Pseudo-Riemannian structure of the noncommutative Standard Model

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Reference

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What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(S)$ of square-integrable spinors and the Dirac operator $D\!\!\!/_M = i\gamma^\mu\left(\partial_\mu + \omega_\mu\right)$ together with the γ_5 grading and the charge conjugation operator.

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Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$

$\mathcal{A}$ is a $*$-algebra represented on Hilbert space $\mathcal{H}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$, J is an antilinear isometry s.th. $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$.

$\mathcal{D}$ is essentially self-adjoint operator with compact resolvent and s.th. $[\mathcal{D}, a]$ is bounded for all $a \in \mathrm{Dom}(\mathcal{D})$ and $\mathcal{D}\gamma = -\gamma\mathcal{D}$.

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Moreover $\mathcal{D}J = \epsilon J\mathcal{D},\ J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$ defining KO-dimension.

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There are additional compatibility conditions for $\mathcal{D}$ and for γ.

Almost-commutative geometry for the Standard Model

$$\big(C^\infty(M)\otimes(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})), L^2(S)\otimes H_f, \not{D}_M\otimes 1+\gamma_5\otimes D_f, \gamma_5\otimes\gamma_f, J_M\otimes J_f\big)$$

$$H_f = H_L\oplus H_R\oplus H_L^c\oplus H_R^c$$

$$D_f\in M_{96}(\mathbb{C})$$

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Expansion of the Euclidean spectral action reproduces the effective action for the SM and allows for the expression of bosonic parameters by fermionic one.

Question

How to include Lorentzian structure on the finite part and what does it imply?

Finite pseudo-Riemannian spectral triple of signature (p, q)

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$$

1. $\mathcal{A}$ is a $*$-algebra represented on an Hilbert space $\mathcal{H}$
2. For $p+q$ even $\gamma^* = \gamma,\ \gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$
3. J is antilinear isometry with $[Ja^*J^{-1}, b] = 0$
4. $\beta = \beta^\dagger, \beta^2 = 1$ commuting with $\mathcal{A}$
5. $\mathcal{D}^\dagger = (-1)^p \beta \mathcal{D} \beta$
6. $[\mathcal{D}, a]$ is bounded
7. $\mathcal{D}\gamma = -\gamma\mathcal{D}$
8. $\mathcal{D}J = \epsilon J\mathcal{D},\ J^2 = \epsilon' \mathrm{id},\ J\gamma = \epsilon'' \gamma J$

$p-q$ mod 8	0	1	2	3	4	5	6	7
ϵ	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ'	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

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9. $\beta\gamma = (-1)^p \gamma\beta,\ \beta J = (-1)^{\frac{p(p-1)}{2}} \epsilon^p J\beta$
10. $\left[JaJ^{-1}, [\mathcal{D}, b]\right] = 0$
11. orientability : there exist $\mathcal{A} \ni a^i, a_0^i, ..., a_n^i,\ i = 1, ..., k$ s.th.

$$\sum_{i=1}^{k} Ja^i J^{-1} a_0^i [\mathcal{D}, a_1^i] ... [\mathcal{D}, a_n^k] = \begin{cases} \gamma, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

12. time-orientation : there exist $\mathcal{A} \ni b^i, b_0^i, ..., b_p^i,\ i = 1, ..., k'$ s.th.

$$\beta = \sum_{i=1}^{k'} Jb^i J^{-1} b_0^i [\mathcal{D}, b_1^i] ... [\mathcal{D}, b_p^k].$$

Motivation

Clifford algebra : $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} 1$

- $\gamma = i^{\frac{p-q}{2}} \gamma_1 ... \gamma_{p+q}$
- there exists unitary B s.th. $B\gamma_i = \epsilon \gamma_i^* B$ and $BB^* = \epsilon'$. Define $J\psi := B\psi^*$.
- $\mathcal{D} = -\sum_j \eta_{jj} \gamma_j \partial_j$
- $B\gamma = \epsilon'' \gamma B$
- $\beta = i^{\frac{1}{2}p(p-1)} \gamma_1 ... \gamma_p$
- $\beta \mathcal{D} \beta = (-1)^p \mathcal{D}^\dagger$

Riemannian from pseudo-Riemannian

$$\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger), \quad \mathcal{D}_- = \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger)$$

We get two Riemannian spectral triples $(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_\pm, J, \gamma)$, that differ by KO-dimensions, with additional selfadjoint grading β s.th.

$$\beta\mathcal{D}_\pm = \pm(-1)^p \mathcal{D}_\pm \beta,$$

$$\beta\gamma = (-1)^p \gamma\beta, \quad \beta J = (-1)^{\frac{1}{2}p(p-1)} \epsilon^p J\beta.$$

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$$\mathcal{D}_E = \mathcal{D}_+ + \mathcal{D}_-$$

$$J_E = J\beta, \quad \text{or} \quad J_E = J\beta\gamma$$

$(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_E, J_E, \gamma)$ is a Riemannian spectral triple of signature $(0, -(p+q))$.

