}^{G_{d}^{-\text{Inv}}} \} = \sum_{\mu_1, \ldots, \mu_d} g_{\mu_1, \ldots, \mu_d}^2, \]

where \( g_{\mu_1, \ldots, \mu_d} \) are the Kronecker coefficients. Branching coefficients in the restriction \( U(N_1 \cdots N_d) \to U(N_1) \times \cdots \times U(N_d) \).

For multimatrix models we have

\[ \dim \{ \mathcal{O}_{n}^{U(N)^{-\text{Inv}}} \} = \sum_{\mu_1^n, \ldots, \nu_d^n} (c_{\nu_1, \ldots, \nu_d}^{\mu})^2, \quad n = n_1 + \cdots + n_d \]
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By means of representation theory (and some work):

$$\dim\{\mathcal{O}^{G_d-\text{Inv}}_n\} = \sum_{|\mu_1|,\ldots,|\mu_d|=n} g_{\mu_1,\ldots,\mu_d}^2 \cdot$$

$$\prod_{k=1}^d \mu_k \leq N_k$$

\(g_{\mu_1,\ldots,\mu_d}\) are the **Kronecker coefficients**. Branching coefficients in the restriction \(U(N_1 \cdots N_d) \to U(N_1) \times \cdots \times U(N_d)\).

For multimatrix models we have

$$\dim\{\mathcal{O}^{U(N)-\text{Inv}}_n\} = \sum_{\mu \vdash n, \nu_1 \vdash n_1, \ldots, \nu_d \vdash n_d} (c^\mu_{\nu_1,\ldots,\nu_d})^2, \quad n = n_1 + \cdots + n_d$$

\(c^\mu_{\nu_1,\ldots,\nu_d}\) are **LR numbers**. Branching coefficients in the restriction \(S_n \to S_{n_1} \times \cdots \times S_{n_d}\)
Matrix and tensor models in the hook sector
Kronecker coefficients do not have yet a general combinatorial
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Kronecker coefficients do not have yet a general combinatorial formula, but using [Liu’17] and [PD’18] we have

\[ g_{\mu(r)\nu\lambda} = \sum_{\gamma |-r} c^\nu_{\rho\gamma} c^\lambda_{\rho\gamma'}, \quad \mu(r) = \begin{array}{c} \text{table} \\ \text{of} \\ \text{size} \\ r+1 \end{array}, \]
Kronecker coefficients do not have yet a general combinatorial formula, but using [Liu’17] and [PD’18] we have

\[ g_{\mu(r)\nu\lambda} = \sum_{\gamma \vdash r} c^\nu_{\rho\gamma} c^\lambda_{\rho\gamma'}, \quad \mu(r) = \underbrace{\gamma|_{n-r}}_{r+1}, \]

At least in the hook sector and for \( d = 3 \) tensor models there is a non-trivial relation between the spectra of matrix and tensor models.
Effective (multi-)matrix theories from tensor theories
Effective (multi-)matrix theories from tensor theories

- Promote the model to QFT: $\Phi_{i_1,\ldots,i_d} \rightarrow \Phi_{i_1,\ldots,i_d}(x)$. 

- Take the same rank for the groups $U(N_1) \times \cdots \times U(N_d) \rightarrow U(N) \times d$.

- The original theory $L(\Phi)$ is invariant under $U(N) \times d$ but we will break the symmetry to $\text{Diag}[U(N)]$.

\[
\int d\Phi d\Phi \exp(i\int d^4x \{L(\Phi(x))\}) \rightarrow \int d\Phi d\Phi \exp(i\int d^4x \{L(\Phi(x)) + i\epsilon \Phi_s(x) \Phi_s(x)\})
\]

\[
\Phi_s(x) \Phi_s(x) = \frac{1}{d!} \sum_{\sigma \in S_d} \Phi_{i_{\sigma}(1)},\ldots,\Phi_{i_{\sigma}(d)}(x) \Phi_{i_1,\ldots,i_d}(x)
\]
Effective (multi-)matrix theories from tensor theories

- Promote the model to QFT: \( \Phi_{i_1, \ldots, i_d} \rightarrow \Phi_{i_1, \ldots, i_d}(x) \).
- Take the same rank for the groups
  \( \mathbb{U}(N_1) \times \cdots \times \mathbb{U}(N_d) \rightarrow \mathbb{U}(N)^\times^d \).
Effective (multi-)matrix theories from tensor theories

