Constraining Two dimensional CFTs with Modular Invariance

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Slogan

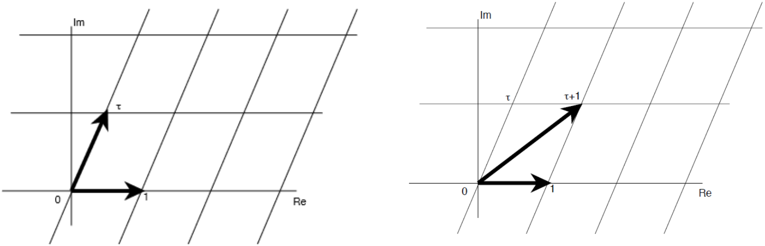
Utilize **Modular Property**

to determine

Torus Partition Function of
Two-Dimensional Conformal Field Theory

Two-Dimensional CFT

- We consider two-dimensional CFT defined on the torus.



- Modular group is generated by $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$.
- The partition function has a Hamiltonian interpretation,

$$Z(q, \bar{q}) = \mathrm{Tr}\left(e^{-\beta H - i\theta J}\right) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right), \qquad q \equiv e^{2\pi i \tau}.$$

Constraints from Modular Property

- Modular Invariance (S-invariance)

$$Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = Z(\tau, \bar{\tau})$$

- With $U(1)$ global symmetry, the partition function read

$$Z(q, \bar{q}, y, \bar{y}) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} y^{J_0} \bar{y}^{\bar{J}_0}\right), \qquad q \equiv e^{2\pi i \tau}, \quad y \equiv e^{2\pi i z},$$

where J_0 is zero mode of conserved current of $U(1)$ symmetry. Then, under the modular transformation,

$$Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \frac{z}{\tau}, \frac{\bar{z}}{\bar{\tau}}) = e^{i\pi k\left(\frac{z^2}{\tau} - \frac{\bar{z}^2}{\bar{\tau}}\right)} Z(\tau, \bar{\tau}, z, \bar{z})$$

Virasoro Character

- Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

- Descendants of the vacuum and primary states are

$$|\Omega\rangle,\ L_{-2}|\Omega\rangle,\ L_{-3}|\Omega\rangle,\ L_{-2}^2|\Omega\rangle,\ L_{-4}|\Omega\rangle,\ L_{-2}L_{-3}|\Omega\rangle,\ L_{-5}|\Omega\rangle,\cdots.$$

$$|h\rangle,\ L_{-1}|h\rangle,\ L_{-2}|h\rangle,\ L_{-1}^2|h\rangle,\ L_{-3}|h\rangle,\ L_{-2}L_{-1}|h\rangle,\ L_{-1}^3|h\rangle,\cdots.$$

The vacuum character and primary character count them.

$$\chi_0(\tau) = q^{\frac{-c+1}{24}}\frac{(1-q)}{\eta(\tau)} = q^{\frac{-c}{24}}(1 + q^2 + q^3 + 2q^4 + 2q^5 + \cdots)$$

$$\chi_h(\tau) = q^{\frac{h-c+1}{24}}\frac{1}{\eta(\tau)} = q^{\frac{h-c}{24}}(1 + q + 2q^2 + 3q^3 + \cdots)$$

Super-Virasoro Algebra

- $\mathcal{N} = 1$ super-Virasoro algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$
$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.$$

L_m and $G_r^{\pm}$ are related to the stress-energy tensor and superconformal current. (r and s are half-integer for **Neveu-Schwarz partition function**)

- The vacuum and primary characters are :

$$\chi_0(\tau) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n} = q^{-\frac{c}{24}} q^{\frac{1}{16}} (1 - q^{\frac{1}{2}}) \frac{\eta(\tau)}{\eta(\frac{\tau}{2})\eta(2\tau)},$$
$$\chi_h(\tau) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n} = q^{h-\frac{c}{24}} q^{\frac{1}{16}} \frac{\eta(\tau)}{\eta(\frac{\tau}{2})\eta(2\tau)}.$$

