Probing Reheating Temperature with primordial GW: its application to Gauss-Bonnet inflation

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We have hundreds of inflation models that are being consistent with the observational data.

\[ n_s = 0.9655 \pm 0.0062 \text{ (68\% CL, Planck TT+lowP)} \]

The simplest scenario is based upon a single field, which is minimally coupled to a gravity, with a flat potential;

\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \]
In light of both the current and future observations, extended models of inflation seem to be more promising!

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S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{1}{2} \xi(\phi) R_{GB}^2 \right],
\]

\[
R_{GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2
\]

In the early universe, approaching the Planck era, it is quite natural to consider corrections like this.

Thus, we are interested in understanding the effects of this additional term
- during inflation and reheating
- its contribution to the Primordial GW spectra
Here, it is important to have: $\xi(\phi) \neq \text{const}.$

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because of the background EoM in flat FRW universe:

$$ds^2 = -dt^2 + a^2 \left( dr^2 + r^2 d\Omega^2 \right),$$

$$H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V + 12 \dot{\xi} H^3 \right),$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ \dot{\phi}^2 - 4 \dot{\xi} H^2 - 4 \dot{\xi} H \left( 2 \dot{H} - H^2 \right) \right],$$

$$\ddot{\phi} + 3H \dot{\phi} + V_\phi + 12 \xi \dot{H} \left( \dot{H} + H^2 \right) = 0,$$

Observable quantities are obtained as,

$$P_\mathcal{S}(k) = P_\mathcal{S}(k_\ast) \left( \frac{k}{k_\ast} \right)^{n_\mathcal{S}-1}, \quad P_T(k) = P_T(k_\ast) \left( \frac{k}{k_\ast} \right)^{n_T}$$

$$n_\mathcal{S} - 1 \simeq -2 \epsilon - \frac{2 \epsilon (2 \epsilon + \eta) - \delta_1 (\delta_2 - \epsilon)}{2 \epsilon - \delta_1}, \quad n_T \simeq -2 \epsilon, \quad r \simeq 8 (2 \epsilon - \delta_1),$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\ddot{H}}{HH}, \quad \zeta \equiv \frac{\dddot{H}}{H^2 H}, \quad \delta_1 \equiv 4 \kappa^2 \dot{\xi} H, \quad \delta_2 \equiv \frac{\dot{\xi}}{\xi H}, \quad \delta_3 = \frac{\ddot{\xi}}{\dot{\xi} H^2}.$$
Here, it is important to have: $\xi(\phi) \neq \text{const}$. 

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{1}{2} \xi(\phi) R_{\text{GB}}^2 \right],$$

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$$\ddot{\phi} + 3H \dot{\phi} + V_\phi + 12 \xi_\phi H^2 \left( \dot{H} + H^2 \right) = 0,$$

Observable quantities are obtained as, 

$$\mathcal{P}_S(k) = \mathcal{P}_S(k_*) \left( \frac{k}{k_*} \right)^{n_S - 1}, \quad \mathcal{P}_T(k_*) \left( \frac{k}{k_*} \right)^{n_T},$$

$$n_S - 1 \simeq -2 \epsilon - \frac{\epsilon(2\epsilon + \eta) - \delta_1(\delta_2 - \epsilon)}{2\epsilon - \delta_1} \simeq 8(2\epsilon - \delta_1),$$

where 

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\dot{H}}{HH}, \quad \zeta \equiv \frac{\ddot{H}}{H}, \quad \delta_1 \equiv 4\kappa^2 \xi H, \quad \delta_2 \equiv \frac{\ddot{\xi}}{\xi H}, \quad \delta_3 \equiv \frac{\dddot{\xi}}{\xi H^2}.$$
\( n_T \) can be either \textbf{negative} or \textbf{positive}. If it is positive (negative), then the tensor power spectrum is called \textit{“blue-tilted”} (\textit{“red-tilted”}).

