Frame-dependence of inflationary observables in scalar-tensor gravity

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Minimal Inflation

Inflation solves the **horizon** and **flatness** problems. When treated quantum-mechanically, it can also provide a mechanism for the generation of the perturbations that have resulted in the anisotropies observed in the CMB.

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] , \qquad (M_{\mathrm{Pl}}^2 \equiv 1)$$

Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{1}{3}\left[\frac{\dot{\phi}^2}{2} + V\right]$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2$$

Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

Slow-roll Approximation (HSRPs)

Slow-roll approximation:

$$V(\phi) \gg \dot{\phi}^2\,, \qquad |\ddot{\phi}| \ll |3H\dot{\phi}|,\, |V'|$$

First **Hubble slow-roll parameter** (HSRP)

$$\epsilon_H = -\frac{\dot{H}}{H^2} = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V}\,, \quad \frac{\ddot{a}}{a} = H^2(1-\epsilon_H)$$

Inflation ends **exactly** when $\epsilon_H = 1$.

Second HSRP

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}}$$

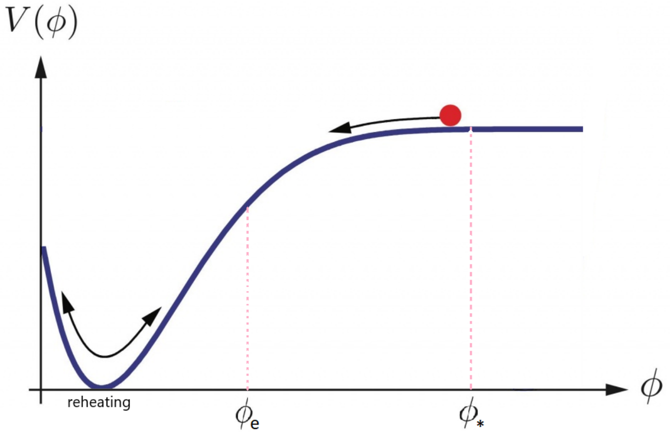
Friedmann and Klein-Gordon equations become

$$H^2 \approx \frac{1}{3}V(\phi)\,, \quad \dot{\phi} \approx -\frac{V'}{3H}\,.$$

Slow-roll Approximation (PSRPs)

The shape of the potential is encoded in the **potential slow-roll parameters**

$$\epsilon_V = \frac{1}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_V = \frac{V''}{V}$$



Number of e-folds and Inflationary Observables

The HSPRs can be related to the PSRPs through the Friedmann and Klein-Gordon equations. One finds Liddle *et al.* astro-ph/9408015

$$\epsilon_V = \epsilon_H \left(\frac{3-\eta_H}{3-\epsilon_H} \right)^2 , \qquad \eta_V = \sqrt{2\epsilon_H} \frac{\eta'_H}{3-\epsilon_H} + \left(\frac{3-\eta_H}{3-\epsilon_H} \right) (\epsilon_H + \eta_H)$$

Taylor expansion (1st order)

$$\epsilon_H \simeq \epsilon_V , \qquad \eta_H \simeq \eta_V - \epsilon_V$$

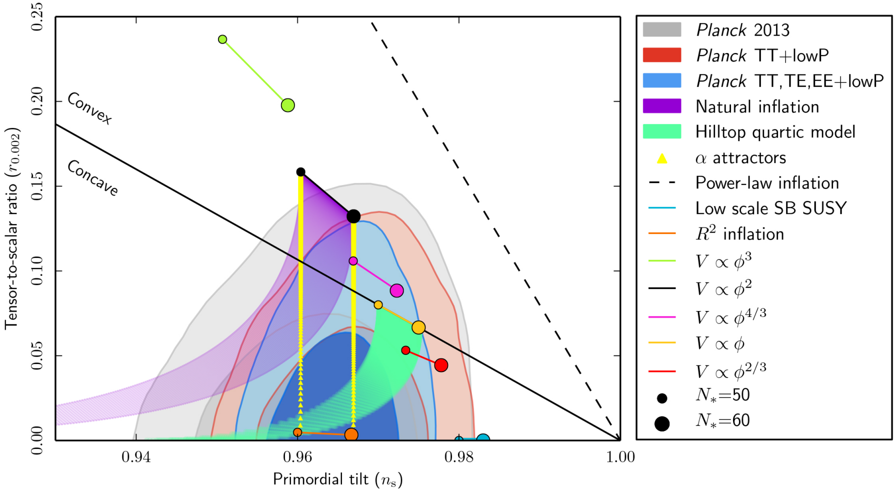
Scalar spectral index and tensor-to-scalar ratio

$$n_S = 1 - 4\epsilon_H + 2\eta_H \simeq 1 - 6\epsilon_V + 2\eta_V , \qquad r = 16\epsilon_H \simeq 16\epsilon_V$$

Number of e-folds

$$N(\phi) = \int_t^{t_{\rm end}} H \mathrm{d}t = \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_H}} \approx \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_V}} \sim 50 - 60$$

Planck 2015 Results



Planck Collaboration 1502.02114

Scalar-Tensor Gravity

$$S=\int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2}\mathcal{A}(\Phi)R-\frac{1}{2}\mathcal{B}(\Phi)g^{\mu\nu}\left(\nabla_\mu\Phi\right)\left(\nabla_\nu\Phi\right)-\mathcal{V}(\Phi)\right\}+S_m\left[e^{2\sigma(\Phi)}g_{\mu\nu},\chi\right]$$

A conformal rescaling and field redefinition can fix two of the model functions to get different parametrizations, e.g.,

