

# Neutrino Mixing in a Rephasing-Invariant Parametrization

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# Outline

- Introducing an alternative parametrization for mixing matrix
- Some applications:
  - ▶ Neutrino mixing in matter
  - ▶ RGE for neutrino parameters
- Summary and Outlook

# Parametrization for $\nu$ mixing matrix

- Neutrino flavor states and the mass eigenstates are mismatched:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = [V_\nu] \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix},$$

where  $V_\nu$  is the PMNS mixing matrix:

$$[V_\nu] = \begin{bmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{bmatrix}$$

- Choices of parametrization for  $V_\nu$  are not unique:
  - ▶ Standard parametrization: 3 mixing angles ( $\theta_{12}$ ,  $\theta_{23}$ ,  $\theta_{13}$ ), and a Dirac phase  $\delta$ .
  - ▶ Absolute values:  $|V_{\alpha j}|$
  - ▶  $\text{Im}(V_{\beta j} V_{\gamma k} V_{\beta k}^* V_{\gamma j}^*) \sim J$  (Jarlskog invariant)
  - ▶ etc.

# A rephasing-invariant parametrization

Consider the  $3 \times 3$  mixing matrix  $V$ , with  $\det V = +1$ :

- A set of six rephasing invariants can be constructed:

$$\boxed{\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}}, \text{ with } (i, j, k) = \text{permutation of } (1, 2, 3)$$

- There exists a matrix  $v$ , satisfying  $\sum V_{ij}v_{ik} = \sum V_{ji}v_{ki} = \delta_{jk}$

$\Rightarrow v_{ij}$  is the cofactor of  $V_{ij}$

- ▶ From  $VV^\dagger = 1 = \det V$

$$\Rightarrow \boxed{V_{ij}^* = v_{ij}}$$

- ▶ For example,  $V_{11}^* = V_{22}V_{33} - V_{23}V_{32}$ ,  
 $V_{12}^* = -(V_{21}V_{33} - V_{23}V_{31})$ , etc.

- Thus, we can relate  $\Gamma_{ijk}$  to  $|V_{lm}|^2$ , e.g.,  
 $|V_{12}|^2 = V_{12}V_{12}^* = V_{12}(-V_{21}V_{33} + V_{23}V_{31}) = \Gamma_{231} - \Gamma_{213}$ 
  - ▶ All the  $|V_{lm}|^2$  are equal to the differences of the  $\Gamma$ 's
  - ▶ All  $\Gamma$ 's have the same imaginary part,  $\text{Im}(\Gamma_{ijk})$ , which can be identified with  $J$
  - ▶  $\boxed{\Gamma_{ijk} = R_{ijk} - iJ}$ ,
- Separate the even and odd permutation of  $R_{ijk}$ :  
 $\text{Re}(\Gamma_{123}, \Gamma_{231}, \Gamma_{312}) = (x_1, x_2, x_3),$   
 $\text{Re}(\Gamma_{132}, \Gamma_{213}, \Gamma_{321}) = (y_1, y_2, y_3).$
- $(x_i, y_j)$  satisfy two constraints:  
 $(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1,$   
 $x_1x_2 + x_2x_3 + x_3x_1 = y_1y_2 + y_2y_3 + y_3y_1.$   
 $\Rightarrow$  Any 4 out of the 6 parameters  $(x_i, y_j)$  can be used as a complete set of parameters for  $V$

# Parametrizing the physical observables

- $|V_{ij}|^2 = W_{ij}$  can be parametrized by  $(x_a, y_b)$ ,

$$W = \begin{pmatrix} |V_{11}|^2 & |V_{12}|^2 & |V_{13}|^2 \\ |V_{21}|^2 & |V_{22}|^2 & |V_{23}|^2 \\ |V_{31}|^2 & |V_{32}|^2 & |V_{33}|^2 \end{pmatrix} \\ = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix}$$

- Cofactors of elements in  $W$  form a matrix  $w$ , with  $w^T W = (\det W)I$ :

$$w = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ x_3 + y_2 & x_1 + y_3 & x_2 + y_1 \\ x_2 + y_3 & x_3 + y_1 & x_1 + y_2 \end{pmatrix}$$

- $J^2 = x_1 x_2 x_3 - y_1 y_2 y_3$
- $(x_i, y_j) \Leftrightarrow (\theta_{ij}, \delta)$