**Class-I:** \( n_T < 0, \quad n_T = -2\epsilon \quad \rightarrow \quad \epsilon > 0, \)

\[ \left\{ \begin{array}{l} \xi_\phi > -\frac{3}{4\kappa^4} \frac{V_\phi}{V^2}, \text{ for } V_\phi > 0, \\ \xi_\phi < -\frac{3}{4\kappa^4} \frac{V_\phi}{V^2}, \text{ for } V_\phi < 0. \end{array} \right. \]

\[ \epsilon = \frac{1}{2\kappa^2} \frac{V_\phi}{V} \left( \frac{V_\phi}{V} + \frac{4}{3} \kappa^4 \xi \phi V \right). \]

**Class-II:** \( n_T > 0, \quad \rightarrow \quad \epsilon < 0, \)

\[ \left\{ \begin{array}{l} \xi_\phi < -\frac{3}{4\kappa^4} \frac{V_\phi}{V^2}, \text{ for } V_\phi > 0, \\ \xi_\phi > -\frac{3}{4\kappa^4} \frac{V_\phi}{V^2}, \text{ for } V_\phi < 0. \end{array} \right. \]

**Model-I:** \( V(\phi) = \frac{V_0}{\kappa^4} (\kappa \phi)^n, \quad \xi(\phi) = \xi_0 (\kappa \phi)^{-n}, \)

**Model-II:** \( V(\phi) = \frac{1}{\kappa^4} \left[ \tanh(\kappa \phi) + \sqrt{\mu} \text{sech}(\kappa \phi) \right]^2, \quad \xi(\phi) = \frac{3}{4} \left[ \frac{\sinh^2(\kappa \phi) - \frac{1}{\sqrt{\mu}} \sinh(\kappa \phi)}{\sqrt{\mu} + \sinh(\kappa \phi)} \right]^2, \)

**Model-I:** \[ V(\phi) = \frac{V_0}{\kappa^4}(\kappa\phi)^n, \quad \xi(\phi) = \xi_0(\kappa\phi)^{-n}, \]

**Model-II:** \[ V(\phi) = \frac{1}{\kappa^4}[\tanh(\kappa\phi) + \sqrt{\mu}\sech(\kappa\phi)]^2, \quad \xi(\phi) = \frac{3}{4} \left[ \sinh^2(\kappa\phi) - \frac{1}{\sqrt{\mu}}\sinh(\kappa\phi) \right] \bigg/ \left[ \sqrt{\mu} + \sinh(\kappa\phi) \right]^2, \]

**Model-I:** \( V(\phi) = \frac{V_0}{\kappa^4}(\kappa\phi)^n, \quad \xi(\phi) = \xi_0(\kappa\phi)^{-n}, \quad \alpha \equiv \frac{4}{3}V_0\xi_0 \)

**Model-II:** \( V(\phi) = \frac{1}{\kappa^4}[\tanh(\kappa\phi) + \sqrt{\mu}\sech(\kappa\phi)]^2, \quad \xi(\phi) = \frac{3}{4}[\sqrt{\mu} + \sinh(\kappa\phi)]^2, \)

\[ n_T \quad \text{where} \quad \text{unity along each line indicates that} \]

Further details of each inflation model can be found in corresponding references II.

We discussed two types of GB inflation models in the previous section. In this section, we plot the theoretical predictions of inflaton potential and the coupling function for this type, which was first introduced in Ref. II.

Among several successful inflationary models that satisfy these conditions II, we get the upper limit of the tensor-to-scalar ratio for Model-I is plotted in dashed line in Figure 1.

The parameter \( n_0 \) is a dimensionless constant and \( \theta(10) \) is positive as long as \( \mu > 0 \), which is necessary condition for inflation to successfully solve from Eq. (1).

The standard single-field slow-roll inflation model is recovered when \( n = 1 \) is assumed.