- Jordan frame Boisseau-Esposito-Farèse-Polarski-Starobinski parametrization

$$\mathcal{A}=F(\phi),\quad \mathcal{B}=1,\quad \mathcal{V}=\mathcal{V}(\phi),\quad \sigma=0,$$

- Jordan frame Brans-Dicke-Bergmann-Wagoner parametrization

$$\mathcal{A}=\Psi,\quad \mathcal{B}=\frac{\omega(\Psi)}{\Psi},\quad \mathcal{V}=\mathcal{V}(\Psi),\quad \sigma=0,$$

- Einstein frame canonical parametrization

$$\mathcal{A}=1,\quad \mathcal{B}=2,\quad \mathcal{V}=\mathcal{V}(\varphi),\quad \sigma=\sigma(\varphi).$$

Outline of the Paper

Scalar-tensor gravity action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2}\mathcal{A}(\Phi)R - \frac{1}{2}\mathcal{B}(\Phi)g^{\mu\nu}\left(\nabla_\mu\Phi\right)\left(\nabla_\nu\Phi\right) - \mathcal{V}(\Phi)\right\} + S_m\left[e^{2\sigma(\Phi)}g_{\mu\nu},\chi\right]$$

What we do:

- Introduce quantities invariant under conformal rescaling <u>and</u> scalar field redefinition
- Write the field equations in terms of these invariants
- Calculate inflationary observables up to third order in the slow-roll approximation
- Express inflationary observables in terms of invariants

Transformation rules

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2}\mathcal{A}(\Phi)R - \frac{1}{2}\mathcal{B}(\Phi)g^{\mu\nu}\left(\nabla_\mu\Phi\right)\left(\nabla_\nu\Phi\right) - \mathcal{V}(\Phi)\right\} + S_m\left[e^{2\sigma(\Phi)}g_{\mu\nu}, \chi\right]$$

- Under conformal rescaling and scalar field reparametrization

$$g_{\mu\nu} = e^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}\,, \qquad \Phi = \bar{f}(\bar{\Phi})\,,$$

  the functions transform as Flanagan gr-qc/0403063

$$\begin{aligned}
\bar{\mathcal{A}}(\bar{\Phi}) &= e^{2\bar{\gamma}(\bar{\Phi})}\mathcal{A}\left(\bar{f}(\bar{\Phi})\right),\\
\bar{\mathcal{B}}(\bar{\Phi}) &= e^{2\bar{\gamma}(\bar{\Phi})}\left[(\bar{f}')^2\mathcal{B}\left(\bar{f}(\bar{\Phi})\right) - 6(\bar{\gamma}')^2\mathcal{A}\left(\bar{f}(\bar{\Phi})\right) - 6\bar{\gamma}'\bar{f}'\mathcal{A}'\right],\\
\bar{\mathcal{V}}(\bar{\Phi}) &= e^{4\bar{\gamma}(\bar{\Phi})}\mathcal{V}\left(\bar{f}(\bar{\Phi})\right),\\
\bar{\sigma}(\bar{\Phi}) &= \sigma\left(\bar{f}(\bar{\Phi})\right) + \bar{\gamma}(\bar{\Phi}),
\end{aligned}$$

- Use these rules to find combinations which remain invariant.

Scalar Invariants

Three basic independent quantities, invariant under rescaling and reparametrization Järv *et al.* 1411.1947:

Matter coupling invariant $$\mathcal{I}_m(\Phi) \equiv \frac{e^{2\sigma(\Phi)}}{\mathcal{A}(\Phi)}$$

Potential invariant $$\mathcal{I}_{\mathcal{V}}(\Phi) \equiv \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^2}$$

Field invariant $$\mathcal{I}_\phi(\Phi) \equiv \int \left(\frac{2\mathcal{A}\mathcal{B} + 3(\mathcal{A}')^2}{4\mathcal{A}^2}\right)^{1/2} \mathrm{d}\Phi$$

- $\mathrm{d}\mathcal{I}_m/\mathrm{d}\Phi \not\equiv 0$ means nonminimal coupling
- $\mathrm{d}\mathcal{I}_{\mathcal{V}}/\mathrm{d}\Phi \not\equiv 0$ means nonvanishing mass or self-interactions
- $\mathrm{d}\mathcal{I}_\phi/\mathrm{d}\Phi \not\equiv 0$ means Φ is dynamical--the integrand can be interpreted as an invariant volume form of the 1-dimensional space of the scalar field. Kuusk *et al.* 1605.07033,1509.02903

Induced gravity inflation:

$$\begin{aligned} \mathcal{A}(\Phi) &= \xi\Phi^2 \\ \mathcal{B}(\Phi) &= 1 \\ \sigma(\Phi) &= 0 \\ \mathcal{V}(\Phi) &= \lambda\left(\Phi^2 - v^2\right)^2 \\ \xi v^2 &= 1 \equiv M_{\rm Pl}^2 \end{aligned}$$

$$\mathcal{I}_\phi = \sqrt{\frac{1+6\xi}{2\xi}}\ln\left(\frac{\Phi}{v}\right)$$

Starobinsky inflation: $f(R) = R + bR^2$

$$\begin{aligned} \mathcal{A}(\Phi) &= \Phi \\ \mathcal{B}(\Phi) &= 0 \\ \sigma(\Phi) &= 0 \\ \mathcal{V}(\Phi) &= \frac{b}{2}\left(\frac{\Phi-1}{2b}\right)^2 \end{aligned}$$