The Standard Model

$$A_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad H_f = (H_l \oplus H_q) \oplus (H_{\bar{l}} \oplus H_{\bar{q}})$$

$$H_l = \langle \{\nu_R, e_R, (\nu_L, e_L)\} \rangle$$

$$H_q = \langle \{u_R, d_R, (u_L, d_L)\}_{c=1,2,3} \rangle$$

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$$\pi(\lambda, h, m) = \lambda \oplus \bar{\lambda} \oplus h \text{ on } H_l \text{ and } H_q$$

$$\pi(\lambda, h, m) = \bar{\lambda} \text{ on } H_{\bar{l}} \text{ and } 1_4 \otimes m \text{ on } H_{\bar{q}}$$

$$D_f = \begin{pmatrix} S & T^\dagger \\ T & \bar{S} \end{pmatrix}, \qquad S = \begin{pmatrix} S_l & \\ & S_q \otimes 1_3 \end{pmatrix}$$

$$T\nu_R = Y_R \bar{\nu}_R$$

The Standard Model

$$S_l = \left[\begin{array}{c|c|c|c} & & Y_\nu^\dagger & \\ \hline & & & Y_e^\dagger \\ \hline Y_\nu & & & \\ \hline & Y_e & & \end{array}\right], \quad S_q = \left[\begin{array}{c|c|c|c} & & Y_u^\dagger & \\ \hline & & & Y_d^\dagger \\ \hline Y_u & & & \\ \hline & Y_d & & \end{array}\right]$$

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- The existence of right neutrinos implies nonorientability of the geometry
- It is well known that the above Dirac operator is not unique within the model-building scheme of noncommutative geometry. Even the introduction of more constraints, like the second-order condition or Hodge-duality does not allow to exclude the terms, which would introduce the couplings between lepton and quarks and lead to the leptoquark fields

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There exists 0-cycle

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$(A_f, H_f, D_f, \gamma_f, J_f, \beta)$ could be seen as a Riemannian restriction of a real even pseudo-Riemannian spectral triple of signature $(0,2)$.

Possible pseudo-Riemannian structures for the Standard Model

Take as a Hilbert space $H \cong F \oplus F^*$ with

$$F \ni v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \in M_4(\mathbb{C}).$$

Vectors from H can be represented as $\begin{bmatrix} v \\ w \end{bmatrix}$, with $v, w \in M_4(\mathbb{C})$.

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We can identify $\mathrm{End}_{\mathbb{C}}(H)$ with $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$ and denote by e_{ij} a matrix with the the 1 in position (i, j) and zero everywhere else.

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Elements of the algebra $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ are represented by

$$\left[\begin{array}{c|c} \begin{matrix}\lambda & \\ & \bar{\lambda}\end{matrix} & 0 \\ \hline 0 & q \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & m \end{array}\right] \otimes e_{22} \otimes 1,$$

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The Dirac operator is of the form

$$D = D_0 + D_1,$$

where $D_1 = JD_0J^{-1}$.

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Let us now take the general form of a Dirac operator that satisfies an order-one condition. We have

$$D_0 = \begin{bmatrix} & M \\ M^\dagger & \end{bmatrix} \otimes e_{11} \otimes e_{11} + \begin{bmatrix} & N \\ N^\dagger & \end{bmatrix} \otimes e_{11} \otimes (1 - e_{11}) + \\ + \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \otimes e_{12} \otimes e_{11} + \begin{bmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{bmatrix} \otimes e_{21} \otimes e_{11},$$

where M, N, A, B are 2×2 complex matrices.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.

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Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
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Finally, with the $\beta = \pi(1, 1, -1) J \pi(1, 1, -1) J^{-1}$ we have no restriction whatsoever for M, N while then $B = 0$ and A needs to satisfy: $A = A \cdot \mathrm{diag}(1, -1)$. That leaves the possibility that A_{11} and A_{21} coefficients are present, providing no significant physical effects, and in particular leading only to terms involving a sterile neutrino.

Summary

- We proposed new definition of the finite pseudo-Riemannian spectral triples
- We proposed an alternative explanation of the observed quarks-leptons symmetry which prevents the $SU(3)$-breaking, as a shadow of the pseudo-Riemannian structure
- We proposed that the consistent model-building for the physical interactions and possible extensions of the Standard Model within the noncommutative geometry framework should use possibly the pseudo-Riemannian extension of finite spectral triples. We demonstrated that the pseudo-Riemannian framework allows for more restrictions and, in the discussed case introduces an extra symmetry grading, which we interpreted as the lepton-quark symmetry