Constraining Two dimensional CFTs with Modular Invariance

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Slogan

Utilize **Modular Property** to determine

**Torus Partition Function of Two-Dimensional Conformal Field Theory**
Two-Dimensional CFT

- We consider two-dimensional CFT defined on the torus.

- Modular group is generated by $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -\frac{1}{\tau}$.

- The partition function has a Hamiltonian interpretation,

$$Z(q, \bar{q}) = \text{Tr} \left( e^{-\beta H - i \theta J} \right) = \text{Tr} \left( q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad q \equiv e^{2\pi i \tau}.$$
Constraints from Modular Property

- Modular Invariance (S-invariance)

\[ Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = Z(\tau, \bar{\tau}) \]

- With $U(1)$ global symmetry, the partition function read

\[ Z(q, \bar{q}, y, \bar{y}) = \text{Tr} \left( q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} y^{J_0} \bar{y}^{\bar{J}_0} \right), \quad q \equiv e^{2\pi i \tau}, \quad y \equiv e^{2\pi i z}, \]

where $J_0$ is zero mode of conserved current of $U(1)$ symmetry. Then, under the modular transformation,

\[ Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \frac{z}{\tau}, \frac{\bar{z}}{\bar{\tau}}\right) = e^{i\pi k \left(\frac{z^2}{\tau} - \frac{\bar{z}^2}{\bar{\tau}}\right)} Z(\tau, \bar{\tau}, z, \bar{z}) \]
Virasoro Character

- Virasoro algebra

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \]

- Descendants of the vacuum and primary states are

\[ |\Omega\rangle, \ L_{-2}|\Omega\rangle, \ L_{-3}|\Omega\rangle, \ L_{-2}^2|\Omega\rangle, \ L_{-4}|\Omega\rangle, \ L_{-2}L_{-3}|\Omega\rangle, \ L_{-5}|\Omega\rangle, \cdots \]
\[ |h\rangle, \ L_{-1}|h\rangle, \ L_{-2}|h\rangle, \ L_{-1}^2|h\rangle, \ L_{-3}|h\rangle, \ L_{-2}L_{-1}|h\rangle, \ L_{-1}^3|h\rangle, \cdots \]

The vacuum character and primary character count them.

\[ \chi_0(\tau) = q^{\frac{c-1}{24}} \frac{(1 - q)}{\eta(\tau)} = q^{\frac{c}{24}} (1 + q^2 + q^3 + 2q^4 + 2q^5 + \cdots) \]

\[ \chi_h(\tau) = q^{\frac{h-c+1}{24}} \frac{1}{\eta(\tau)} = q^{\frac{h-c}{24}} (1 + q + 2q^2 + 3q^3 + \cdots) \]
Super-Virasoro Algebra

- The \( \mathcal{N} = 1 \) super-Virasoro algebra is given by

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}
\]

\[
[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}
\]

\[
\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\]

\( L_m \) and \( G_r^\pm \) are related to the stress-energy tensor and superconformal current. (\( r \) and \( s \) are half-integer for Neveu-Schwarz partition function)

- The vacuum and primary characters are:

\[
\chi_0(\tau) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1 + q^{n-\frac{1}{2}}}{1 - q^n} = q^{-\frac{c}{24}} q^{\frac{1}{16}} (1 - q^\frac{1}{2}) \frac{\eta(\tau)}{\eta(\frac{\tau}{2})\eta(2\tau)},
\]

\[
\chi_h(\tau) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1 + q^{n-\frac{1}{2}}}{1 - q^n} = q^{h-\frac{c}{24}} q^{\frac{1}{16}} \frac{\eta(\tau)}{\eta(\frac{\tau}{2})\eta(2\tau)}.
\]
Character Decomposition

- In general, the torus partition function is written as,

\[ Z(\tau, \bar{\tau}) = \chi_0(q)\bar{\chi}_0(\bar{q}) + \sum_{h, \bar{h}} d_{h, \bar{h}} \chi_h(q)\bar{\chi}_{\bar{h}}(\bar{q}) \]

\[ + \sum_j d_j (\chi_j(q)\bar{\chi}_0(\bar{q}) + \chi_0(q)\bar{\chi}_j(\bar{q})) \]

We will consider the modular constraint \( Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = Z(\tau, \bar{\tau}) \).

