Non-orientable Surfaces and Electric-magnetic Duality

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based on arXiv: 1804.11343 [hep-th]

I. Topological sectors of 4d gauge theory

gauge group G : compact semisimple Lie group

$\pi_1(G)$ is a finite Abelian group.

spacetime X : orientable 4-manifold

G-bundles over X are topologically classified by

$H^4(X, \pi_3(G)) \ni$ instanton number

$H^2(X, \pi_1(G)) \ni$ discrete flux 't Hooft (1978) suppose $X^4 = T^1 \times Y^3$ is a space-time splitting

$$H^2(X, \pi_1(G)) = \underset{\substack{\Downarrow \\ m}}{H^2(Y, \pi_1(G))} \underset{+}{\oplus} \underset{\substack{\Downarrow \\ a}}{H^1(Y, \pi_1(G))}.$$

$m \in H^2(Y, \pi_1(G))$ classifies the topology of the G-bundle over a time slice Y^3.

It is the discrete magnetic flux.

$Z(G)$, the centre of G, is a finite Abelian group.

$H^1(Y, Z(G))$ is a discrete symmetry of the theory.
$g \in H^1(Y, Z(G))$ modifies the holonomy of A along a loop γ in Y by $g([\gamma]) \in Z(G)$.

The quantum theory consists of sectors labelled by $e \in H^1(Y, Z(G))^\vee$. For A = Abel. group, $A^\vee = \mathrm{Hom}(A, U(1))$.

These are the discrete electric fluxes

When G is exchanged with the Langlands dual ${}^L G$

(or the Goddard-Nuyts-Olive magnetic group)

we have $H^1(Y, Z(G))^\vee \cong H^2(Y, \pi_1({}^L G))$, $H^2(Y, \pi_1(G)) \cong H^1(Y, Z({}^L G))^\vee$.

$\begin{pmatrix} \text{elec fluxes} \\ \text{in } G\text{-theory} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{mag fluxes} \\ \text{in } {}^L G\text{-theory} \end{pmatrix}$, vice versa

This is consistent with the electric-magnetic duality (or S-duality, Montonen-Olive duality).

II. Reduction to 2d along orientable surfaces

Kapustin-Witten (2006)

a twisted $N=4$ Yang-Mills

$\Downarrow$

sigma-model on Σ with target space $\mathcal{M}_H(C, G)$

$$X^4 = \Sigma^2 \times C^2$$

(large) (small)

$\mathcal{M}_H(C, G)$ = Hitchin's moduli space (hyper Kähler)

I, J, K: $\omega_I, \omega_J, \omega_K$

Assume $\Sigma^2 = T^1 \times S^1$. Then $X^4 = T^1 \times Y^3$, where $Y^3 = S^1 \times C^2$.

4d $\left\{ \begin{array}{l} H^2(Y, \pi_1(G)) = H^2(C, \pi_1(G)) \oplus H^1(C, \pi_1(G)) \\ \quad m \quad = \quad m_0 \quad + \quad m_1 \\ H^1(Y, Z(G))^\vee = H^1(C, Z(G))^\vee \oplus H^0(C, Z(G))^\vee \\ \quad e \quad = \quad e_1 \quad + \quad e_0 \end{array} \right.$

2d $\left\{ \begin{array}{ll} m_0 \in \pi_0(\mathcal{M}_H(C, G)) & \text{connected components} \\ m_1 \in \pi_1(\mathcal{M}_H(C, G)) & \text{winding} \end{array} \right.$

2d (cont'd) $\begin{cases} e_1 \text{ are the momenta of the discrete translations } H^1(C, Z(G)) \text{ on } \mathcal{M}_H(C,G) \\ e_0 \text{ label flat } B\text{-fields on } \mathcal{M}_H(C,G). \end{cases}$

4d S-duality $\Rightarrow$ 2d mirror symmetry

$\mathcal{M}_H(C,G)$ and $\mathcal{M}_H(C,{}^LG)$ are mirrors

Hausel-Thaddeus (03), Donagi-Pantev (12)

III. Reduction along possibly non-orientable surfaces

$\mathbb{Z}_2 \curvearrowright C$ orientable surface

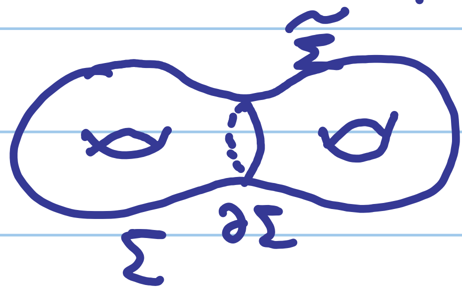
$\pi \downarrow$ 2:1 cover

C' non-orientable

e.g. $C = S^2$, $C' = \mathbb{RP}^2$

$\mathbb{Z}_2 \curvearrowright \tilde{\Sigma}$ orientable surface

has fixed points



$\tilde{\Sigma}/\mathbb{Z}_2 = \Sigma$

with $\partial\Sigma$.

