A Dispersion Relation for Conformal Theories

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based on:
1703.00278;
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work in progress with:
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- Alex Maloney, Zarah Zarahee, Yan Gobeil

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Outline

1. A dispersion relation for CFT
   - Kramers-Kronig relations
   - ‘Absorptive part’

2. Applications to 3D critical Ising model

   1/J expansion

3. Application to AdS/CFT

   strongly coupled theories and bulk locality

Classical Kramers-Kronig relation:

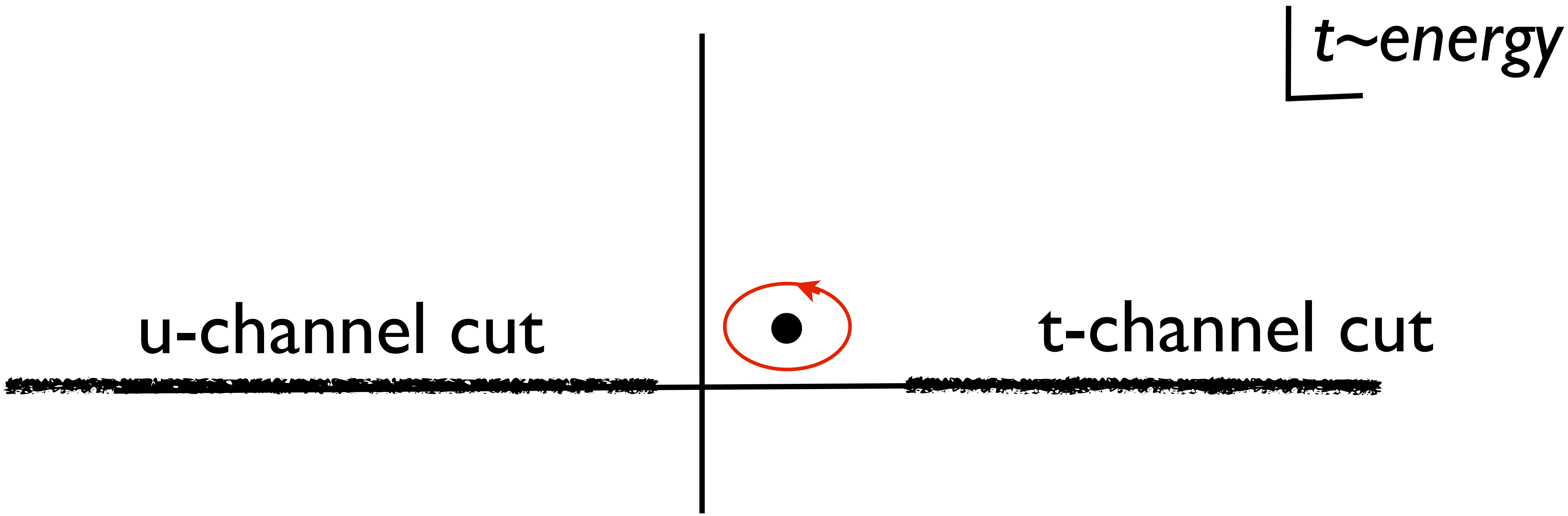
$$\epsilon(\omega) = 1 + \int_{-\infty}^{\infty} \frac{d\omega' \, \mathrm{Im}\, \epsilon(\omega')}{\pi(\omega' - \omega - i0)}$$