- We mainly use the reduced character for convenience.

\[ \hat{\chi}_0(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_0(\tau), \quad \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_h(\tau) \]

The reduced partition function is polynomial of \( q \)!
The Modular Constraints

- **$T$- transformation**: All states should have integer spin.
- **$S$- transformation**: $Z(\tau, \bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

\[
Z_0(\tau, \bar{\tau}) + \sum_{h, \bar{h}} d(h, \bar{h}) Z_{h, \bar{h}}(\tau, \bar{\tau}) + \sum_{j=1} \left[ d(j) Z_j(\tau, \bar{\tau}) + \tilde{d}(j) Z_{\bar{j}}(\tau, \bar{\tau}) \right] = 0
\]

where $Z_\lambda(\tau, \bar{\tau})$ is defined as $\chi_\lambda(\tau) \bar{\chi}_\lambda(\bar{\tau}) - \chi_\lambda(-\frac{1}{\tau}) \bar{\chi}_\lambda(-\frac{1}{\bar{\tau}})$.

- In case of supersymmetric theory, we focus on the NS partition function. Corresponding modular group is,

\[
\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad b \equiv 0 \text{ (mod 2)} \right\},
\]

generated by $\tau \rightarrow \tau + 2$ and $\tau \rightarrow -\frac{1}{\tau}$. 
The Modular Bootstrap Program

• Apply $\alpha \left[ \hat{Z}(z, \bar{z}) \right] \equiv \sum_{m,n}^{\mathbb{m} + n = N} \alpha_{m,n} \partial_z^m \partial_{\bar{z}}^n \hat{Z}(z, \bar{z})$ to constraint. Reparametrize $\tau \equiv i e^z$, the crossing point is $z = 0$.

$$\alpha \left[ \hat{Z}_0(z, \bar{z}) \right] + \sum_{j=1}^{j_{max}} \left( d(j) \alpha \left[ \hat{Z}^j(z, \bar{z}) \right] + d(j) \alpha \left[ \hat{Z}^j(z, \bar{z}) \right] \right) + \sum_{h, \bar{h} \in \mathcal{P}} d(h, \bar{h}) \alpha \left[ \hat{Z}^h, \bar{h}(z, \bar{z}) \right] = 0.$$  

• Find $\alpha_{m,n}$ such that,

$$\alpha \left[ \hat{Z}_0(z, \bar{z}) \right] > 0,$$

and $\alpha \left[ \hat{Z}^j(z, \bar{z}) \right] \geq 0$, $\alpha \left[ \hat{Z}^j(z, \bar{z}) \right] \geq 0$ for $j \in \mathbb{Z}$,

and $\alpha \left[ \hat{Z}^h, \bar{h}(z, \bar{z}) \right] \geq 0$ for $(h, \bar{h}) \in \mathcal{P}$

If we find such $\alpha_{m,n}$, then we conclude that the theory with spectrum $\mathcal{P}$ is not modular invariant.
Semi-Definite Programming

- Using the Hilbert theorem,

\[ f(x) \geq 0 \text{ for the polynomial } f(x) \text{ and } x \in \mathbb{R}_+ \]

\[ \iff f(x) = \left\{ \sum_{n=0}^{d_1} b_n x^n \right\}^2 + x \cdot \left\{ \sum_{n=0}^{d_2} c_n x^n \right\}^2 \]

\[ = [x]_{d_1}^T \cdot B^T \cdot B \cdot [x]_{d_1} + x \left( [x]_{d_2}^T \cdot C^T \cdot C \cdot [x]_{d_2} \right), \]

where \([x]_d = (1, x, x^2, \cdots, x^d)^T\) and \(B, C \succeq 0\).

our problem is converted to the semi-definite programming.

- We can make following assumptions on the \((h, \bar{h}) \in \mathcal{P}\).

  Scalar Gap : \(\Delta \geq \Delta_s\) for \(\ell = 0\), \(\Delta \geq \ell\) for \(\ell \neq 0\)

  Twist Gap : \(\Delta \geq \Delta_t + \ell\), for \(\forall \ell\)

We mainly consider the twist gap problem, with the presence of holomorphic currents.
The numerical bound \((c \leq 10)\) with the Twist Gap.
Boundary Theories

- Central charge and conformal weight of states in WZW model with affine Lie algebra $\hat{g}$ and level-$k$ are given by,

$$ c = \frac{k \dim \hat{g}}{k + h^\vee}, \quad h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}. $$

- The result of twist gap problem realize nine level-1 WZW models on the numerical boundary!