# Is this parametrization useful?

- Physical measurables can always be expressed in terms of  $(x_i, y_j)$ , e.g.,  $W_{\alpha i} = |V_{\alpha i}|^2 = x_a - y_b$ , etc.
- In many cases, the formulations are symmetric and simple
  - ▶ e.g.,  $P(\nu_\alpha \rightarrow \nu_\beta) = -4[\Lambda_{\gamma 3} \sin^2 \Phi_{21} + \Lambda_{\gamma 2} \sin^2 \Phi_{31} + \Lambda_{\gamma 1} \sin^2 \Phi_{32}] + 2J[\sin 2\Phi_{21} + \sin 2\Phi_{13} + \sin 2\Phi_{32}]$ ,
    - ▶  $\Phi_{ij} = \Phi_i - \Phi_j = (L/4E)(D_i - D_j)$
    - ▶  $\Lambda_{\gamma k} = \frac{1}{2}(W_{\alpha i}W_{\beta j} + W_{\alpha j}W_{\beta i} - W_{\gamma k})$
  - ▶ Easier to look for certain flavor mixing symmetry under permutation of generations
- Applications:  $\nu$  mixing in matter, RGE of  $\nu$  mass matrix, etc.
  - ▶ NOTE: Valid for Dirac  $\nu$ , or lepton number conserving processes, e.g.,  $\nu$  oscillations

## $\nu$ mixing in matter

- When  $\nu$  propagate in matter,  $D_i = m_i^2$  and  $V_{\alpha i}$  are functions of  $A = \sqrt{2}G_F n_e E$

- $\frac{dD_i}{dA} = |V_{ei}|^2 = W_{ei}, \quad \frac{dV_{\alpha i}}{dA} = \sum_{k \neq i} \frac{V_{\alpha k} V_{ei}}{D_i - D_k} V_{ek}^*$

- $\frac{d}{dA} W_{\alpha i} = \frac{d}{dA} (V_{\alpha i}^* V_{\alpha i}) = 2 \sum_{k \neq i} \frac{1}{D_i - D_k} \text{Re}(\Pi_{ik}^{\alpha e})$

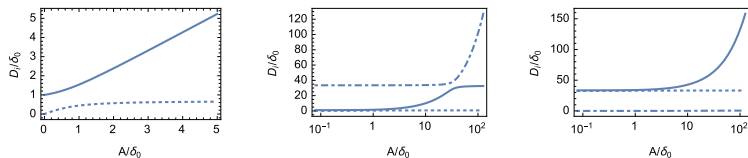
(with  $\Pi_{ik}^{\alpha e} \equiv V_{\alpha i} V_{\alpha k}^* V_{ek} V_{ei}^*$ )

►  $\frac{1}{2} \frac{d}{dA} W_{ei} = \sum_{k \neq i} \frac{W_{ei} W_{ek}}{D_i - D_k},$

$$\frac{1}{2} \frac{d}{dA} W_{\alpha i} = \sum_{k \neq i \neq j} \frac{\Lambda_{\beta j}}{D_i - D_k}, \text{ with } (\alpha, \beta) = (\mu, \tau)$$

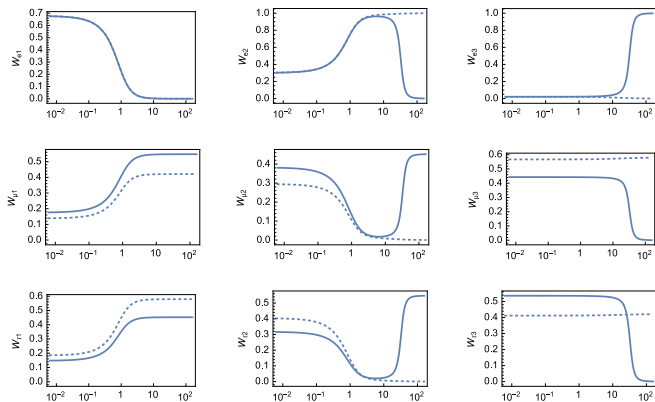
- $\frac{d}{dA} (\ln J) = \frac{-W_{e1} + W_{e2}}{D_1 - D_2} + \frac{-W_{e2} + W_{e3}}{D_2 - D_3} + \frac{-W_{e3} + W_{e1}}{D_3 - D_1}$

# Numerical Solutions of $D_i$



**Figure:** The evolution of  $D_1$  (dotted),  $D_2$  (solid), and  $D_3$  (dot-dashed) for the  $\nu$ -sector in matter under normal (left and middle) and inverted (right) orderings.

# Numerical Solutions of $W_{\alpha i}$



**Figure:** Evolution of  $W_{\alpha i}$  as functions of  $A/\delta_0$  under the normal ordering (solid) and the inverted ordering (dashed).