Character Decomposition

- In general, the torus partition function is written as,

$$Z(\tau,\bar{\tau}) = \underbrace{\chi_0(q)\bar{\chi}_0(\bar{q})}_{\text{vacuum}} + \sum_{h,\bar{h}} d_{h,\bar{h}} \underbrace{\chi_h(q)\bar{\chi}_{\bar{h}}(\bar{q})}_{\text{primary}}$$
$$+ \sum_j d_j \underbrace{(\chi_j(q)\bar{\chi}_0(\bar{q}) + \chi_0(q)\bar{\chi}_j(\bar{q}))}_{\textbf{holomorphic currents}}$$

We will consider the modular constraint $Z(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}) = Z(\tau,\bar{\tau})$.

- We mainly use the reduced character for convenience.

$$\hat{\chi}_0(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_0(\tau), \quad \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_h(\tau)$$

The reduced partition function is **polynomial of q!**

The Modular Constraints

- T- transformation : All states should have **integer spin**.
- S- transformation : $Z(\tau,\bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

$$\boxed{\mathcal{Z}_0(\tau,\bar{\tau}) + \sum_{h,\bar{h}} d(h,\bar{h})\mathcal{Z}_{h,\bar{h}}(\tau,\bar{\tau}) + \sum_{j=1}\left[d(j)\mathcal{Z}_j(\tau,\bar{\tau}) + \tilde{d}(j)\mathcal{Z}_{\bar{j}}(\tau,\bar{\tau})\right] = 0}$$

where $\mathcal{Z}_\lambda(\tau,\bar{\tau})$ is defined as $\chi_\lambda(\tau)\bar{\chi}_\lambda(\bar{\tau}) - \chi_\lambda(-\frac{1}{\tau})\bar{\chi}_\lambda(-\frac{1}{\bar{\tau}})$.

- In case of supersymmetric theory, we focus on the NS partition function. Corresponding modular group is,

$$\Gamma^0(2) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad b \equiv 0 \pmod 2 \right\},$$

generated by $\tau \to \tau + 2$ and $\tau \to -\frac{1}{\tau}$.

The Modular Bootstrap Program

[Collier, Lin, Yin]

- Apply $\alpha\left[\hat{\mathcal{Z}}(z,\bar{z})\right] \equiv \sum_{m,n}^{m+n=N} \alpha_{m,n}\partial_z^m\partial_{\bar{z}}^n\hat{\mathcal{Z}}(z,\bar{z})$ to constraint.

Reparametrize $\tau \equiv \mathrm{i}e^z$, the crossing point is $z=0$.

$$\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right] + \sum_{j=1}^{j_{max}}\left(d(j)\ \alpha\left[\hat{\mathcal{Z}}^j(z,\bar{z})\right] + \bar{d}(j)\ \alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right]\right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right] = 0.$$

- Find $\alpha_{m,n}$ such that,

$$\alpha\left[\hat{\mathcal{Z}}_0(z,\bar{z})\right] > 0,$$

$$\text{and}\quad \alpha\left[\hat{\mathcal{Z}}^j(z,\bar{z})\right] \geq 0,\ \alpha\left[\hat{\mathcal{Z}}^{\bar{j}}(z,\bar{z})\right] \geq 0 \quad \text{for}\quad j\in\mathbb{Z},$$

$$\text{and}\quad \alpha\left[\hat{\mathcal{Z}}^{h,\bar{h}}(z,\bar{z})\right] \geq 0 \quad \text{for}\quad (h,\bar{h})\in\mathcal{P}$$

If we find such $\alpha_{m,n}$, then **we conclude that the theory with spectrum $\mathcal{P}$ is not modular invariant**.