<table>
<thead>
<tr>
<th>Central Charge</th>
<th>Lowest Primary</th>
<th>Expected CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = 1$</td>
<td>$\Delta_t = 1/2$</td>
<td>$SU(2)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 2$</td>
<td>$\Delta_t = 2/3$</td>
<td>$SU(3)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 14/5$</td>
<td>$\Delta_t = 4/5$</td>
<td>$(G_2)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 4$</td>
<td>$\Delta_t = 1$</td>
<td>$SO(8)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 26/5$</td>
<td>$\Delta_t = 6/5$</td>
<td>$(F_4)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 6$</td>
<td>$\Delta_t = 4/3$</td>
<td>$(E_6)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 7$</td>
<td>$\Delta_t = 3/2$</td>
<td>$(E_7)_1$ WZW model</td>
</tr>
<tr>
<td>$c = 8$</td>
<td>$\Delta_t = 2$</td>
<td>$(E_8)_1$ WZW model</td>
</tr>
</tbody>
</table>
The numerical bound \((c \leq 55)\) with the Twist Gap.

\[ E_8 \times E_8 \]

\[ \Delta_{\text{gap}} \]

\[ k=1 \text{ ECFT} \]

\[ k=2 \text{ ECFT} \]

Conserved Currents with \(j \geq 1\)

Conserved Currents with \(j \geq 2\)

Conserved Currents with \(j \geq 3\)
Boundary Theories

- When \( j \geq 1 \) holomorphic currents are included,

<table>
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<tr>
<th>Central Charge</th>
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<th>Expected CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = 16 )</td>
<td>( \Delta_t = 2 )</td>
<td>( (E_8 \times E_8)_1 ) WZW model</td>
</tr>
</tbody>
</table>

- When \( j \geq 2 \) holomorphic currents are included,

<table>
<thead>
<tr>
<th>Central Charge</th>
<th>Lowest Primary</th>
<th>Expected CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = 24 )</td>
<td>( \Delta_t = 4 )</td>
<td>Monster CFT</td>
</tr>
<tr>
<td>( c = 48 )</td>
<td>( \Delta_t = 6 )</td>
<td>“( c = 48 ) ECFT”</td>
</tr>
<tr>
<td>( c = 8 )</td>
<td>( \Delta_t = 1 )</td>
<td>CFT with ( O_{10}^+(2) )</td>
</tr>
<tr>
<td>( c = 16 )</td>
<td>( \Delta_t = 2 )</td>
<td>CFT with ( O_{10}^+(2) )</td>
</tr>
<tr>
<td>( c = 47/2 )</td>
<td>( \Delta_t = 3 )</td>
<td>Baby Monster CFT</td>
</tr>
</tbody>
</table>
Modular Differential Equation [Mathur, Muhki, Sen]

- Idea: $n$ characters of rational conformal field theory (RCFT) are the solutions to the $n$-th order holomorphic differential equation.

\[ D^n_\tau \chi(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) D^k_\tau \chi(\tau) = 0, \quad D_\tau = \partial_\tau - \frac{i \pi r}{6} E_2 \]

- The solution of below differential equation,

\[ D^2_\tau \chi(\tau) + \hat{\mu} E_4(\tau) \chi(\tau) = 0, \]

with an ansatz $\chi(\hat{\lambda})(q) = q^\alpha (a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots)$ is identified to the vacuum character of $n = 2$ RCFT.

- The coefficients $\{a_0, a_1, a_2, \cdots\}$ are positive integer only for

\[ c \in \left\{ \frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8 \right\}. \]

[Mathur, Muhki, Sen], [Tuite]
Modular Differential Equation

- The solution of below differential equation,

\[ D_\tau^3 \chi(\tau) + \mu_1 E_4(\tau) D_\tau \chi(\tau) + \mu_2 E_6(\tau) \chi(\tau) = 0, \]

with an ansatz \( \chi(\hat{q}) = q^\alpha (a_0 + a_2 q^2 + a_3 q^3 + \cdots) \) is identified to the vacuum character of \( n = 3 \) RCFT.