In addition, \( n_T < 0 \), and \( n_T > 0 \).

\[ \text{Planck2015 TT+lowP} \]

\[ 0 < \mu \leq \theta(10) \]

\[ 0 \leq \alpha \leq 1 \]

\[ n = 1, n = 2, n = 4 \]
Part-II: Primordial GW spectrum
The pGWs are described by the tensor part of the pert. metric

\[ ds^2 = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right], \]

The strength of GW is characterized by their energy spectrum

\[ \Omega_{GW}(k) = \frac{1}{\rho_{\text{crit}}} \frac{d\rho_{GW}}{d \ln k}, \]

\[ \Omega_{GW}(k) = \frac{k^2}{12H_0^2} P_T(k), \]

\[ \rho_{GW} = \frac{M_p^2}{4} \int d\ln k \left( \frac{k}{a} \right)^2 \frac{k^3}{\pi^2} \sum_{\lambda} \langle h^\dagger_{\lambda, k} h_{\lambda, k} \rangle. \]

\[ \rho_{\text{crit}} = 3H_0^2 M_p^2 \]

\[ P_T \equiv \frac{k^3}{\pi^2} \sum_{\lambda} \langle h^\dagger_{\lambda, k} h_{\lambda, k} \rangle = T^2(k) P_T(k). \]

\[ P_T(k) = \mathcal{P}_T(k_*) \left( \frac{k}{k_*} \right)^{n_T + \frac{\alpha T}{2} \ln(k/k_*)}, \]

\[ h''_{\lambda, k} + 2 \frac{z_T}{z_T} h'_{\lambda, k} + k^2 c_T^2 h_{\lambda, k} = 0, \]

\[ h_0^2 \Omega_{GW} = \frac{3h_0^2}{32\pi^2 H_0^2 f^2} \Omega_m^2 T_1^2 \left( \frac{f}{f_{\text{eq}}} \right) T_2^2 \left( \frac{f}{f_{\text{th}}} \right) r \mathcal{P}_S \left( \frac{f}{f_\star} \right)^{n_T + \frac{\alpha T}{2} \ln(f/f_\star)}, \]
**Numerical results:**

\[
h_0^2\Omega_{GW} = \frac{3h_0^2}{32\pi^2H_0^2\tau_0^4f^2}\Omega_m^2 T_1^2 \left(\frac{f}{f_{\text{eq}}}\right) T_2^2 \left(\frac{f}{f_{\text{th}}}\right) r\mathcal{P}_S \left(\frac{f}{f_*}\right)^{n_T + \frac{\alpha_T}{2}\ln(f/f_*)},
\]

**Model-I:** \[V(\phi) = \frac{V_0}{\kappa^4}(\kappa\phi)^n, \quad \xi(\phi) = \xi_0(\kappa\phi)^{-n}, \quad \text{with} \quad \alpha \equiv \frac{4}{3}V_0\xi_0\]

**Model-II:** \[V(\phi) = \frac{1}{\kappa^4}[\tanh(\kappa\phi) + \sqrt{\mu}\sech(\kappa\phi)]^2, \quad \xi(\phi) = \frac{3\left[\sinh^2(\kappa\phi) - \frac{1}{\sqrt{\mu}}\sinh(\kappa\phi)\right]}{4\left[\sqrt{\mu} + \sinh(\kappa\phi)\right]^2},\]

**Graphs:**
- **Model-I:**
  - \(\alpha = 0.0, 0.1, 0.3, 0.7\)
  - Correlated DECIGO
  - Ultimate DECIGO

- **Model-II:**
  - \(\mu = 10^{-4}, 10, 40\)
  - Correlated DECIGO
  - Ultimate DECIGO
\[ h_0^2 \Omega_{GW} = \frac{3 h_0^2}{32 \pi^2 H_0^2 \tau_0^4 f^2} \Omega_m^2 T_1^2 \left( \frac{f}{f_{eq}} \right) T_2^2 \left( \frac{f}{f_{th}} \right) r \mathcal{P}_S \left( \frac{f}{f_*} \right)^{n_T + \frac{\alpha T}{2} \ln(f/f_*)}, \]