Re(ε) ~ phase velocity of light

Im(ε) ~ absorption by medium

Absorptive part determines propagation

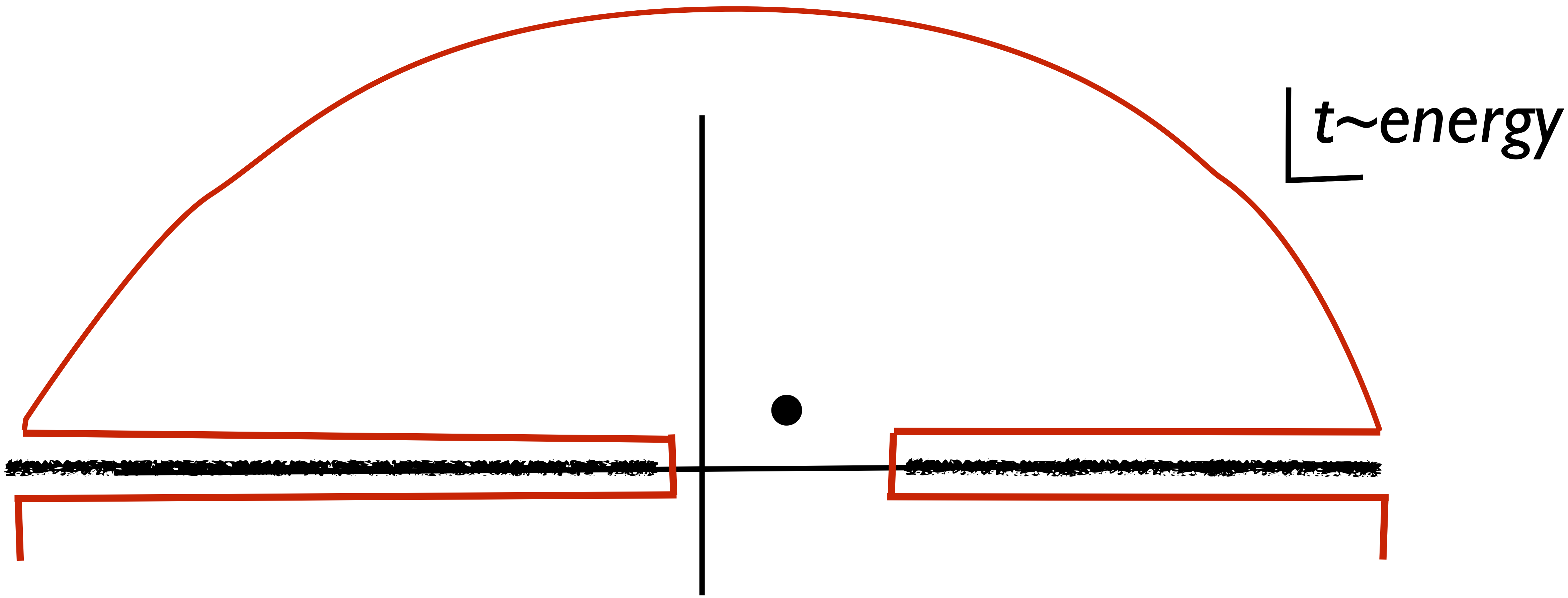
2->2 scattering « Dispersion relation »



simplest scenario: analytic in E-plane outside two cuts

$$\mathcal{M}(s,t) = \frac{1}{2\pi i}\oint \frac{dt'}{t'-t}\mathcal{M}(s,t')$$

2->2 scattering



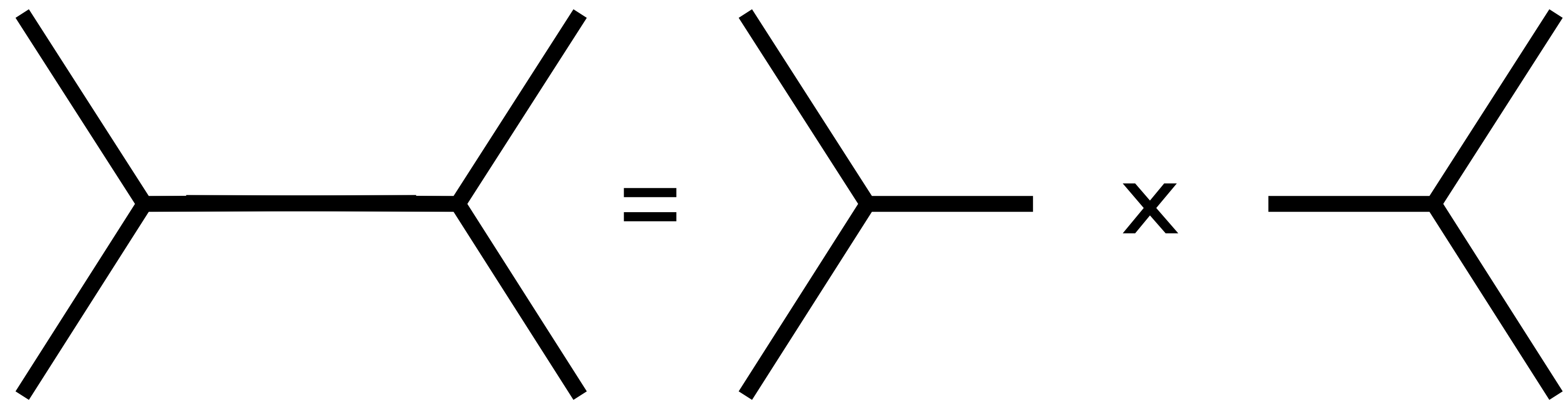
$$\Rightarrow \mathcal{M}(s,t) = \mathrm{Poly}_s(t) + \int_{t_{\min}}^{\infty} \frac{dt'}{\pi(t-t')} \mathrm{Disc}\, \mathcal{M}(s,t')$$