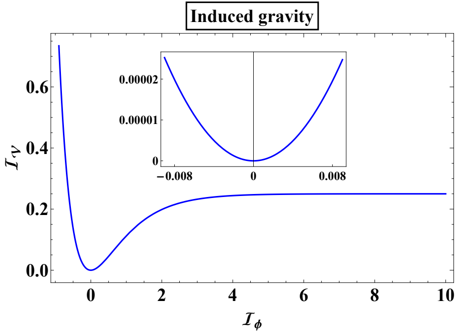
$$\mathcal{I}_\phi = \frac{\sqrt{3}}{2}\ln\Phi$$

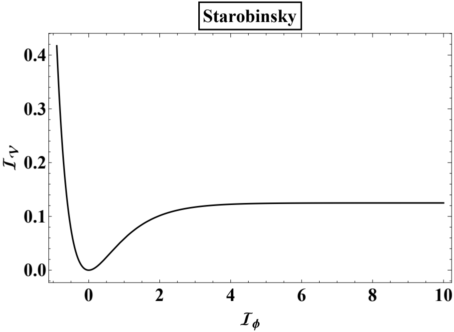
Classification of Inflationary Models

AK, Pappas, Tamvakis 1707.00984

$$\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\phi) = \frac{\lambda}{\xi^2}\left(1 - e^{-\sqrt{\frac{8\xi}{1+6\xi}}\mathcal{I}_\phi}\right)^2$$

$$\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\phi) = \frac{1}{8\,b}\left(1 - e^{-\frac{2}{\sqrt{3}}\mathcal{I}_\phi}\right)^2$$





Tensorial Invariants and FLRW Cosmology

Use the transformation properties of the model functions to define tensorial invariants:

Inv. Einstein: $\hat{g}_{\mu\nu} \equiv \mathcal{A}(\Phi) g_{\mu\nu}$, **Inv. Jordan:** $\bar{g}_{\mu\nu} \equiv e^{2\sigma(\Phi)} g_{\mu\nu} = \mathcal{I}_m \hat{g}_{\mu\nu}$

FLRW: $\mathrm{d}s^2 \equiv g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + (a(t))^2 \, \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \, .$

Due to homogeneity and isotropy, the scalar field and the invariants may only depend on the cosmic time, i.e. $\mathcal{I}_i = \mathcal{I}_i(t)$ etc. We can also put the invariant metric $\hat{g}_{\mu\nu}$ also in a FLRW form:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{t}} \equiv \frac{1}{\sqrt{\mathcal{A}}} \frac{\mathrm{d}}{\mathrm{d}t} \, , \qquad \hat{a}(\hat{t}) \equiv \sqrt{\mathcal{A}} \, a(t) \, .$$

An analogous transformation, namely

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}} = \frac{1}{\sqrt{\mathcal{I}_m}} \frac{\mathrm{d}}{\mathrm{d}\hat{t}} \, , \quad \bar{a}(\bar{t}) = \sqrt{\mathcal{I}_m} \hat{a}(\hat{t}) \, , \quad \bar{H} \equiv \frac{1}{\bar{a}} \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{t}} = \frac{1}{\sqrt{\mathcal{I}_m}} \left(\hat{H} + \frac{1}{2} \frac{\mathrm{d} \ln \mathcal{I}_m}{\mathrm{d}\hat{t}} \right)$$

takes the invariant metric $\bar{g}_{\mu\nu}$ to the FLRW form as well, but with different time parameter.

Slow-roll in the Einstein Frame

The field equations in terms of the invariants in the EF have the form

$$\hat{H}^2 = \frac{1}{3}\left[\left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}}\right)^2 + \mathcal{I}_\mathcal{V}\right], \quad \frac{\mathrm{d}\hat{H}}{\mathrm{d}\hat{t}} = -\left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}}\right)^2, \quad \frac{\mathrm{d}^2\mathcal{I}_\phi}{\mathrm{d}\hat{t}^2} = -3\hat{H}\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}} - \frac{1}{2}\frac{\mathrm{d}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi}$$

The standard HSRPs are

$$\hat{\epsilon}_0 \equiv -\frac{1}{\hat{H}^2}\frac{\mathrm{d}\hat{H}}{\mathrm{d}\hat{t}}, \qquad \hat{\eta} \equiv -\left(\hat{H}\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}}\right)^{-1}\frac{\mathrm{d}^2\mathcal{I}_\phi}{\mathrm{d}\hat{t}^2}$$

Introduce new series of Hubble slow-roll parameters:

$$\hat{\kappa}_0 \equiv \frac{1}{\hat{H}^2}\left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}}\right)^2, \quad \hat{\kappa}_1 \equiv \frac{1}{\hat{H}\hat{\kappa}_0}\frac{\mathrm{d}\hat{\kappa}_0}{\mathrm{d}\hat{t}} = 2\left(-\hat{\eta} + \hat{\epsilon}_0\right), \quad \hat{\kappa}_{i+1} \equiv \frac{1}{\hat{H}\hat{\kappa}_i}\frac{\mathrm{d}\hat{\kappa}_i}{\mathrm{d}\hat{t}}$$

Slow-roll in the Jordan Frame

The field equations in terms of the invariants in the JF have the form

$$\bar{H}^2 = \frac{1}{3}\left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}}\right)^2 + \bar{H}\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}} - \frac{1}{4}\left(\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}}\right)^2 + \frac{1}{3}\frac{\mathcal{I}_\mathcal{V}}{\mathcal{I}_m},$$

$$\frac{\mathrm{d}^2\mathcal{I}_\phi}{\mathrm{d}\bar{t}^2} = \left(-3\bar{H} + \frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}}\right)\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}} - \frac{1}{2\mathcal{I}_m}\frac{\mathrm{d}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi},$$

$$\frac{\mathrm{d}\bar{H}}{\mathrm{d}\bar{t}} = -\frac{1}{2}\bar{H}\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}} + \frac{1}{4}\left(\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}}\right)^2 - \left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}}\right)^2 + \frac{1}{2}\frac{\mathrm{d}^2\ln\mathcal{I}_m}{\mathrm{d}\bar{t}^2}$$