Semi-Definite Programming

- Using the Hilbert theorem,

$f(x) \geq 0$ for the polynomial $f(x)$ and $x \in \mathbb{R}_+$

$$\Leftrightarrow f(x) = \left\{ \sum_{n=0}^{d_1} b_n x^n \right\}^2 + x \cdot \left\{ \sum_{n=0}^{d_2} c_n x^n \right\}^2$$
$$= [x]_{d_1}^T \cdot \mathcal{B}^T \cdot \mathcal{B} \cdot [x]_{d_1} + x\Big([x]_{d_2}^T \cdot \mathcal{C}^T \cdot \mathcal{C} \cdot [x]_{d_2}\Big),$$

where $[x]_d = (1, x, x^2, \cdots, x^d)^T$ and $\mathcal{B}, \mathcal{C} \succeq 0$.

our problem is converted to the semi-definite programming.

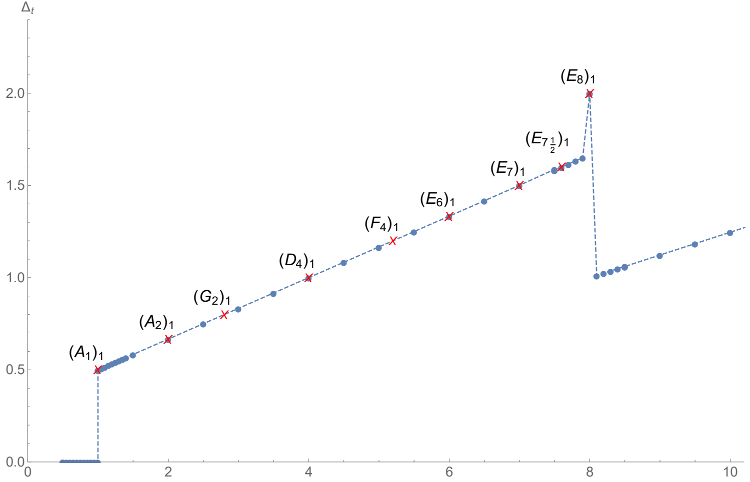
- We can make following assumptions on the $(h, \bar{h}) \in \mathcal{P}$.

$$\text{Scalar Gap} : \Delta \geq \Delta_s \text{ for } \ell = 0,\ \Delta \geq \ell \text{ for } \ell \neq 0$$
$$\text{Twist Gap} : \Delta \geq \Delta_t + \ell,\ \text{for } {}^\forall \ell$$

We mainly consider the **twist gap problem**, with the presence of **holomorphic currents**.

Numerical Bound With Twist Gap (1)

- The numerical bound ($c \leq 10$) with the Twist Gap.



Boundary Theories

- Central charge and conformal weight of states in WZW model with affine Lie algebra $\hat{\mathfrak{g}}$ and level-k are given by,

$$c = \frac{k \ \dim\hat{\mathfrak{g}}}{k + h^\vee}, \qquad h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}.$$

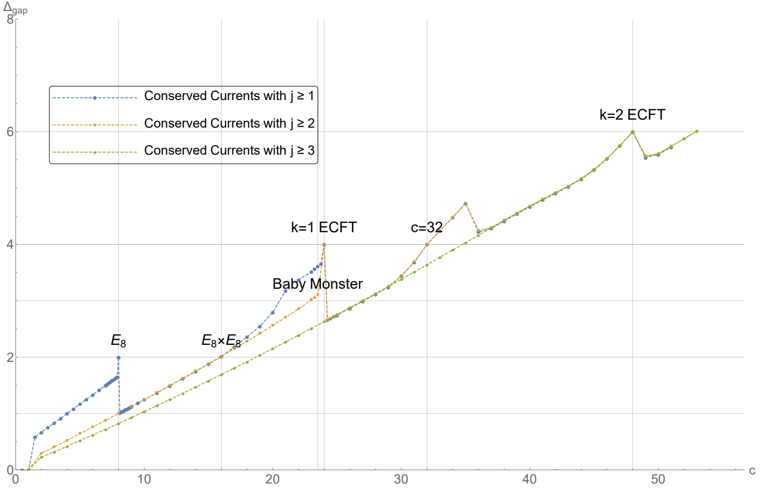
- **The result of twist gap problem realize nine level-1 WZW models on the numerical boundary!**