$$+(t \leftrightarrow u)$$

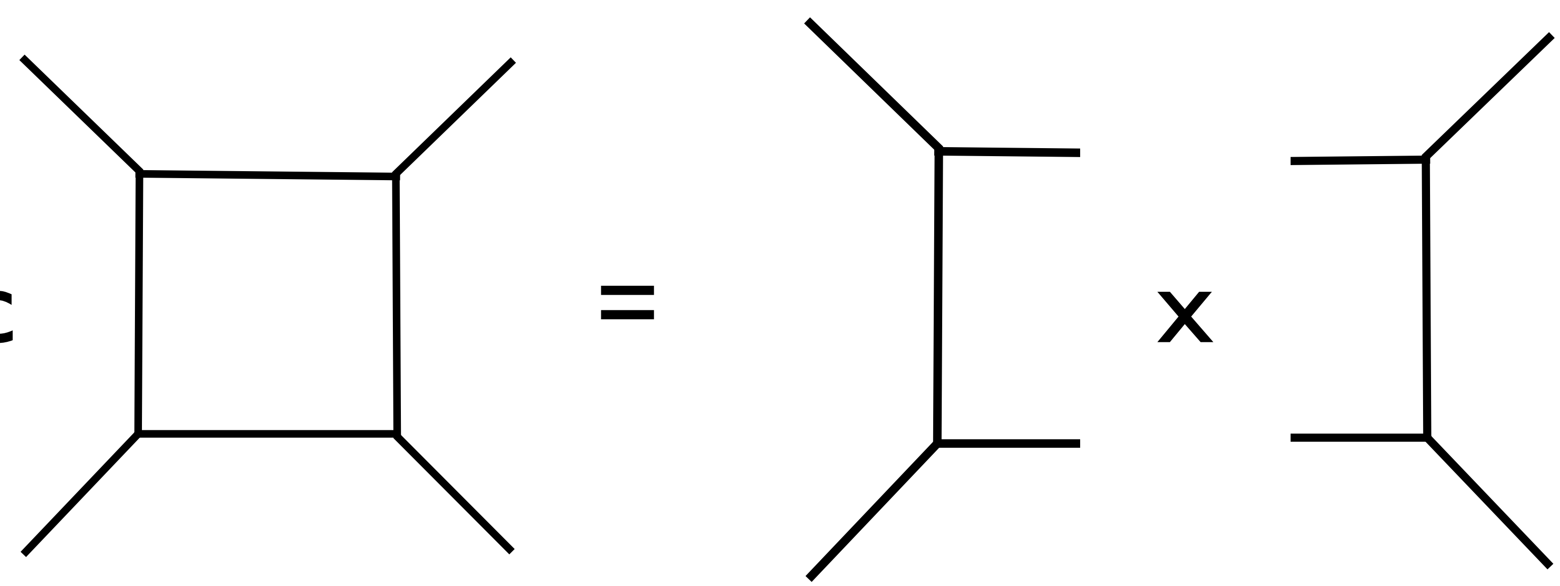
What is it good for?