- The coefficients \( \{a_0, a_2, a_3, \cdots\} \) are positive integer only for

\[ c \in \left\{ \frac{44}{5}, 8, 16, \frac{47}{2}, 24, 32, \frac{164}{5}, \frac{236}{7}, 40 \right\}. \]

- The primary characters have a form of

\[ \chi_{h\pm}(\tau) = q^{h\pm-\frac{c}{24}} \left[ b_0 + b_1 q + b_2 q^2 + \cdots \right] \]

with \( h_{\pm}(c) = \frac{c+4}{16} \pm \frac{\sqrt{368+24c-c^2}}{16\sqrt{31}}. \)
Finding Partition Function

From the modular differential equation at $c = \frac{26}{5}$,

$$f_0(\tau) = 1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \cdots,$$

$$f_{\frac{3}{5}}(\tau) = q^{\frac{3}{5}} \left( 26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \cdots \right),$$

Partition function may have the structure of

$$Z(q, \bar{q}) = f_0(\tau)f_0(\bar{\tau}) + \alpha f_{\frac{3}{5}}(\tau)f_{\frac{3}{5}}(\bar{\tau}).$$

From the modular bootstrap (Extremal Functional Method), [Paulos, El-Showk]

$$\hat{Z}(q, \bar{q}) = 676\hat{\chi}_{\frac{3}{5}}(\tau)\hat{\chi}_{\frac{3}{5}}(\bar{\tau}) + 7098\hat{\chi}_{\frac{3}{5}}(\tau)\hat{\chi}_{\frac{8}{5}}(\bar{\tau}) + 7098\hat{\chi}_{\frac{8}{5}}(\tau)\hat{\chi}_{\frac{3}{5}}(\bar{\tau})$$

$$+ 2704\hat{\chi}_{1}(\tau)\hat{\chi}_{1}(\bar{\tau}) + 16848\hat{\chi}_{2}(\tau)\hat{\chi}_{1}(\bar{\tau}) + 16848\hat{\chi}_{1}(\tau)\hat{\chi}_{2}(\bar{\tau}) + \cdots.$$ 

The degeneracies of first few low-lying state are enough to fix $\alpha!$

$$Z_{F_4}(q, \bar{q}) = |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2$$
Results of Modular Constraint \( (1) \)

- The partition functions of \((G_2)_1\), \((F_4)_1\) and \((E_7)_1\) WZW model are, \[ \text{[Gannon]} \]

\[
\begin{align*}
Z_{G_2}(q, \bar{q}) &= |f_0^{c=14/5}(q)|^2 + |f_1^{c=14/5}(q)|^2 \\
Z_{F_4}(q, \bar{q}) &= |f_0^{c=26/5}(q)|^2 + |f_1^{c=26/5}(q)|^2 \\
Z_{E_7}(q, \bar{q}) &= |f_0^{c=7}(q)|^2 + |f_1^{c=7}(q)|^2
\end{align*}
\]

and for \((E_6)_1\) WZW model,

\[
Z_{E_6}(q, \bar{q}) = f_0^{c=6}(q) \bar{f}_0^{c=6}(\bar{q}) + 2f_1^{c=6}(q) \bar{f}_1^{c=6}(\bar{q})
\]

- The partition function of \(c = 24\) extremal CFT is,

\[
c = 24 : Z_{c=24}(q, \bar{q}) = J(q) \bar{J}(\bar{q}).
\]
Results of Modular Constraint (2)

• The modular constraint completely determine the partition function of $c = 8$ CFT without spin-one conserved current.

$$Z_{c=8} = f_{h=0}^{c=8} (\tau) f_{h=0}^{c=8} (\bar{\tau}) + 496 f_{h=1/2}^{c=8} (\tau) f_{h=1/2}^{c=8} (\bar{\tau}) | a_0 = 1 + 33728 f_{h=1}^{c=8} (\tau) f_{h=1}^{c=8} (\bar{\tau}) | a_1 = 1.$$

$$= 1 + \underbrace{496} \frac{q^2}{1+155+340} + \underbrace{17856} \frac{q^3}{2 \times 155 + 2 \times 868 + 15810} + \underbrace{33728} q\bar{q} + \underbrace{539648} q^2 \bar{q} + \cdots.$$