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- Take the same rank for the groups $U(N_1) \times \cdots \times U(N_d) \rightarrow U(N)^\times d$.
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$\int d\Phi d\Phi \exp(i \int d^4x \{ \mathcal{L}(\Phi(x)) + i \epsilon \Phi_s(x) \Phi_s(x) \})$
Effective (multi-)matrix theories from tensor theories

- Promote the model to QFT: $\Phi_{i_1,\ldots,i_d} \rightarrow \Phi_{i_1,\ldots,i_d}(x)$.
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$$
\int d\Phi d\overline{\Phi} \exp \left( i \int d^4x \{ \mathcal{L}(\Phi(x)) \} \right)
$$

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- $\Phi^s(x)\Phi^s(x) = \frac{1}{d!} \sum_{\sigma \in S_d} \Phi_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}(x)\Phi^{i_1,\ldots,i_d}(x)$
How the symmetric term transform
How the symmetric term transform

\[
\Phi_{j_1j_2...j_d} = \sum_{i_1,...,i_d} (g_1)_{j_1}^{i_1} \cdot (g_d)_{j_d}^{i_d} \Phi_{i_1...i_d}, \quad g_k \in U(N)
\]

\[
\Phi^s_{j_1j_2...j_d} = \frac{1}{d!} \sum_{i_1,...,i_d} g_{j_1}^{i_1} \cdots g_{j_d}^{i_d} \Phi_{i_{\sigma(1)}...i_{\sigma(d)}}, \quad g \in U(N)
\]
SSB and degrees of freedom

▶ \[ G = U(N^2) \times d \rightarrow H = \text{Diag}[U(N^2)]. \]

▶ # of Goldstone bosons
\[ N_{GB} = (d^2 - 1)N^2. \]

▶ Each collection of \( N^2 \) GB is seen to transform in an irrep of \( \text{Diag}[U(N^2)] \).

\[ B_a(x) = i(\Phi^s(x)^T a \Phi^s(x) - \Phi^s(x)^T k a \Phi^s(x)), \quad a = 1, \ldots, N^2. \]

▶ So they group into \( d - 1 \) multiplets of \( N^2 \) elements each.

▶ The multiplets organize into matrices transforming in the adjoint:
\[ Z_{ij}(x) = \sum_a B_a(x)(T_a)_{ij}. \]
SSB and degrees of freedom

➤ SSB: \( G = U(N)^{\times d} \rightarrow H = \text{Diag}[U(N)]. \)
SSB and degrees of freedom

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- # of Goldstone bosons

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N_G - N_H = (d - 1)N^2
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SSB and degrees of freedom

- SSB: $G = U(N)^{\times d} \rightarrow H = \text{Diag}[U(N)]$.
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$$N_G - N_H = (d - 1)N^2$$

- Each collection of $N^2$ GB is seen to transform in an irrep of $\text{Diag}[U(N)]$.

$$B_a(x) = i \left( \Phi^s(x) T_a \Phi^s(x) - \Phi^s(x) T_a^k \Phi^s(x) \right), \quad a = 1, \ldots, N^2.$$ 
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SSB and degrees of freedom

- **SSB:** \( G = U(N)^{\times d} \to H = \text{Diag}[U(N)]. \)
- # of Goldstone bosons

\[ N_G - N_H = (d - 1)N^2 \]

- Each collection of \( N^2 \) GB is seen to transform in an irrep of \( \text{Diag}[U(N)]. \)

\[ B_a(x) = i\left( \Phi_s^s(x) T_a \Phi^s(x) - \Phi^s(x) T^k_a \Phi^s(x) \right), \quad a = 1, \ldots, N^2. \]

So they group into \( d - 1 \) multiplets of \( N^2 \) elements each.

- The multiplets organize into matrices transforming in the adjoint:

\[ Z^i_j(x) = \sum_a B_a(x)(T_a)^i_j. \]
Summary of SSB and applications
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\[ U(N)^\times d \rightarrow \Phi_{i_1,...,i_d} \] (melonic diagrams, solvable, but unclear physics)
Summary of SSB and applications

\[ U(N)^\times d \quad \rightarrow \quad \Phi_{i_1,\ldots,i_d} \quad \text{(melonic diagrams, solvable, but unclear physics)} \]

\[ \text{Diag}[U(N)] \quad \rightarrow \quad \Phi_{i_1,\ldots,i_d}^s(x) + \{(Z_1)^i_j(x), \ldots, (Z_{d-1})^i_j(x)\} \]

+ Diag\[U(N)] – singlets

(non-solvable, holographic interpretation)
Thanks!