absorptive part Disc M is often
easier to compute/measure

Tree-level: Disc



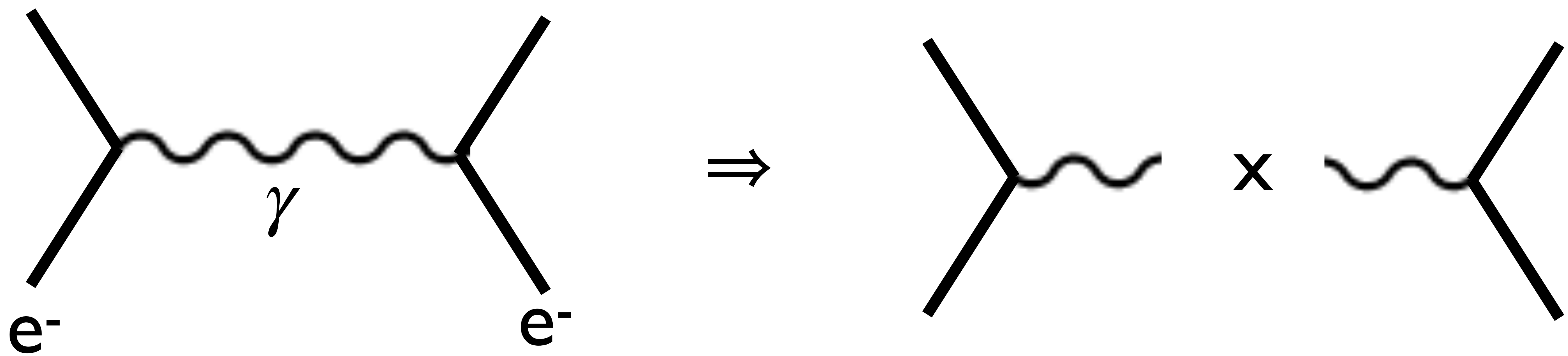
One-loop: Disc



Physical input: causality (⇒analyticity at complex energy)

Recall: = why forces must come from exchanging particles
(doesn't allow instantaneous interactions at a distance!)

Dispersion relations: reconstructs forces from exchanged stuff



cf many state-of-the-art amplitude techniques
(BCFW recursion, generalized unitarity…)

Conformal Field Theories

- Describe scale-invariant systems
  (ie. near phase transitions)

- Many interesting theories are near-conformal
  (ie. QCD at high energies)

- AdS/CFT: define quantum gravity in AdS

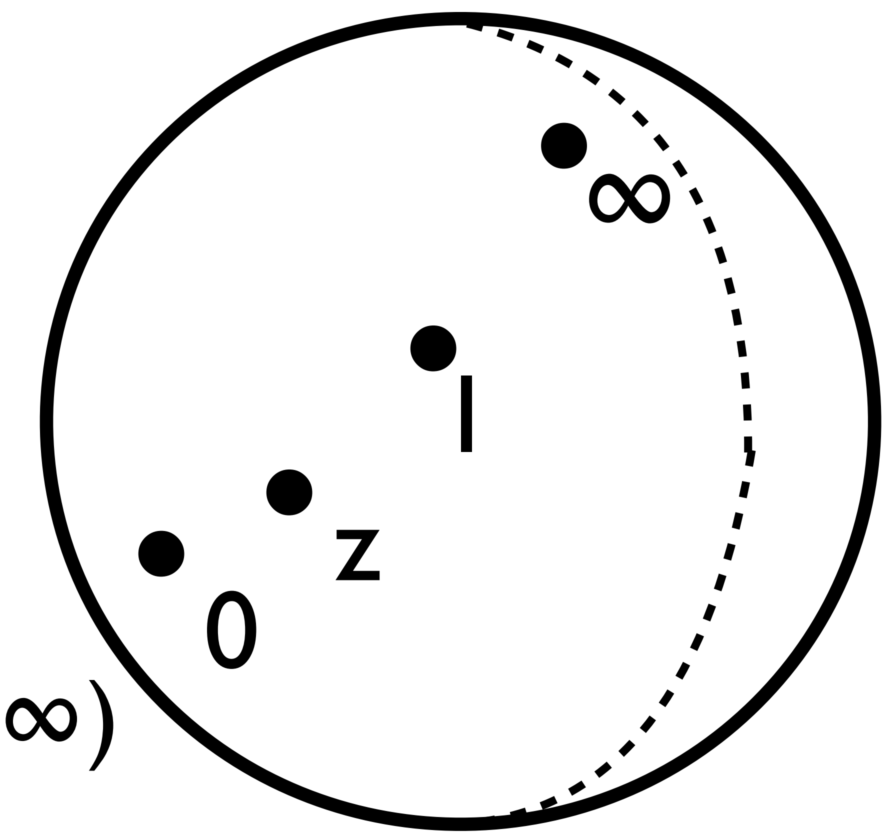
Conformal 2- and 3-point correlators: pure numbers