$X^4 = C \times_{\mathbb{Z}_2} \tilde{\Sigma}$ i.e. $(C \times \tilde{\Sigma})/\mathbb{Z}_2$

X is a smooth, orientable 4-manifold.

$$X^4 = \tilde{\Sigma}^2 \times_{\mathbb{Z}_2} C^2 \xrightarrow{\pi_X} \Sigma \ni \sigma$$

$$\pi_X^{-1}(\sigma) = \begin{cases} C & \text{if } \sigma \in \text{interior of } \Sigma \\ C' & \text{if } \sigma \in \partial\Sigma . \end{cases}$$

If $\Sigma, \tilde{\Sigma}$ are large, C', C are small, gauge theory on X reduces to

$$\begin{array}{ccc} \partial\Sigma & \xrightarrow{\phi'} & \mathcal{M}_H(C', G) \\ \downarrow & & \downarrow p \\ \Sigma & \xrightarrow{\phi} & \mathcal{M}_H(C, G) \end{array}$$

Hitchin's moduli space for a non-orient. surface.

Ho-Wilkin-W. arXiv: 1211.0746

Description of the map $p : \mathcal{M}_H(C', G) \to \mathcal{M}_H(C, G)$.

- by pulling back fields from C' to C.
- the image $N \subset \mathcal{M}_H(C, G)^{\mathbb{Z}_2} \subset \mathcal{M}_H(C, G)$

  holomorphic in J

  Lagrangian in ω_I, ω_K.

- $$\begin{array}{c} \mathcal{M}_H(C', G) \\ \downarrow \\ N \end{array}$$ is a $Z(G)_{[2]}$-sheeted finite cover.

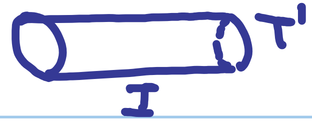
For A = Abelian group, $A_{[2]} = \{ a \in A : 2a = 0 \}$.

For $e_2 \in Z(G)^\vee_{[2]} = \mathrm{Hom}(Z(G)_{[2]}, U(1))$, we get a flat line bundle $\ell^{e_2} = \mathcal{M}_H(C', G) \times_{e_2} \mathbb{C}$ over $\mathcal{N}$.

$$\mathcal{M}_H(C', G) = \coprod_{m_2 \in \pi_1 G / 2\pi_1 G} \mathcal{M}_H^{m_2}(C', G)$$ decomposition into connected components.

$\mathcal{N}^{m_2} = \varphi(\mathcal{M}_H^{m_2})$, $\quad \mathcal{B}^{e_2, m_2} = (\mathcal{N}^{m_2}, \ell^{e_2})$.

branes of type (A, B, A)

Consider $\tilde{\Sigma} = T^1 \times \overset{\mathbb{Z}_2 \circlearrowright}{S^1}$. $\Sigma = \tilde{\Sigma}/\mathbb{Z}_2 =$ 

cylinder

relative winding $\quad m_1 \in \dfrac{H^1(C, \pi_1 G)}{\pi^* H^1(C', \pi_1 G)}$

brane reduces the $H^1(C, Z(G))$ symmetry to $\pi^* H^1(C', Z(G))$.

discrete momenta $\quad e_1 \in \left(\pi^* H^1(C', Z(G))\right)^\vee$.

To summarise, the sectors of the 2d theory are labelled by $\quad m_1, e_1 ; m_2, e_2$.

Calculations from 4d gauge theory: exact sequences

$$0 \to \frac{H^1(C, \pi_1 G)}{\pi^* H^1(C', \pi_1 G)} \longrightarrow H^2(Y, \pi_1 G) \longrightarrow (\pi_1 G / 2\pi_1 G)^{\oplus 2} \longrightarrow 0$$

$$m_1 \qquad\qquad m \qquad\qquad (m_2^L, m_2^R)$$

$$0 \to (\pi^* H^1(C', Z(G)))^\vee \longrightarrow H^1(Y, Z(G))^\vee \longrightarrow (Z(G)^\vee_{[2]})^{\oplus 2} \longrightarrow 0$$

$$e_1 \qquad\qquad e \qquad\qquad (e_2^L, e_2^R)$$

Perfect matching!

Mirror Symmetry $\qquad m_1 \longleftrightarrow e_1 \,, \quad m_2 \longleftrightarrow e_2$

For example,
$$\frac{H^1(C, \pi_1 G)}{\pi^* H^1(C', \pi_1 G)} \cong \left(\pi^* H^1(C', Z({}^L G))\right)^\vee$$
$$\pi_1 G / 2\pi_1 G \cong Z({}^L G)^\vee_{[2]} \,, \quad \text{etc.}$$

B^{e_2, m_2} and ${}^L B^{m_2, e_2}$ are related by a fibrewise Fourier-Mukai transform

$$\begin{tikzcd}
\mathcal{M}_H(C, G) \arrow[dr, "h"'] & & \mathcal{M}_H(C, {}^L G) \arrow[dl, "{}^L h"] \\
& \mathbb{B} &
\end{tikzcd}$$

Unfortunately, this is just an approximation.

In reality, we may not be able to fix m and e simultaneously in 4d gauge theory.

$g \in H^1(Y, Z(G))$ changes $m \in H^2(Y, \pi_1 G)$ by $\delta(g)$, where δ is the connecting homomorphism in alg. topology

But things can be fixed. See arXiv: 1804.11343 and W. Theor. Math. Phys. (2015)

Thank you very much!