The standard HSRPs in the JF have the form

$$\bar{\epsilon}_0 \equiv -\frac{1}{\bar{H}^2}\frac{\mathrm{d}\bar{H}}{\mathrm{d}\bar{t}}, \qquad \bar{\eta} \equiv -\left(\bar{H}\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}}\right)^{-1}\frac{\mathrm{d}^2\mathcal{I}_\phi}{\mathrm{d}\bar{t}^2}$$

Introduce new series of Hubble slow-roll parameters:

$$\bar{\kappa}_0 \equiv \frac{1}{\bar{H}^2}\left(\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}}\right)^2, \quad \bar{\kappa}_1 \equiv \frac{1}{\bar{H}\bar{\kappa}_0}\frac{\mathrm{d}\bar{\kappa}_0}{\mathrm{d}\bar{t}} = 2\left(-\bar{\eta} + \bar{\epsilon}_0\right), \quad \bar{\kappa}_{i+1} \equiv \frac{1}{\bar{H}\bar{\kappa}_i}\frac{\mathrm{d}\bar{\kappa}_i}{\mathrm{d}\bar{t}}$$

$$\bar{\lambda}_0 \equiv \frac{1}{2\bar{H}}\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\bar{t}}, \quad \bar{\lambda}_1 \equiv \frac{1}{\bar{H}\bar{\lambda}_0}\frac{\mathrm{d}\bar{\lambda}_0}{\mathrm{d}\bar{t}}, \quad \bar{\lambda}_{i+1} \equiv \frac{1}{\bar{H}\bar{\lambda}_i}\frac{\mathrm{d}\bar{\lambda}_i}{\mathrm{d}\bar{t}}$$

Invariant Potential Slow-roll Parameters

The potential slow-roll parameters are manifestly invariant

$$\epsilon_V = \frac{1}{4\mathcal{I}_\mathcal{V}^2}\left(\frac{\mathrm{d}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi}\right)^2 , \quad \eta_V = \frac{1}{2\mathcal{I}_\mathcal{V}}\left(\frac{\mathrm{d}^2\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi^2}\right)$$

Higher-order parameters can be encoded in

$${}^n\beta_V \equiv \left(\frac{1}{2\mathcal{I}_\mathcal{V}}\right)^n \left(\frac{\mathrm{d}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi}\right)^{n-1} \left(\frac{\mathrm{d}^{(n+1)}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi^{(n+1)}}\right)$$

$$\zeta_V^2 = \frac{1}{4\mathcal{I}_\mathcal{V}^2}\left(\frac{\mathrm{d}\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi}\right)\left(\frac{\mathrm{d}^3\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi^3}\right)$$

$$\rho_V^3 = \frac{1}{8\mathcal{I}_\mathcal{V}^3}\left(\frac{\mathrm{d}^2\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi^2}\right)\left(\frac{\mathrm{d}^4\mathcal{I}_\mathcal{V}}{\mathrm{d}\mathcal{I}_\phi^4}\right)$$

Einstein Frame Results

Using the Green's function method of Stewart, Gong astro-ph/0101225
Scalar spectral index:

$$\begin{aligned}\hat{n}_S =& 1 - 2\hat{\kappa}_0 - \hat{\kappa}_1 - 2\hat{\kappa}_0^2 + \alpha\hat{\kappa}_1\hat{\kappa}_2 + (2\alpha-3)\hat{\kappa}_0\hat{\kappa}_1 - 2\hat{\kappa}_0^3 + (6\alpha - 17 + \pi^2)\hat{\kappa}_0^2\hat{\kappa}_1 \\ &+ \left(-2 + \frac{\pi^2}{4}\right)\hat{\kappa}_1^2\hat{\kappa}_2 + \left(-\frac{\alpha^2}{2} + \frac{\pi^2}{24}\right)\hat{\kappa}_1\hat{\kappa}_2^2 + \left(-\alpha^2 + 3\alpha - 7 + \frac{7\pi^2}{12}\right)\hat{\kappa}_0\hat{\kappa}_1^2 \\ &+ \left(-\frac{\alpha^2}{2} + \frac{\pi^2}{24}\right)\hat{\kappa}_1\hat{\kappa}_2\hat{\kappa}_3 + \left(-\alpha^2 + 4\alpha - 7 + \frac{7\pi^2}{12}\right)\hat{\kappa}_0\hat{\kappa}_1\hat{\kappa}_2\end{aligned}$$

Tensor-to-scalar ratio:

$$\hat{r} = 16\hat{\kappa}_0\left[1 - \alpha\hat{\kappa}_1 + \left(-\alpha + 5 - \frac{\pi^2}{2}\right)\hat{\kappa}_0\hat{\kappa}_1 + \left(\frac{\alpha^2}{2} + 1 - \frac{\pi^2}{8}\right)\hat{\kappa}_1^2 + \left(\frac{\alpha^2}{2} - \frac{\pi^2}{24}\right)\hat{\kappa}_1\hat{\kappa}_2\right]$$

where $\alpha \equiv (2 - \ln 2 - \gamma) \simeq 0.729637$ and $\gamma \simeq 0.577216$.