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}}$$

« critical exponent »

$$\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle \propto f_{ijk}$$

« OPE coefficients »

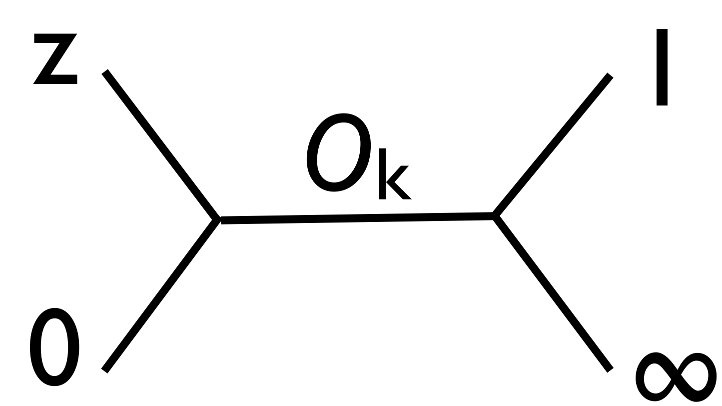


Key object is 4-point correlator

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = G(z,\bar{z})$$

(can always conformally map 3 points to 0,1,∞)

OPE: $G(z,\bar{z}) = \sum_k f_{12k} f_{34k} G_{J_k,\Delta_k}(z,\bar{z})$



CFTs don't have stable particle nor S-matrix.
We can't use standard dispersion relations.

Claim:

$$c(J,\Delta) = \int_{\Diamond} \left[\text{special function}\right] \times \left[\text{dDisc}\, G\right]$$