• Likewise, the partition function of $c = 16$ CFT without spin-one conserved current read:

$$Z_{c=16} = f_{h=0}^{c=16} (\tau) f_{h=0}^{c=16} (\bar{\tau}) + 134912 f_{h=1}^{c=16} (\tau) f_{h=1}^{c=16} (\bar{\tau}) | b_0 = 1 + 32505856 f_{h=3/2}^{c=16} (\tau) f_{h=3/2}^{c=16} (\bar{\tau}) | b_1 = 1.$$

$$= 1 + \underbrace{2296} \frac{q^2}{2 \times 1 + 186 + 2108} + \underbrace{65536} \frac{q^3}{2 \times 1 + 186 + 14756 + 50592} + \underbrace{134912} q\bar{q} + \cdots.$$
‘Dual’ CFT Description [Mukhi]

• Ising model and Babymonster CFT

<table>
<thead>
<tr>
<th>Ising model</th>
<th>$c$</th>
<th>$h_1$</th>
<th>$h_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ising model</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>Baby monster CFT</td>
<td>$\frac{47}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{31}{16}$</td>
</tr>
<tr>
<td>Sum</td>
<td>24</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ j(\tau) - 744 = \chi_{\text{VB}^{(0)}_{(0)}}(\tau)\chi_{\text{Ising}}(\tau) + \chi_{\text{VB}^{(1)}_{(1)}}(\tau)\chi_{h=\frac{1}{2}}(\tau) + \chi_{\text{VB}^{(3)}_{(3)}}(\tau)\chi_{h=\frac{1}{16}}(\tau) \]

• $c = 8$ and $c = 16$ CFTs are related by bilinear relation:

\[ j(\tau) - 744 = (f_{c=8|h=0}^{c=8}) (f_{c=16|h=0}^{c=16}) + \left( f_{h=1|h=0}^{c=8} \right) \left( f_{h=1\|c=16}^{c=16} \right) \left( f_{h=3/2|h=0}^{c=16} \right) \]

\[ + \left( f_{h=1/2|h=0}^{c=8} \right) \left( f_{h=3/2|h=0}^{c=16} \right) \]
Twist Gap With $\mathcal{N} = 1$ Supersymmetry

- The numerical bound ($c \leq 26$) with the Twist Gap.
Modular Differential Equation with $\Gamma^0(2)$

- Modular forms of the congruence subgroups are,

$$M_2(q) \equiv \vartheta_{01}(q)^4 - \vartheta_{10}(q)^4,$$
$$M_4^{(1)}(q) \equiv \vartheta_{00}(q)^8, \quad M_4^{(2)}(q) \equiv \vartheta_{10}(q)^8 + \vartheta_{01}(q)^8,$$
$$M_6^{(1)}(q) = \vartheta_{01}(q)^{12} - \vartheta_{10}(q)^{12} + \vartheta_{10}(q)^8 \vartheta_{01}(q)^4 - \vartheta_{10}(q)^4 \vartheta_{01}(q)^8,$$
$$M_6^{(2)}(q) = \vartheta_{01}(q)^{12} - \vartheta_{10}(q)^{12} + \vartheta_{10}(q)^4 \vartheta_{01}(q)^8 - \vartheta_{10}(q)^8 \vartheta_{01}(q)^4.$$  

- Using above modular forms, we construct the modified modular differential equation of order two and three.

\[
\mathcal{D}_\tau^2 f(\tau) + \mu_1 M_2(\tau) \mathcal{D}_\tau f(\tau) + \mu_2 M_4^{(1)}(\tau) f(\tau) + \mu_3 M_4^{(2)}(\tau) f(\tau) = 0.
\]

\[
\mathcal{D}_\tau^3 f(\tau) + \mu_1 M_2 \mathcal{D}_\tau^2 f(\tau) + \mu_2 M_4^{(1)}(\tau) \mathcal{D}_\tau f(\tau) + \mu_3 M_4^{(2)}(\tau) \mathcal{D}_\tau f(\tau) + \mu_4 M_6^{(1)} f(\tau) + \mu_5 M_6^{(2)} f(\tau) = 0.
\]
Solutions of the MDE [Höhn]

