Three new roads to the Planck scale

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Abstract

We present three new heuristic derivations of the Planck scale based on basic phenomena of relativistic gravity and quantum physics. The Planck scale quantities thus obtained are within one order of magnitude of the "standard" ones.

1 Introduction

The Planck scale, introduced by Max Planck [1] in 1899, predates the Planck law of blackbody radiation. The importance of the Planck units was realized by Eddington [2] and the idea that gravitation and quantum mechanics should be taken into account simultaneously at this scale was popularized by Wheeler [3, 4]. The themes that a fundamental system of units exists in nature, and that the values of these units can perhaps be derived in a super-theory, have been the subject of a large literature (see [5] for a popular exposition).

All derivations of the Planck scale more or less correspond to taking various combinations of Newton's constant G, the speed of light c, and the reduced Planck constant $\hbar$. Usually the Planck scale is deduced, following Planck, on a purely dimensional basis [1] or it is derived using the concept of a black hole in conjunction with that of a matter wave. By combining G, c, and $\hbar$ one obtains a unique quantity with the dimensions of a length, the "standard" Planck length

$$l_{\text{pl}} = \sqrt{\frac{G\hbar}{c^3}} = 1.6 \cdot 10^{-35}\,\text{m}\,. \tag{1}$$

By combining l_{pl} with G and c one then obtains the Planck mass

$$m_{\text{pl}} = \frac{l_{\text{pl}} c^2}{G} = \sqrt{\frac{\hbar c}{G}} = 2.2 \cdot 10^{-8}\,\text{kg}\,, \tag{2}$$

the Planck energy

$$E_{\text{pl}} = m_{\text{pl}} c^2 = \sqrt{\frac{\hbar c^5}{G}} = 1.3 \cdot 10^{19}\,\text{GeV}\,, \tag{3}$$

the Planck mass density

$$\rho_{\text{pl}} = \frac{m_{\text{pl}}}{l_{\text{pl}}^3} = \frac{c^2}{l_{\text{pl}}^2 G} = \frac{c^5}{\hbar G^2} = 5.2 \cdot 10^{96}\,\text{kg} \cdot \text{m}^{-3}\,, \tag{4}$$

and the Planck temperature

$$T_{\rm pl} = \frac{E_{\rm pl}}{k_{\rm B}} = \frac{l_{\rm pl} c^4}{k_{\rm B} G} = \sqrt{\frac{\hbar c^5}{G k_{\rm B}^2}} = 1.4 \cdot 10^{32}\,{\rm K}\,, \tag{5}$$

where $k_{\rm B}$ is the Boltzmann constant. We denote with $x_{\rm pl}$ the Planck scale value of a quantity x determined by dimensional analysis as in the above. Two alternative derivations appear in the literature, and six slightly more complicated roads to the Planck scale are discussed in [15]. Here we present three new ones [6].

1.1 A Planck size black hole

In what is probably the most popular derivation of the Planck scale, one postulates that a particle of mass m and Compton wavelength $\lambda = h/(mc)$, which has Planck energy, collapses to a black hole of radius $R_{\rm S} = 2Gm/c^2$ (the Schwarzschild radius of a spherical static black hole of mass m [7, 8]). Equating the Compton wavelength of this mass m to its black hole radius gives

$$m = \sqrt{\frac{hc}{2G}} = \sqrt{\pi}\, m_{\rm pl} \simeq 1.77\, m_{\rm pl}\,. \tag{6}$$

1.2 A universe of size comparable with its Compton wavelength

It is not compulsory to restrict to black holes. Why not use a relativistic universe instead of a black hole? This approach is followed in the following argument proposed in the popular book [11], but it does not appear in the technical literature.

Consider a spatially homogeneous and isotropic universe which, for simplicity, will be taken to be a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with line element

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)\,, \tag{7}$$

and with scale factor $a(t)$ and Hubble parameter[1] $H(t) \equiv \dot{a}/a$. The size of the observable universe is its Hubble radius cH^{-1} which is also, in order of magnitude, the radius of curvature (in the sense of four-dimensional curvature) of this spacetime. The mass m enclosed in a Hubble sphere is given by

$$mc^2 = \frac{4\pi}{3}\,\rho\left(H^{-1}c\right)^3 = \frac{H^{-1}c^5}{2G}\,, \tag{8}$$

where ρ is the cosmological energy density and we used the Friedmann equation [7, 8]

$$H^2 = \frac{8\pi G}{3c^2}\,\rho \tag{9}$$

[1] An overdot denotes differentiation with respect to the comoving time t.