s-channel
OPE data

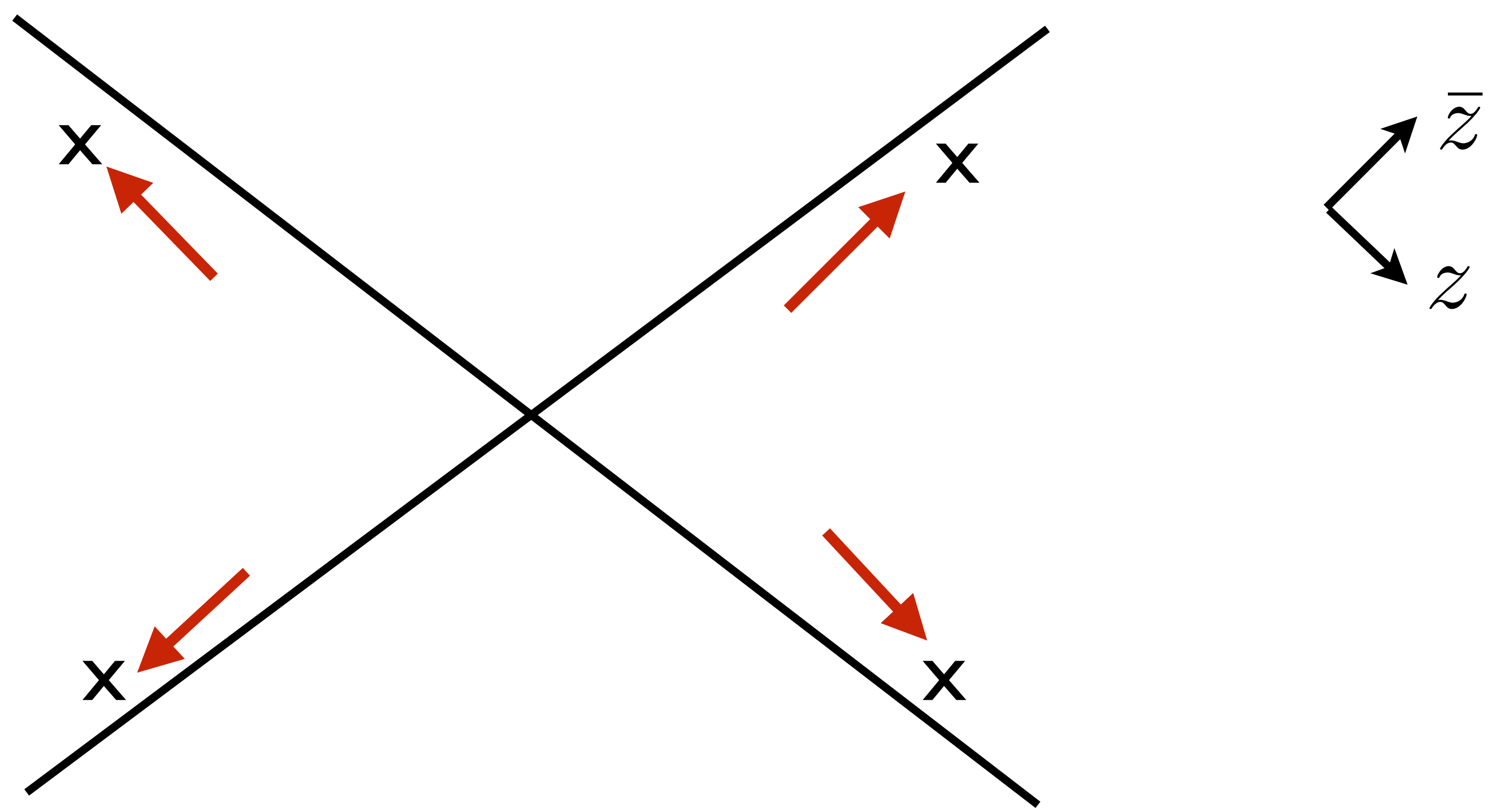
absorptive part

[SCH, '17]

see also: [Simmons-Duffin, Stanford&Witten '17]
[Simmons-Duffin& Kravchuk '18]

We'll study Lorentzian 4-point correlator in CFT_d

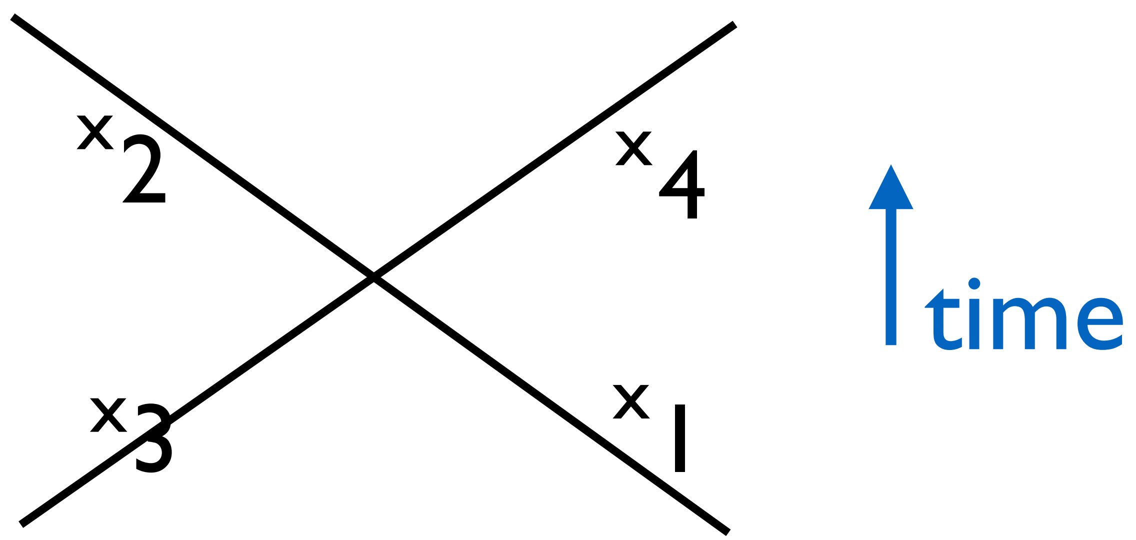
Crux is large (real/complex) energy $\Rightarrow$ large boost