Jordan Frame Results

Scalar spectral index:

$$
\begin{aligned}
\bar{n}_S =& 1 - 2\bar{\kappa}_0 - \bar{\kappa}_1 - 2\bar{\kappa}_0^2 - 2\bar{\lambda}_0\bar{\lambda}_1 + \alpha\bar{\kappa}_1\bar{\kappa}_2 - \bar{\kappa}_1\bar{\lambda}_0 - 4\bar{\kappa}_0\bar{\lambda}_0 + (2\alpha - 3)\bar{\kappa}_1\bar{\kappa}_0 - 2\bar{\kappa}_0^3 \\
&- 8\bar{\lambda}_0\bar{\kappa}_0^2 - 6\bar{\lambda}_0^2\bar{\kappa}_0 + (6\alpha - 17 + \pi^2)\bar{\kappa}_0^2\bar{\kappa}_1 - \bar{\kappa}_1\bar{\lambda}_0^2 + \left(-2 + \frac{\pi^2}{4}\right)\bar{\kappa}_1^2\bar{\kappa}_2 \\
&- 4\bar{\lambda}_0^2\bar{\lambda}_1 + 2\alpha\bar{\lambda}_0\bar{\lambda}_1^2 + \left(-\frac{\alpha^2}{2} + \frac{\pi^2}{24}\right)\bar{\kappa}_1\bar{\kappa}_2^2 + \left(-\alpha^2 + 3\alpha - 7 + \frac{7\pi^2}{12}\right)\bar{\kappa}_0\bar{\kappa}_1^2 \\
&+ 2\alpha\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2 + (6\alpha - 9)\bar{\lambda}_0\bar{\kappa}_0\bar{\kappa}_1 + (4\alpha - 4)\bar{\lambda}_0\bar{\lambda}_1\bar{\kappa}_0 + (\alpha + 1)\bar{\kappa}_1\bar{\lambda}_0\bar{\lambda}_1 + 2\alpha\bar{\lambda}_0\bar{\kappa}_1\bar{\kappa}_2 \\
&+ \left(-\alpha^2 + 4\alpha - 7 + \frac{7\pi^2}{12}\right)\bar{\kappa}_0\bar{\kappa}_1\bar{\kappa}_2 + \left(-\frac{\alpha^2}{2} + \frac{\pi^2}{24}\right)\bar{\kappa}_1\bar{\kappa}_2\bar{\kappa}_3
\end{aligned}
$$

Tensor-to-scalar ratio:

$$
\begin{aligned}
\bar{r} =& 16\,\bar{\kappa}_0\Bigg[1 + 2\bar{\lambda}_0 - \alpha\bar{\kappa}_1 + 3\bar{\lambda}_0^2 - 2\alpha\bar{\lambda}_0\bar{\lambda}_1 - 3\alpha\bar{\lambda}_0\bar{\kappa}_1 + \left(-\alpha + 5 - \frac{\pi^2}{2}\right)\bar{\kappa}_0\bar{\kappa}_1 \\
&+ \left(\frac{\alpha^2}{2} + 1 - \frac{\pi^2}{8}\right)\bar{\kappa}_1^2 + \left(\frac{\alpha^2}{2} - \frac{\pi^2}{24}\right)\bar{\kappa}_1\bar{\kappa}_2\Bigg]
\end{aligned}
$$

Equivalence of the Frames

Expanding the EF slow-roll parameters up to third order in the JF slow-roll parameters:

$$\hat{\kappa}_0 \approx \bar{\kappa}_0 + 2\bar{\kappa}_0\bar{\lambda}_0 + 3\bar{\kappa}_0\bar{\lambda}_0^2, \quad \hat{\kappa}_1 \approx \bar{\kappa}_1 + \bar{\kappa}_1\bar{\lambda}_0 + \bar{\kappa}_1\bar{\lambda}_0^2 + 2\bar{\lambda}_0\bar{\lambda}_1 + 4\bar{\lambda}_0^2\bar{\lambda}_1,$$

$$\hat{\kappa}_1\hat{\kappa}_2 \approx \bar{\kappa}_1\bar{\kappa}_2 + 2\bar{\kappa}_1\bar{\kappa}_2\bar{\lambda}_0 + \bar{\kappa}_1\bar{\lambda}_0\bar{\lambda}_1 + 2\bar{\lambda}_0\bar{\lambda}_1^2 + 2\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2,$$

$$\hat{\kappa}_0\hat{\kappa}_1\hat{\kappa}_2 \approx \bar{\kappa}_0\bar{\kappa}_1\bar{\kappa}_2\,, \quad \hat{\kappa}_1\hat{\kappa}_2\hat{\kappa}_3 \approx \bar{\kappa}_1\bar{\kappa}_2\bar{\kappa}_3.$$

Then, plugging the above in the EF expressions for the indices we find

$$\hat{n}_S = \bar{n}_S,$$

$$\hat{r} = \bar{r}.$$

Therefore, the spectral indices calculated in the EF and JF coincide. Since the Green's function method is valid up to arbitrary order in the slow-roll expansion, we expect the equivalence between the spectral indices in the JF and EF to also hold to all orders.