Central Charge	Lowest Primary	Expected CFT
$c = 1$	$\Delta_t = 1/2$	$SU(2)_1$ WZW model
$c = 2$	$\Delta_t = 2/3$	$SU(3)_1$ WZW model
$c = 14/5$	$\Delta_t = 4/5$	$(G_2)_1$ WZW model
$c = 4$	$\Delta_t = 1$	$SO(8)_1$ WZW model
$c = 26/5$	$\Delta_t = 6/5$	$(F_4)_1$ WZW model
$c = 6$	$\Delta_t = 4/3$	$(E_6)_1$ WZW model
$c = 7$	$\Delta_t = 3/2$	$(E_7)_1$ WZW model
$c = 8$	$\Delta_t = 2$	$(E_8)_1$ WZW model

Deligne's Exceptional Series

Numerical Bound With Twist Gap (2)

- The numerical bound ($c \leq 55$) with the Twist Gap.



Boundary Theories

- When $j \geq 1$ holomorphic currents are included,

Central Charge	Lowest Primary	Expected CFT
$c = 16$	$\Delta_t = 2$	$(E_8 \times E_8)_1$ WZW model

- When $j \geq 2$ holomorphic currents are included,

Central Charge	Lowest Primary	Expected CFT
$c = 24$	$\Delta_t = 4$	Monster CFT
$c = 48$	$\Delta_t = 6$	"$c = 48$ ECFT"
$c = 8$	$\Delta_t = 1$	CFT with $O^+_{10}(2)$
$c = 16$	$\Delta_t = 2$	CFT with $O^+_{10}(2)$
$c = 47/2$	$\Delta_t = 3$	Baby Monster CFT

Modular Differential Equation [Mathur, Muhki, Sen]

- Idea : n characters of rational conformal field theory(RCFT) are the solutions to the n-th order holomorphic differential equation.

$$D_\tau^n \chi(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) D_\tau^k \chi(\tau) = 0, \quad D_\tau = \partial_\tau - \frac{i\pi r}{6} E_2$$

- The solution of below differential equation,

$$D_\tau^2 \chi(\tau) + \hat{\mu} E_4(\tau)\chi(\tau) = 0,$$

with an ansatz $\chi_{\hat{\lambda}}(q) = q^\alpha(a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots)$ is identified to the vacuum character of $n = 2$ RCFT.

- The coefficients $\{a_0, a_1, a_2, \cdots\}$ are **positive integer** only for

$$\boxed{c \in \{\frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8\}}.$$

[Mathur, Muhki, Sen], [Tuite]

Modular Differential Equation

- The solution of below differential equation,

$$D_\tau^3 \chi(\tau) + \mu_1 E_4(\tau) D_\tau \chi(\tau) + \mu_2 E_6(\tau) \chi(\tau) = 0,$$

with an ansatz $\chi_{\hat{\lambda}}(q) = q^\alpha (a_0 + a_2 q^2 + a_3 q^3 + \cdots)$ is identified to the vacuum character of $n = 3$ RCFT.

- The coefficients $\{a_0, a_2, a_3, \cdots\}$ are **positive integer** only for

$$\boxed{c \in \{-\frac{44}{5}, 8, 16, \frac{47}{2}, 24, 32, \frac{164}{5}, \frac{236}{7}, 40\}}.$$

- The primary characters have a form of

$$\chi_{h_\pm}(\tau) = q^{h_\pm - \frac{c}{24}} \left[b_0 + b_1 q + b_2 q^2 + \cdots \right]$$

with $h_\pm(c) = \frac{c+4}{16} \pm \frac{\sqrt{368+24c-c^2}}{16\sqrt{31}}$.