• The solutions of the 2\textsuperscript{nd} modular differential equation with $\Gamma^0(2)$ are,

<table>
<thead>
<tr>
<th>$c$</th>
<th>Solution of 2\textsuperscript{nd} order MDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>$f(\tau) = 1 + 3q^{\frac{1}{2}} + 3q + 4q^{\frac{3}{2}} + 9q^2 + \cdots$</td>
</tr>
<tr>
<td>$2$</td>
<td>$f(\tau) = 1 + 4q^{\frac{1}{2}} + 6q + 8q^{\frac{3}{2}} + 17q^2 + \cdots$</td>
</tr>
<tr>
<td>$\frac{5}{2}$</td>
<td>$f(\tau) = 1 + 5q^{\frac{1}{2}} + 10q + 15q^{\frac{3}{2}} + 30q^2 + \cdots$</td>
</tr>
<tr>
<td>$3$</td>
<td>$f(\tau) = 1 + 6q^{\frac{1}{2}} + 15q + 26q^{\frac{3}{2}} + 51q^2 + \cdots$</td>
</tr>
<tr>
<td>$\frac{17}{2}$</td>
<td>$f(\tau) = 1 + 255q + 221q^{\frac{3}{2}} + 4216q^2 + 4114q^{\frac{5}{2}} + \cdots$</td>
</tr>
<tr>
<td>$9$</td>
<td>$f(\tau) = 1 + 261q + 456q^{\frac{3}{2}} + 4500q^2 + 8424q^{\frac{5}{2}} + \cdots$</td>
</tr>
</tbody>
</table>

• And, the solutions of the 3\textsuperscript{rd} modular differential equation is given by

<table>
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<tr>
<th>$c$</th>
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<tbody>
<tr>
<td>$16$</td>
<td>$f(\tau) = 1 + 7936q^{\frac{3}{2}} + 2296q^2 + 412672q^{\frac{5}{2}} + \cdots$</td>
</tr>
<tr>
<td>$\frac{33}{2}$</td>
<td>$f(\tau) = 1 + 7766q^{\frac{3}{2}} + 11220q^2 + 408507q^{\frac{5}{2}} + \cdots$</td>
</tr>
</tbody>
</table>
The Results

- $f(\tau)$ at $c = \frac{3}{2}, 2, \cdots, \frac{15}{2}$ are same with the partition function of $k$-copies free-fermion.

$$f_{c=\frac{3}{2}, k}(\tau) = \left( q^{1/48} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) \right)^k$$

- The following theories are realized on the numerical boundary!

<table>
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<tr>
<td>A</td>
<td>$c = \frac{3}{2}$</td>
<td>$\Delta_t = 1$</td>
<td>Three-fermion</td>
</tr>
<tr>
<td>B</td>
<td>$c = 2$</td>
<td>$\Delta_t = 1$</td>
<td>Four-fermion</td>
</tr>
<tr>
<td>C</td>
<td>$c = \frac{5}{2}$</td>
<td>$\Delta_t = 1$</td>
<td>Five-fermion</td>
</tr>
<tr>
<td>D</td>
<td>$c = 3$</td>
<td>$\Delta_t = 1$</td>
<td>Six-fermion</td>
</tr>
<tr>
<td>E</td>
<td>$c = 9$</td>
<td>$\Delta_t = 2$</td>
<td>?</td>
</tr>
<tr>
<td>F</td>
<td>$c = \frac{33}{2}$</td>
<td>$\Delta_t = 3$</td>
<td>?</td>
</tr>
</tbody>
</table>
Conclusion and Outlook

- The twist gap problem with holomorphic currents \((j \geq 1)\) successfully realize two-character RCFTs and three-character RCFTs on the numerical bound.

- We discover the partition function of \(c = 8, c = 16\) CFTs without Kac-Moody symmetry via modular bootstrap.

- The twist gap with \(\mathcal{N} = 1\) supersymmetry realize the system of free-fermion on the numerical boundary. Is there any underlying symmetry at \(c = 9\) and \(c = \frac{33}{2}\)?

- The modular bootstrap program with \(\mathcal{N} = 2\) supersymmetry can provide further information: Hellerman bound, Charge bound, Weak gravity conjecture ...