Invariant Expressions for the Inflationary Observables

Scalar spectral index:

$$n_S = 1 - 6\epsilon_V + 2\eta_V + \left(24\alpha - \frac{10}{3}\right)\epsilon_V^2 - (16\alpha + 2)\,\epsilon_V\eta_V + \frac{2}{3}\eta_V^2 + \left(2\alpha + \frac{2}{3}\right)\zeta_V^2$$
$$- \left(90\alpha^2 - \frac{104}{3}\alpha + \frac{3734}{9} - \frac{87\pi^2}{2}\right)\epsilon_V^3 + \left(90\alpha^2 + \frac{4}{3}\alpha + \frac{1190}{3} - \frac{87\pi^2}{2}\right)\epsilon_V^2\eta_V$$
$$- \left(16\alpha^2 + 12\alpha + \frac{742}{9} - \frac{28\pi^2}{3}\right)\epsilon_V\eta_V^2 - \left(12\alpha^2 + 4\alpha + \frac{98}{3} - 4\pi^2\right)\epsilon_V\zeta_V^2$$
$$+ \left(\alpha^2 + \frac{8}{3}\alpha + \frac{28}{3} - \frac{13\pi^2}{2}\right)\eta_V\zeta_V^2 + \frac{4}{9}\eta_V^3 + \left(\alpha^2 + \frac{2}{3}\alpha + \frac{2}{9} - \frac{\pi^2}{12}\right)\rho_V^3$$

Tensor-to-scalar ratio:

$$r = 16\epsilon_V\left[1 - \left(4\alpha + \frac{4}{3}\right)\epsilon_V + \left(2\alpha + \frac{2}{3}\right)\eta_V + \left(16\alpha^2 + \frac{28}{3}\alpha + \frac{356}{9} - \frac{14\pi^2}{3}\right)\epsilon_V^2\right.$$
$$- \left(14\alpha^2 + 10\alpha + \frac{88}{3} - \frac{7\pi^2}{2}\right)\epsilon_V\eta_V + \left(2\alpha^2 + 2\alpha + \frac{41}{9} - \frac{\pi^2}{2}\right)\eta_V^2$$
$$\left. + \left(\alpha^2 + \frac{2}{3}\alpha + \frac{2}{9} - \frac{\pi^2}{12}\right)\zeta_V^2\right]$$

Number of e-folds

The number of e-folds is usually defined in the EF as

$$\mathrm{d}\hat{N} \equiv \hat{H}\mathrm{d}\hat{t} = \mathrm{d}\ln\hat{a} = -\frac{1}{\sqrt{\hat{\kappa}_0}}\,\mathrm{d}\mathcal{I}_\phi = -\frac{1}{\sqrt{\hat{\epsilon}_0}}\,\mathrm{d}\mathcal{I}_\phi = -\frac{1}{\sqrt{\epsilon_H}}\,\mathrm{d}\mathcal{I}_\phi .$$

The number of e-folds in the JF becomes

$$\mathrm{d}\bar{N} = \mathrm{d}\hat{N} + \frac{1}{2}\,\mathrm{d}\ln\mathcal{I}_m = \left(-\frac{1}{\sqrt{\epsilon_H}} + \frac{1}{2}\frac{\mathrm{d}\ln\mathcal{I}_m}{\mathrm{d}\mathcal{I}_\phi}\right)\mathrm{d}\mathcal{I}_\phi .$$

Non-minimal Coleman-Weinberg Inflation

The model functions are Kannike, Racioppi, Raidal 1509.05423

$$\mathcal{A}(\Phi) = \xi\Phi^2\,, \quad \mathcal{B}(\Phi) = 1\,, \quad \sigma(\Phi) = 0$$

$$\mathcal{V}(\Phi) = \Lambda^4 + \frac{1}{8}\beta_{\lambda_\Phi}\left(\ln\frac{\Phi^2}{v_\Phi^2} - \frac{1}{2}\right)\Phi^4\,, \quad 1 = \xi v_\Phi^2$$

Minimize and rewrite the potential as

$$\mathcal{V}(\Phi) = \Lambda^4\left\{1 + \left[2\ln\left(\frac{\Phi^2}{v_\Phi^2}\right) - 1\right]\frac{\Phi^4}{v_\Phi^4}\right\}\,, \qquad \mathcal{I}_\phi = \sqrt{\frac{1+6\xi}{2\xi}}\ln\left(\frac{\Phi}{v_\Phi}\right)$$

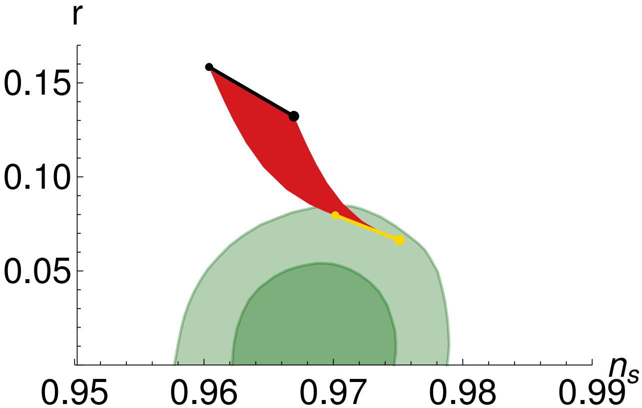
Invariant potential $\mathcal{I}_\mathcal{V}$ in terms of $\mathcal{I}_\phi$

$$\mathcal{I}_\mathcal{V} = \Lambda^4\left(4\sqrt{\frac{2\xi}{1+6\xi}}\,\mathcal{I}_\phi + e^{-4\sqrt{\frac{2\xi}{1+6\xi}}\mathcal{I}_\phi} - 1\right)$$