[we'll stay inside Rindler wedges]

Intuition: Lorentzian correlator

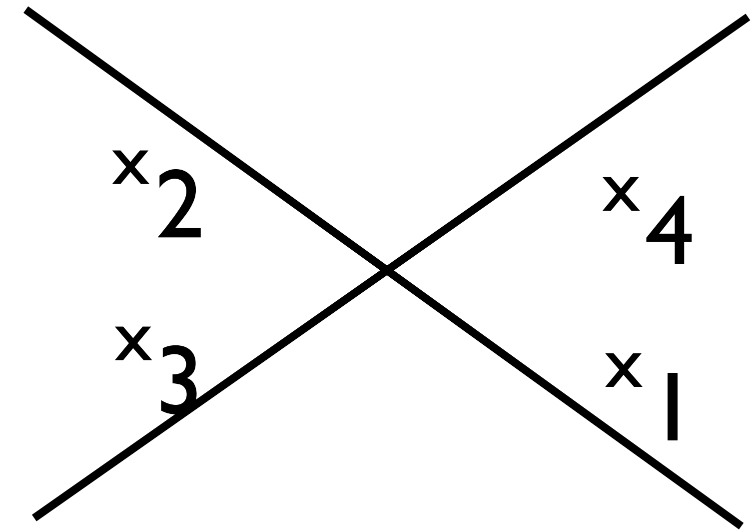
= amplitude for 13 to scatter to 24 final state



Bounded by 'amplitude without scattering':

$$|G| \leq G_E$$

What's 'absorptive part'?



$$\langle 0|T\phi_1\cdots\phi_4|0\rangle \equiv G = G_E + i\mathcal{M}$$

$$\langle 0|\bar{T}\phi_1\cdots\phi_4|0\rangle \equiv G^* = G_E - i\mathcal{M}^*$$

$$\langle 0|\phi_2\phi_3\phi_1\phi_4|0\rangle \equiv G_E$$

$$\mathrm{dDiscG} \equiv G_E - \tfrac{1}{2}G - \tfrac{1}{2}G^* = \text{“}\mathrm{Im}\,\mathcal{M}\text{”}$$

equal to double-commutator:

$$\mathrm{dDisc}\,G \equiv \tfrac{1}{2}\langle 0|[\phi_2,\phi_3][\phi_1,\phi_4]|0\rangle$$

Positive & bounded

cf: [Maldacena, Shenker&Stanford 'bound on chaos']
[Hartman,Kundu&Tajdini 'proof of ANEC']

What do we extract? OPE data

partial waves: $$a_j(s) = \int_{-1}^{1} d\cos\theta \, P_j(\cos(\theta)) \mathcal{M}(s, t(\cos\theta))$$

+

disp. relation: $$\mathcal{M}(s,t) = \int \frac{dt'}{\pi(t-t')} \mathrm{Im}\, \mathcal{M}(s,t') \quad +(t \leftrightarrow u)$$

=

analyticity in spin $$a_j(s) = \int_{1}^{\infty} d\cosh\eta \, Q_j(\cosh(\eta)) \mathrm{Im}\, \mathcal{M} \quad +(-1)^j (t \leftrightarrow u)$$



[Froissart-Gribov ~60]

backbone of Regge theory

‘special function’: fill the missing box

	Euclidean		Lorentzian	

Taylor series:

	E^J		E^{-J}	

Rotation symmetry:

SO(2)	$\cos(j\theta)$	$\longrightarrow$	$e^{-j\eta}$	SO(1,1)
SO(3)	$P_j(\cos\theta)$	$\longrightarrow$	$Q_j(\cosh\eta)$	SO(2,1)

Conformal symmetry:

SO(d+1,1)	$G_{j,\Delta}(z,\bar{z})$	$\longrightarrow$	$G_{\Delta+1-d,J+d-3}(z,\bar{z})$	SO(d,2)