Finding Partition Function

From the modular differential equation at $c = \frac{26}{5}$,

$$f_0(\tau) = 1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \cdots\,,$$

$$f_{\frac{3}{5}}(\tau) = q^{\frac{3}{5}}\left(26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \cdots\right),$$

Partition function may have the structure of

$$Z(q,\bar{q}) = f_0(\tau)f_0(\bar{\tau}) + \alpha f_{\frac{3}{5}}(\tau)f_{\frac{3}{5}}(\bar{\tau}).$$

From the modular bootstrap (Extremal Functional Method), [Paulos, El-Showk]

$$\hat{Z}(q,\bar{q}) = 676\hat{\chi}_{\frac{3}{5}}(\tau)\bar{\hat{\chi}}_{\frac{3}{5}}(\bar{\tau}) + 7098\hat{\chi}_{\frac{3}{5}}(\tau)\bar{\hat{\chi}}_{\frac{8}{5}}(\bar{\tau}) + 7098\hat{\chi}_{\frac{8}{5}}(\tau)\bar{\hat{\chi}}_{\frac{3}{5}}(\bar{\tau})$$
$$+ 2704\hat{\chi}_1(\tau)\bar{\hat{\chi}}_1(\bar{\tau}) + 16848\hat{\chi}_2(\tau)\bar{\hat{\chi}}_1(\bar{\tau}) + 16848\hat{\chi}_1(\tau)\bar{\hat{\chi}}_2(\bar{\tau}) + \cdots\,.$$

The degeneracies of first few low-lying state are enough to fix α!

$$Z_{F_4}(q,\bar{q}) = |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2$$

Results of Modular Constraint (1)

- The partition functions of $(G_2)_1, (F_4)_1$ and $(E_7)_1$ WZW model are, [Gannon]

$$Z_{G_2}(q,\bar{q}) = |f_0^{c=14/5}(q)|^2 + |f_1^{c=14/5}(q)|^2$$
$$Z_{F_4}(q,\bar{q}) = |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2$$
$$Z_{E_7}(q,\bar{q}) = |f_0^{c=7}(q)|^2 + |f_1^{c=7}(q)|^2$$

and for $(E_6)_1$ WZW model,

$$Z_{E_6}(q,\bar{q}) = f_0^{c=6}(q)\bar{f}_0^{c=6}(\bar{q}) + 2f_1^{c=6}(q)\bar{f}_1^{c=6}(\bar{q})$$

- The partition function of $c = 24$ extremal CFT is,

$$c = 24 : Z_{c=24}(q,\bar{q}) = J(q)\bar{J}(\bar{q}).$$

Results of Modular Constraint (2)

- The modular constraint completely determine the partition function of $c = 8$ CFT without spin-one conserved current.

$$Z_{c=8} = f^{c=8}_{h=0}(\tau)\bar{f}^{c=8}_{h=0}(\bar{\tau}) + 496 f^{c=8}_{h=1/2}(\tau)\bar{f}^{c=8}_{h=1/2}(\bar{\tau})|_{a_0=1} + 33728 f^{c=8}_{h=1}(\tau)\bar{f}^{c=8}_{h=1}(\bar{\tau})|_{a_1=1}.$$

$$= 1 + \underbrace{496}_{1+155+340} q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}} + \underbrace{17856}_{2\times 155+2\times 868+15810} q^{\frac{3}{2}}\bar{q}^{\frac{1}{2}} + \underbrace{33728}_{2108+31620} q\bar{q} + \underbrace{539648}_{539648} q^2\bar{q} + \cdots$$

- Likewise, the partition function of $c = 16$ CFT without spin-one conserved current read :

$$Z_{c=16} = f^{c=16}_{h=0}(\tau)\bar{f}^{c=16}_{h=0}(\bar{\tau})$$
$$+ 134912 f^{c=16}_{h=1}(\tau)\bar{f}^{c=16}_{h=1}(\bar{\tau})|_{b_0=1} + 32505856 f^{c=16}_{h=3/2}(\tau)\bar{f}^{c=16}_{h=3/2}(\bar{\tau})|_{b_1=1}.$$
$$= 1 + \underbrace{2296}_{2\times 1+186+2108} q^2 + \underbrace{65536}_{2\times 1+186+14756+50592} q^3 + \underbrace{134912}_{186+340+868+22858+110670} q\bar{q} + \cdots .$$