Non-minimal Coleman-Weinberg Inflation

Interpolates between quadratic and linear inflation (which is an attractor solution AK, Marzola, Racioppi, Pappas, Tamvakis 1711.09861)

$$\mathcal{I}_\mathcal{V}|_{\xi\to 0} \sim 16\,\xi\,\Lambda^4\,\mathcal{I}_\phi^2\,, \qquad \mathcal{I}_\mathcal{V}|_{\xi\to\infty} \sim \frac{4}{\sqrt{3}}\,\Lambda^4\,\mathcal{I}_\phi.$$

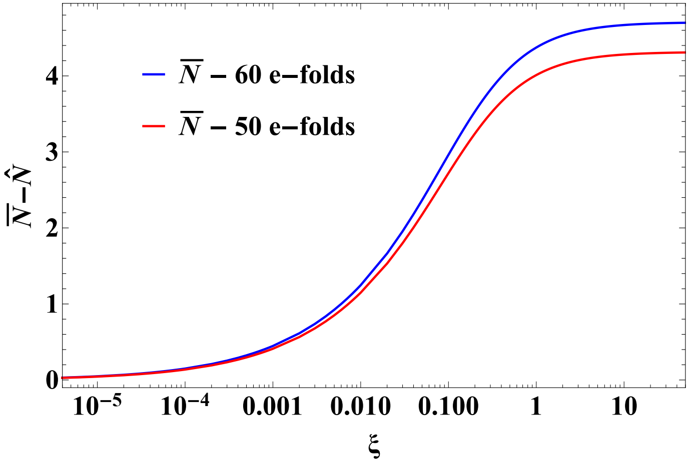


Non-minimal Coleman-Weinberg Inflation

	$n_S^{(\mathrm{I})}$	$n_S^{(\mathrm{III})}$	$r^{(\mathrm{I})}$	$r^{(\mathrm{III})}$	ξ
$\hat{N} = 60$	0.96702	0.96712	0.12782	0.12552	10^{-5}
$\bar{N} = 60$	0.96699	0.96709	0.12792	0.12562	10^{-5}
$\hat{N} = 60$	0.96935	0.96956	0.09655	0.09466	10^{-3}
$\bar{N} = 60$	0.96911	0.96933	0.09736	0.09544	10^{-3}
$\hat{N} = 60$	0.97451	0.97477	0.06796	0.06675	0.1
$\bar{N} = 60$	0.97320	0.97348	0.07148	0.07013	0.1
$\hat{N} = 60$	0.97482	0.97507	0.06716	0.06597	10
$\bar{N} = 60$	0.97276	0.97305	0.07264	0.07125	10

CORE 1612.08270 and LiteBIRD 1311.2847 will measure r with accuracy of 0.001.

Non-minimal Coleman-Weinberg Inflation



Related Results

Other observables/features computed in terms of invariants

- Parametrized post-Newtonian parameters for point mass and cosmological de Sitter attractor solutions Järv, Kuusk, Saal, Vilson 1411.1947
- Parametrized post-Newtonian parameters for homogeneous sphere Hohmann, Schärer 1708.07851
- Correspondence of frames even at the $\omega_{BD} \to 0$ limit Järv, Kuusk, Saal, Vilson 1504.02686

The formalism of invariants has been generalized to

- Multiscalar-tensor gravity fields Kuusk, Järv, Vilson 1509.02903
- Scalar-tensor gravity in the Palatini approach Kozak, Borowiec 1808.05598
- Scalar-torsion gravity Hohmann 1801.06531
- Higher-dimensional scalar-tensor gravity AK, Lykkas, Tamvakis 1803.04960

In progress...

- Anatomy of inflationary attractors Järv, AK, Rünkla, Saal

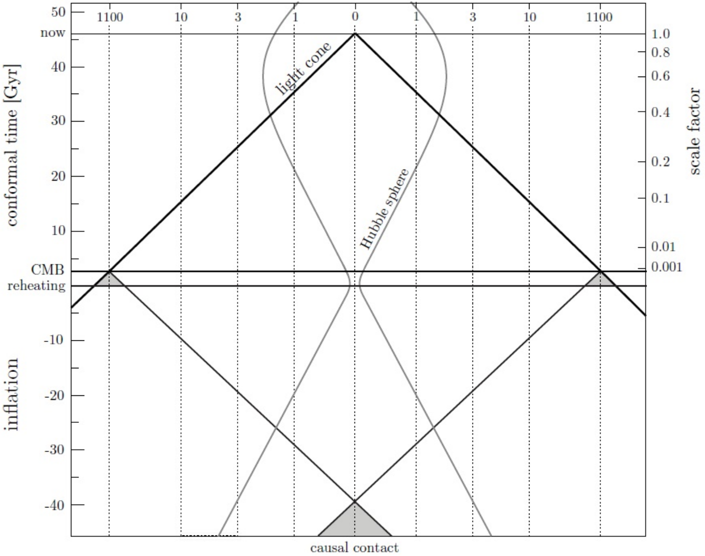
Summary and Conclusions

- Non-minimally coupled field can also drive inflation
- Such models belong to the class of scalar-tensor theories
- Frame issue can be circumvented by the introduction of invariants
- We calculated the inflationary observables up to third-order in the slow-roll approximation in both frames and showed their equivalence
- We expressed the observables in terms of PSRPs which are invariant
- However, number of e-folds differs in the two frames. We regard the Jordan e-folds as the fundamental one since it can be expressed in terms of all three principal invariants and also includes the Einstein definition

Thank you!