(following standard notation, $\rho_{\rm pl}$ and ρ denote a *mass density* and an *energy density*, respectively). The Planck scale is reached when the Compton wavelength of the mass m is comparable with the Hubble radius, *i.e.*, when

$$\frac{c}{H} \sim \lambda = \frac{h}{mc} \,. \tag{10}$$

The expression (8) of m then gives

$$H^2 = \frac{c^5}{2Gh} \,. \tag{11}$$

Using again Eq. (9) yields the energy density

$$\rho \sim \frac{3c^7}{16\pi G^2 h} = \frac{3c^2}{32\pi^2}\,\rho_{\rm pl} \simeq 10^{-2} c^2 \rho_{\rm pl} \,, \tag{12}$$

from which the other Planck quantities (1)-(5) can be deduced by dimensional analysis:

$$l = \frac{c}{\sqrt{G\rho}} \simeq 10\, l_{\rm pl} \,, \qquad m = \frac{lc^2}{G} \simeq 10\, m_{\rm pl} \,, \tag{13}$$

$$E = mc^2 \simeq 10\, E_{\rm pl} \,, \qquad T = \frac{E}{k_{\rm B}} \simeq 10\, T_{\rm pl} \,. \tag{14}$$

2 Pair creation of particle-universes

A new approach to the Planck scale is the following. The idea of a universe which is quantum-mechanical in nature has been present in the literature for a long time and the use of the uncertainty principle to argue something about the universe goes back to Tryon's 1973 proposal that the universe may have originated as a vacuum fluctuation [12]. This notion of creation features prominently also in recent popular literature [13]. Consider now universes so small that they are ruled by quantum mechanics and regard the mass-energies contained in them as elementary particles. At high energies there could be production of pairs of such "particle-antiparticle universes". The Heisenberg uncertainty principle $\Delta E \Delta t \geq \hbar/2$ can be used by assuming that ΔE is the energy contained in a FLRW causal bubble of radius $R \sim H^{-1}c$ containing the energy $\Delta E \simeq 4\pi\rho R^3/3$. Setting $\Delta t \sim H^{-1}$ (the age of this very young universe), $\Delta E \Delta t \simeq \hbar/2$ gives

$$\frac{4\pi}{3}\,\rho \left(H^{-1}c\right)^3 H^{-1} \simeq \frac{\hbar}{2} \tag{15}$$

which can be rewritten as

$$\frac{8\pi G}{3}\,\rho\,\frac{c^3}{GH^4} = \hbar \,. \tag{16}$$

Equation (9) then yields the mass density

$$\frac{\rho}{c^2} \simeq \frac{3c^5}{8\pi G^2 \hbar} = \frac{3}{8\pi}\,\rho_{\rm pl} \,, \tag{17}$$

one order of magnitude smaller than the "standard" Planck mass density (4). The other Planckian quantities can then be derived from ρ and G, c, and h.

3 Scattering of a matter wave off the background curvature of spacetime

The second new alternative road to the Planck scale comes from the fact that, in general, waves propagating on a curved background spacetime scatter off it [14, 15, 16, 17]. This phenomenon is well known and can be interpreted as if these waves had an effective mass induced by the spacetime curvature. It is experienced by waves with wavelength λ comparable with, or larger than, the radius of curvature L of spacetime. High frequency waves do not "feel" the larger scale inhomogeneities of the spacetime curvature and, as is intuitive, essentially propagate as if they were in flat spacetime [15, 16, 7, 17]. The phenomenon is not dissimilar from the scattering experienced by a wave propagating through an inhomogeneous medium when its wavelength is comparable with the typical size of the inhomogeneities. We extrapolate the backscattering of a test-field wave by the fixed background curvature to a new regime in which this wave packet gravitates and, at the Planck scale, impedes its own propagation.