'Dual' CFT Description [Mukhi]

- Ising model and Babymonster CFT

	c	h_1	h_2
Ising model	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$
Baby monster CFT	$\frac{47}{2}$	$\frac{3}{2}$	$\frac{31}{16}$
Sum	24	2	2

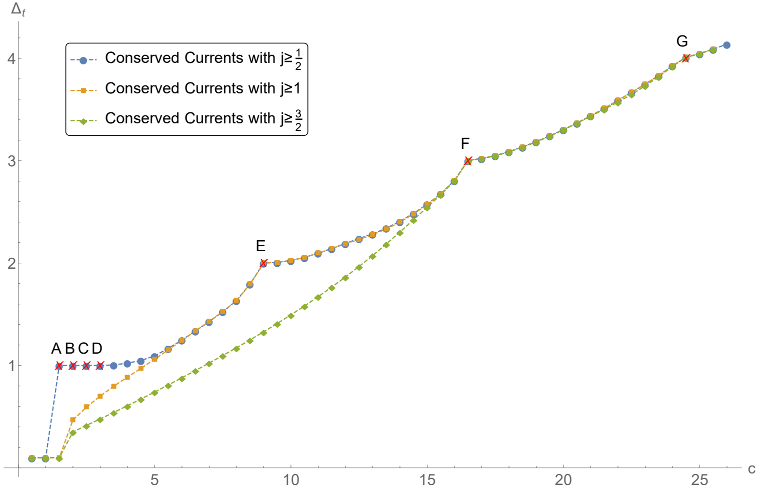
$$j(\tau) - 744 = \chi_{\mathrm{VB}^\natural_{(0)}}(\tau)\chi^{\mathrm{Ising}}_{\mathrm{vac}}(\tau) + \chi_{\mathrm{VB}^\natural_{(1)}}(\tau)\chi^{\mathrm{Ising}}_{h=\frac{1}{2}}(\tau) + \chi_{\mathrm{VB}^\natural_{(3)}}(\tau)\chi^{\mathrm{Ising}}_{h=\frac{1}{16}}(\tau)$$

- $c = 8$ and $c = 16$ CFTs are related by bilinear relation :

$$j(\tau) - 744 = \left(f^{c=8}_{h=0}\right)\left(f^{c=16}_{h=0}\right) + \left(f^{c=8}_{h=1}\Big|_{a_1=\sqrt{33728}}\right)\left(f_{h=1}\Big|^{c=16}_{b_0=\sqrt{134912}}\right)$$
$$+ \left(f^{c=8}_{h=1/2}\Big|_{a_0=\sqrt{496}}\right)\left(f^{c=16}_{h=3/2}\Big|_{b_1=\sqrt{32505856}}\right)$$

Twist Gap With $\mathcal{N} = 1$ Supersymmetry

- The numerical bound ($c \leq 26$) with the Twist Gap.



Modular Differential Equation with $\Gamma^0(2)$

- Modular forms of the congruence subgroups are,

$$
\begin{aligned}
M_2(q) &\equiv \vartheta_{01}(q)^4 - \vartheta_{10}(q)^4, \\
M_4^{(1)}(q) &\equiv \vartheta_{00}(q)^8, \quad M_4^{(2)}(q) \equiv \vartheta_{10}(q)^8 + \vartheta_{01}(q)^8, \quad . \\
M_6^{(1)}(q) &= \vartheta_{01}(q)^{12} - \vartheta_{10}(q)^{12} + \vartheta_{10}(q)^8\vartheta_{01}(q)^4 - \vartheta_{10}(q)^4\vartheta_{01}(q)^8 \\
M_6^{(2)}(q) &= \vartheta_{01}(q)^{12} - \vartheta_{10}(q)^{12} + \vartheta_{10}(q)^4\vartheta_{01}(q)^8 - \vartheta_{10}(q)^8\vartheta_{01}(q)^4
\end{aligned}
$$

- Using above modular forms, we construct the modified modular differential equation of order two and three.