**Numerical results:**

- **Model-I:**
  - without running
  - with running

- **Model-II:**
  - without running
  - with running

\[ T_{th} = 10^6 \text{ GeV}, \quad T_{th} = 10^8 \text{ GeV}, \quad T_{th} = 10^{10} \text{ GeV} \]

\[ \text{correlated-DECIGO}, \quad \text{ultimate-DECIGO} \]
**Numerical results:**

\[
h_0^2 \Omega_{GW} = \frac{3h_0^2}{32\pi^2 H_0^2 \tau_0^4 f^2} \Omega_m^2 T_1^2 \left( \frac{f}{f_{\text{eq}}} \right) T_2^2 \left( \frac{f}{f_{\text{th}}} \right) r P_S \left( \frac{f}{f_*} \right)^{n_T + \frac{\alpha T}{2} \ln(f/f_*)},
\]

where \( \Omega_m \) is the matter density of the universe, \( T_0^2 \) is the reheating temperature, and \( \alpha = 0.0, 0.1, 0.3, 0.7 \).

\[
T_1^2 \left( \frac{k}{k_{\text{eq}}} \right) = 1 + 1.65 \left( \frac{k}{k_{\text{eq}}} \right) + 1.92 \left( \frac{k}{k_{\text{eq}}} \right)^2,
\]

\[
T_2^2 \left( \frac{k}{k_{\text{th}}} \right) = \left[ 1 + \gamma \left( \frac{k}{k_{\text{th}}} \right)^2 + \sigma \left( \frac{k}{k_{\text{th}}} \right)^2 \right]^{-1},
\]

**Model-I:**

- \( \alpha = 0.0 \)
- \( \alpha = 0.1 \)
- \( \alpha = 0.3 \)
- \( \alpha = 0.7 \)

**Model-II:**

- \( \mu = 10^{-4} \)
- \( \mu = 10 \)
- \( \mu = 40 \)

**Once the pGWs from inflationary origin are detected:**

\[
k_{\text{th}} = 1.7 \times 10^{13} \text{ Mpc}^{-1} \left( \frac{g* s(T_{\text{th}})}{106.75} \right)^{1/6} \left( \frac{T_{\text{th}}}{10^6 \text{ GeV}} \right)
\]
Part-III: Constraints on Reheating
By assuming,

- constant equation-of-state during reheating, \( \omega_{\text{th}} = \text{const} \),
- no entropy production after the end of reheating

we calculate the duration of reheating and the thermalization temperature at the end of reheating,

\[
N_{\text{th}} = \frac{4}{3\omega_{\text{th}} - 1} \left[ \ln \left( \frac{k}{a_0 T_0} \right) + \frac{1}{3} \ln \left( \frac{11 g_{*s}}{43} \right) + \frac{1}{4} \ln \left( \frac{30 \lambda_{\text{end}}}{\pi^2 g_*} \right) + \frac{1}{4} \ln \left( \frac{V_{\text{end}}}{H_*^4} \right) + N_* \right].
\]

\[
T_{\text{th}} = \left( \frac{30 \lambda_{\text{end}} V_{\text{end}}}{\pi^2 g_*} \right)^{\frac{1}{4}} e^{-\frac{3}{4}(1+\omega_{\text{th}})N_{\text{th}}}. \text{ where } \lambda_{\text{end}} = \frac{6}{6 - 2\epsilon - \delta_1(5 - 2\epsilon + \delta_2)} \bigg|_{\phi = \phi_{\text{end}}}
\]

In our numerical study, we consider following models:

**Model-I:** \( V(\phi) = \frac{V_0}{\kappa^4} (\kappa \phi)^n \), \( \xi(\phi) = \xi_0 (\kappa \phi)^{-n} \), \( \alpha \equiv \frac{4}{3} V_0 \xi_0 \)

**Model-II:** \( V(\phi) = \frac{1}{\kappa^4} \left[ \tanh (\kappa \phi) + \sqrt{\mu} \sech (\kappa \phi) \right]^2 \), \( \xi(\phi) = \frac{3 \left[ \sinh^2 (\kappa \phi) - \frac{1}{\sqrt{\mu}} \sinh (\kappa \phi) \right]}{4 \left[ \sqrt{\mu} + \sinh (\kappa \phi) \right]^2} \),
Numerical results:

**Model-I:** \[ V(\phi) = \frac{V_0}{\kappa^4}(\kappa \phi)^n, \quad \xi(\phi) = \xi_0(\kappa \phi)^{-n}, \quad \text{with} \quad \alpha \equiv \frac{4}{3}V_0\xi_0 \]

From solid to dotted: \( \omega_{th} = -1/3; 0; 1/4 \) and 1.
\textbf{Numerical results:}

\textbf{Model-I:} \quad V(\phi) = \frac{V_0}{\kappa^4}(\kappa \phi)^n, \quad \xi(\phi) = \xi_0(\kappa \phi)^{-n}, \quad \text{with} \quad \alpha \equiv \frac{4}{3}V_0\xi_0

\begin{equation*}
N_{th} = \frac{4}{3\omega_{th} - 1} \left[ -60.0085 + \frac{1}{4} \ln \left( \frac{3\lambda_{end}}{100\pi^2} \right) + \frac{1}{4} \ln \left( \frac{V_{end}}{H_*^4} \right) + N_* \right]
\end{equation*}

\begin{equation*}
\lambda_{end} = \left. \frac{6}{6 - 2\epsilon - \delta_1(5 - 2\epsilon + \delta_2)} \right|_{\phi = \phi_{end}} > 0; \quad 0 \leq \alpha \leq 1 \quad \implies \quad 0 \leq \alpha < \frac{n}{2n + 2}.
\end{equation*}

\textit{For} \quad T_{th} = 10^6 \text{ GeV} \quad \text{and} \quad n_s = 0.9655:

\begin{align*}
\text{Model-I with n=1} & \quad 0 \leq \alpha < 1/4. \\
\text{Model-I with n=2} & \quad 0 \leq \alpha < 1/3.
\end{align*}
**Numerical results:**

**Model-II:** \[ V(\phi) = \frac{1}{\kappa^4} [\tanh(\kappa \phi) + \sqrt{\mu} \text{sech}(\kappa \phi)]^2, \]
\[ \xi(\phi) = \frac{3 \left( \sinh^2(\kappa \phi) - \frac{1}{\sqrt{\mu}} \sinh(\kappa \phi) \right)}{4 \left( \sqrt{\mu} + \sinh(\kappa \phi) \right)^2}, \]

10^{-4} \leq \mu \leq 0.3517.
we are interested in understanding the effects of this additional term during inflation and reheating its contribution to the Primordial GW spectra

**CONCLUSION:**

- Inflationary models with a Gauss–Bonnet term are consistent with observational data,
- These models predict both red- and blue-tilted inflationary tensor power spectrum,
- Primordial GW spectrum suppresses (enhances) for \( n_T < 0 \) (\( n_T > 0 \)),
- Once pGWs are detected (DECIGO), \( T_{th} \) can be determined hence the other parameter of reheating including \( N_{th} \) and \( \omega_{th} \) can also be determined.
- \( T_{th} \) significantly increases in the presence of the GB term
- Moreover, reheating can be used as an additional constraint to inflationary models!
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- \( T_{th} \) significantly increases in the presence of the GB term
- Moreover, reheating can be used as an additional constraint to inflationary models!

THANK YOU FOR YOUR KIND ATTENTION!