# Patterns of mixing elements in matter

Example: Normal ordering

$$W_0 \sim \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}, W_l \sim \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

$$W_i \sim \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}, W_h \sim \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix},$$

$$W_d \sim \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

0=vacuum,  $l$ =lower resonance,  $i$ =intermediate range ( $A \sim 10\delta_0$  here),  $h$ =higher resonance, and  $d$ =dense matter ( $A \rightarrow \infty$ ).

# Approximate Solutions

- The behavior of  $W_{\alpha i}$  in matter can be solved numerically
- How about analytical solutions?  
 $\Rightarrow$  Different pole terms in  $dW_{\alpha i}/dA$  dominate at different ranges of  $A$ .
- One may thus solve for the approximate solutions for  $W_{\alpha i}$  at various regions of  $A$  [PRD 97, 055026 (2018)].
- Example: The term  $\propto 1/(D_2 - D_1)$  dominates at small  $A$ 
  - ▶  $\Delta_{21} \cong [A^2 - \frac{2}{3}\delta_0 A + \delta_0^2]^{1/2}$
  - ▶  $W_{e1} \cong \frac{1}{2}[1 - (A - \frac{1}{3}\delta_0)/\Delta_{21}]$
  - ▶  $W_{e2} \cong \frac{1}{2}[1 + (A - \frac{1}{3}\delta_0)/\Delta_{21}]$

# Invariants in matter

- Matter invariant, e.g.,

- ▶  $J(D_1 - D_2)(D_2 - D_3)(D_3 - D_1) = \text{constant}$

- ▶  $\frac{d}{dA}[J^2/(x_1 - y_1)(x_2 - y_2)(x_3 - y_3)] = 0$

$$\Rightarrow \frac{J^2}{|V_{e1}|^2|V_{e2}|^2|V_{e3}|^2} \text{ is constant in matter}$$

- “partial matter invariants”

- ▶  $(D_1 - D_2)^2|V_{e1}|^2|V_{e2}|^2 \cong \text{constant}, \quad (\text{low-A})$

$$J^2(D_1 - D_2)^2 \cong \text{constant}.$$

- ▶  $(D_2 - D_3)^2|V_{e2}|^2|V_{e3}|^2 \cong \text{constant}, \quad (\text{high-A})$

$$J^2(D_2 - D_3)^2 \cong \text{constant}.$$

# Renormalisation Group Equations (RGE)

- As the energy scale changes, one expects the pattern of regularity to evolve according to the RGE
  - ⇒ RGEs bridge the physics in high and low energy scales
- Parameters are all measured in low energies
  - ⇒ Necessary to bring RGE effects into the picture
- RGEs are simple when formulated in full matrices, but are very complicated in terms of physical observables
  - ⇒ a large number of superfluous degrees of freedom must be stripped away

# Parametrization for RGE

- In terms of  $(x_i, y_j)$  and  $W_{ij}$ , RGE for neutrinos [PLB 760, 544 (2016)] and quarks [PRD 93, 093006 (2016)] mixing parameters can be cast in a compact and simple form
- These equations exhibit manifest symmetry under flavour permutations.
- Some RGE invariants and approximate solutions of the parameters can be derived

## Example: RGE Invariants for $\nu$ Parameters

We can show that

- $\boxed{\frac{d}{dt}[\ln J^2(\sinh^2 \ln r_{12})(\sinh^2 \ln r_{23})(\sinh^2 \ln r_{31})] = 0},$

with  $t = \ln(\mu/M_W)$ ,  $r_{ij} = m_i/m_j$ , and

$$\sinh^2 \ln r_{ij} = \frac{1}{4} \frac{(m_i^2 - m_j^2)^2}{m_i^2 m_j^2}$$

- $\boxed{(|V_{\alpha i}|^2 |V_{\beta j}|^2 |V_{\alpha j}|^2 |V_{\beta i}|^2 - \Lambda_{\gamma k}^2) [\Pi_{ij}(\sinh \ln r_{ij}^2)] = \text{constant}},$

with  $\Lambda_{\gamma k} = \frac{1}{2}(|V_{\alpha i}|^2 |V_{\beta j}|^2 + |V_{\alpha j}|^2 |V_{\beta i}|^2 - |V_{\gamma k}|^2)$

# Summary and Outlook

- An alternative, rephasing-invariant parametrization to mixing matrices
  - Highly symmetric evolution equations, invariants, and approximate solutions of physical parameters can be derived:
    - ▶ Neutrinos propagating in matter
    - ▶ RGE evolutions of Dirac neutrino and quark parameters
- ⇒ There may exist certain flavor mixing symmetry under permutation of generations [arXiv: 1805.05600]
- Outlook
    - ▶ Correlation of leptons and quarks
    - ▶ Evolution, textures, and symmetries of mass matrices
    - ▶ etc.....