$$
\boxed{\mathcal{D}_\tau^2 f(\tau) + \mu_1 M_2(\tau)\mathcal{D}_\tau f(\tau) + \mu_2 M_4^{(1)}(\tau) f(\tau) + \mu_3 M_4^{(2)}(\tau) f(\tau) = 0.}
$$

$$
\begin{aligned}
&\mathcal{D}_\tau^3 f(\tau) + \mu_1 M_2 \mathcal{D}_\tau^2 f(\tau) + \mu_2 M_4^{(1)}(\tau)\mathcal{D}_\tau f(\tau) \\
&\quad + \mu_3 M_4^{(2)}(\tau)\mathcal{D}_\tau f(\tau) + \mu_4 M_6^{(1)} f(\tau) + \mu_5 M_6^{(2)} f(\tau) = 0.
\end{aligned}
$$

Solutions of the MDE [Höhn]

- The solutions of the 2nd modular differential equation with $\Gamma^0(2)$ are,

	Solution of 2nd order MDE
$c = \frac{3}{2}$	$f(\tau) = 1 + 3q^{\frac{1}{2}} + 3q + 4q^{\frac{3}{2}} + 9q^2 + \cdots$
$c = 2$	$f(\tau) = 1 + 4q^{\frac{1}{2}} + 6q + 8q^{\frac{3}{2}} + 17q^2 + \cdots$
$c = \frac{5}{2}$	$f(\tau) = 1 + 5q^{\frac{1}{2}} + 10q + 15q^{\frac{3}{2}} + 30q^2 + \cdots$
$c = 3$	$f(\tau) = 1 + 6q^{\frac{1}{2}} + 15q + 26q^{\frac{3}{2}} + 51q^2 + \cdots$
$c = \frac{17}{2}$	$f(\tau) = 1 + 255q + 221q^{\frac{3}{2}} + 4216q^2 + 4114q^{\frac{5}{2}} + \cdots$
$c = 9$	$f(\tau) = 1 + 261q + 456q^{\frac{3}{2}} + 4500q^2 + 8424q^{\frac{5}{2}} + \cdots$

- And, the solutions of the 3rd modular differential equation is given by

	Solution of 3rd order MDE
$c = 16$	$f(\tau) = 1 + 7936q^{\frac{3}{2}} + 2296q^2 + 412672q^{\frac{5}{2}} + \cdots$
$c = \frac{33}{2}$	$f(\tau) = 1 + 7766q^{\frac{3}{2}} + 11220q^2 + 408507q^{\frac{5}{2}} + \cdots$

The Results

- $f(\tau)$ at $c = \frac{3}{2}, 2, \cdots, \frac{15}{2}$ are same with the partition function of k-copies free-fermion.

$$f_{c=\frac{1}{2}\cdot k}(\tau) = \left(q^{\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) \right)^k$$

- The following theories are realized on the numerical boundary!

Label	Central Charge	Lowest Primary	Expected CFT
A	$c = 3/2$	$\Delta_t = 1$	Three-fermion
B	$c = 2$	$\Delta_t = 1$	Four-fermion
C	$c = 5/2$	$\Delta_t = 1$	Five-fermion
D	$c = 3$	$\Delta_t = 1$	Six-fermion
E	$c = 9$	$\Delta_t = 2$	?
F	$c = 33/2$	$\Delta_t = 3$	?

Conclusion and Outlook

• The twist gap problem with holomorphic currents ($j \geq 1$) successfully realize two-character RCFTs and three-character RCFTs on the numerical bound.

• We discover **the partition function of c = 8, c = 16 CFTs without Kac-Moody symmetry** via modular bootstrap.

• The twist gap with $\mathcal{N} = 1$ supersymmetry realize the system of free-fermion on the numerical boundary. Is there any underlying symmetry at $c = 9$ and $c = \frac{33}{2}$?

• The modular bootstrap program with $\mathcal{N} = 2$ supersymmetry can provide further information : Hellerman bound, Charge bound, Weak gravity conjecture ...