☺

Horizon Problem



Flatness Problem

The Friedmann equation (with zero cosmological constant),

$$H^2 = \frac{1}{3}\rho - \frac{\mathcal{K}}{a^2}\,, \qquad \Rightarrow \quad 1 - \Omega(t) = \frac{-\mathcal{K}}{(aH)^2}$$

For some earlier time t_i we can write

$$1 - \Omega(t_i) = [1 - \Omega(t_0)] \left(\frac{\dot{a}(t_0)}{\dot{a}(t_i)} \right)^2 .$$

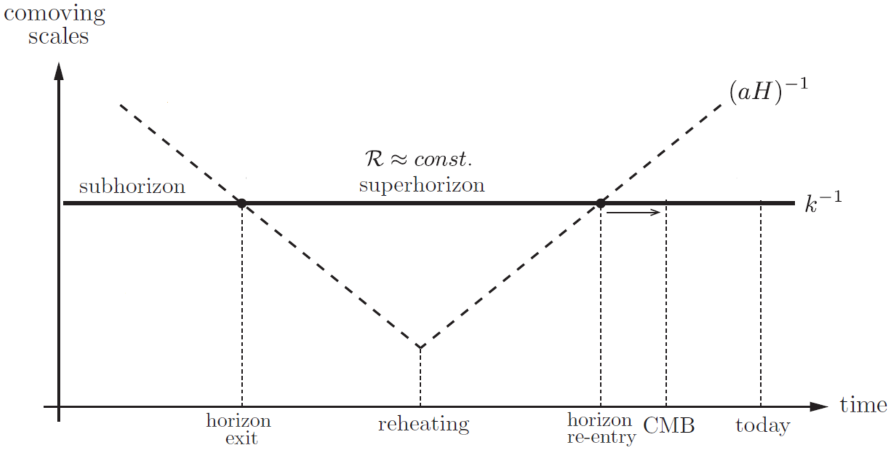
If we go back to the Planck time, $t_{\rm Pl} \sim 5 \times 10^{-44}{\rm s}$, we find

$$1 - \Omega(t_{\rm Pl}) < 10^{-64} .$$

During inflation, the Hubble parameter H_I is almost constant and the scale factor grows as

$$a(t) \simeq a_{\rm end} \exp\left[H_I\left(t - t_{\rm end}\right)\right] = \exp\left[-N(t)\right]\,, \quad 1 - \Omega \propto e^{-2N} \to 0$$

Horizon Evolution



Mukhanov-Sasaki Equation

The evolution of linear (tensor and scalar) curvature cosmological perturbations in a flat FLRW background and in the presence of a scalar inflaton field is governed by the *Mukhanov-Sasaki equation* (MSE) which reads

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}\tau^2} + \left(k^2 - \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2}\right)\nu = 0, \qquad \nu \equiv z\mathcal{R}_k$$

For tensor and scalar perturbations:

$$z = \frac{\bar{a}}{\sqrt{\mathcal{I}_m}} = \hat{a}, \qquad z = \sqrt{\frac{2}{\mathcal{I}_m}}\frac{\bar{a}}{\bar{H}\left(1-\bar{\lambda}_0\right)}\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\bar{t}} = \sqrt{2}\frac{\hat{a}}{\hat{H}}\frac{\mathrm{d}\mathcal{I}_\phi}{\mathrm{d}\hat{t}}.$$

Asymptotic solutions for ν (subhorizon and superhorizon limit)

$$\nu \to \begin{cases} \dfrac{1}{\sqrt{2k}}e^{-ik\tau} & \text{as} \quad -k\tau \to \infty, \\ A_k z & \text{as} \quad -k\tau \to 0. \end{cases}$$

Green's Function Method

By introducing the dimensionless variable $x \equiv -k\tau$ and redefining the field as $y \equiv \sqrt{2k}\nu$, the asymptotic solutions become

$$y \to \begin{cases} e^{-ix} & \text{as} \quad x \to \infty, \\ \sqrt{2k} A_k z & \text{as} \quad x \to 0. \end{cases}.$$

Also, by assuming the following ansatz for z:

$$z = \frac{1}{x} f(\ln x),$$

we can recast the MSE in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(1 - \frac{2}{x^2}\right) y = \frac{1}{x^2} g(\ln x) y,$$

where the function g is defined through

$$g(\ln x) = \frac{1}{f(\ln x)} \left[-3 \frac{\mathrm{d} f(\ln x)}{\mathrm{d} \ln x} + \frac{\mathrm{d}^2 f(\ln x)}{\mathrm{d}(\ln x)^2}\right].$$

Green's Function Method

The homogeneous solution with the appropriate asymptotic behavior at $x \to \infty$ is

$$y_0(x) = \left(1 + \frac{i}{x}\right) e^{ix}.$$

We can rewrite the MSE as an integral equation

$$y(x) = y_0(x) + \frac{i}{2} \int_x^{\infty} \mathrm{d}u \frac{1}{u^2} g(\ln u) y(u) \left[y_0^*(u) y_0(x) - y_0^*(x) y_0(u) \right]$$

Taylor-expanding xz around $x = 1$ in the following way:

$$xz = f(\ln x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} (\ln x)^n, \qquad f_n = \frac{\mathrm{d}^n (xz)}{\mathrm{d}(\ln x)^n}$$

The second-order power spectrum is then given in terms of the coefficients f_0, f_1 and f_2 as

$$P(k) = \frac{k^2}{(2\pi)^2} \frac{1}{f_0^2} \left[1 - 2\alpha \frac{f_1}{f_0} + \left(3\alpha^2 - 4 + \frac{5\pi^2}{12}\right) \left(\frac{f_1}{f_0}\right)^2 + \left(-\alpha^2 + \frac{\pi^2}{12}\right) \frac{f_2}{f_0} \right]$$