Consider now *a matter wave* associated with a particle of mass m and Compton wavelength $\lambda = h/(mc)$ scattering off the curvature of spacetime. The Planck scale can be pictured as that at which the spacetime curvature is caused by the mass m itself and the radius of curvature of spacetime due to this mass is comparable with the Compton wavelength. Essentially, high frequency waves do not backscatter but, at the Planck scale, there can be no waves shorter than the background curvature radius. Dimensionally, the length scale L associated with the mass m (the radius of curvature of spacetime) is given by $m = Lc^2/G$ and quantum and gravitational effects become comparable when $\lambda \sim L$, which gives

$$\frac{h}{(Lc^2/G)\,c} \sim L \tag{18}$$

or

$$L = \sqrt{\frac{Gh}{c^3}} = \sqrt{2\pi}\, l_{\mathrm{pl}} \simeq 2.51\, l_{\mathrm{pl}}\,. \tag{19}$$

In other words, if we pack enough energy into a matter wave so that it curves spacetime, the curvature induced by this wave will impede its own propagation when the Planck scale is reached. When the energy of this wave becomes too compact, the propagation of the matter wave is affected drastically.

4 Hawking evaporation of a black hole in a single burst

Hawking's discovery that, quantum mechanically, black holes emit a thermal spectrum of radiation allowed for the development of black hole thermodynamics by assigning a non-zero temperature to black holes [9]. In the approximation of a fixed black hole background and of a test quantum field in this spacetime, a spherical static black hole of mass m emits a thermal spectrum at the Hawking temperature

$$T_{\rm H} = \frac{\hbar c^3}{8\pi G k_{\rm B} m} \,. \tag{20}$$

As is well known, the emitted radiation peaks at a wavelength $\lambda_{\rm max}$ larger than the horizon radius $R_{\rm S} = 2Gm/c^2$. In fact, using Wien's law of displacement for blackbodies

$$\lambda_{\rm max} T_{\rm H} = b = \frac{hc}{4.9651 k_{\rm B}} \simeq 2.8978 \cdot 10^{-3}\,{\rm m}\cdot{\rm K} \tag{21}$$

and Eq. (20), one obtains

$$\lambda_{\rm max} = \frac{b}{T_{\rm H}} = \frac{8\pi^2}{4.9651}\,\frac{2Gm}{c^2} \simeq 15.90 R_{\rm S} \,. \tag{22}$$

Therefore, most of the thermal radiation is emitted at wavelengths comparable to, or larger than, the black hole horizon, giving a fuzzy image of the black hole.

Heuristically, one can extrapolate Hawking's prediction to a Planck regime in which the loss of energy is comparable with the black hole mass. Then the Planck scale is reached when the entire black hole mass m is radiated in a single burst of N particles of wavelength $\sim \lambda_{\rm max}$ and energy

$$E = \frac{hc}{\lambda_{\rm max}} \sim \frac{hc}{16 R_{\rm S}} = \frac{hc^3}{32 G m} \,. \tag{23}$$

This procedure provides a Planck scale of the same order of magnitude as the other procedures considered. Assuming N of order unity (say, $N=2$) and equating this energy with the black hole energy mc^2 yields

$$m \simeq \frac{NE}{c^2} \simeq \sqrt{\frac{hc}{16G}} = \sqrt{\frac{\pi}{8}}\, m_{\rm pl} \sim 0.627\, m_{\rm pl} \,. \tag{24}$$

5 Discussion

Although black holes are a most striking prediction of Einstein's theory of gravity [7, 8], they do not constitute the entire phenomenology of general relativity and there is no need to limit oneself to the black hole concept in heuristic derivations of the Planck scale. One can consider cosmology as well, which is appropriate since cosmology can only be discussed in the context of relativistic gravity. This approach leads to Barrow's derivation of the Planck scale [11] by considering, in a FLRW universe, a Hubble sphere with size comparable to the Compton wavelength of the mass it contains. Alternatively, one can consider the pair creation of causal bubbles so small that they can be treated as particles, or one can derive the Planck scale using the scattering of waves off the background curvature of spacetime which leads again, in order of magnitude, to the Planck scale when applied to matter waves. Alternatively, one can consider a black hole that evaporates completely in a single burst at the Planck scale. Of course, other approaches to the Planck scale are in principle conceivable. The exercice of imagining new heuristic avenues to the Planck scale can be quite stimulating.

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