CFT Froissart-Gribov formula

$$c(J,\Delta) = \int_{\diamondsuit} \left[\text{Inverse block}\right] \times \left[\text{dDisc}\, G\right]$$

s-channel
OPE coefficients

block with
J and Δ
exchanged

absorptive
part

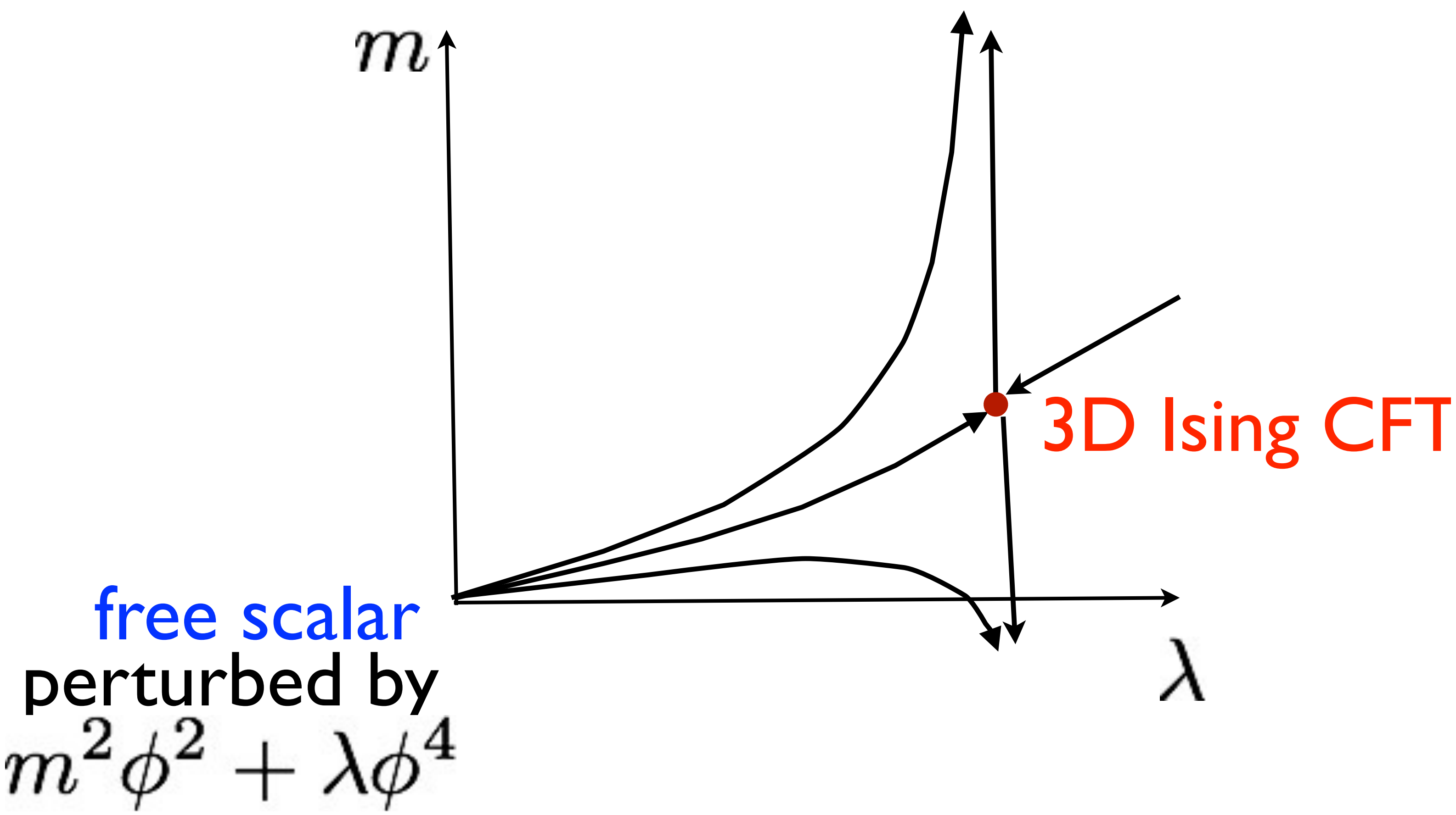
converges for J>1 (boundedness in Regge limit)

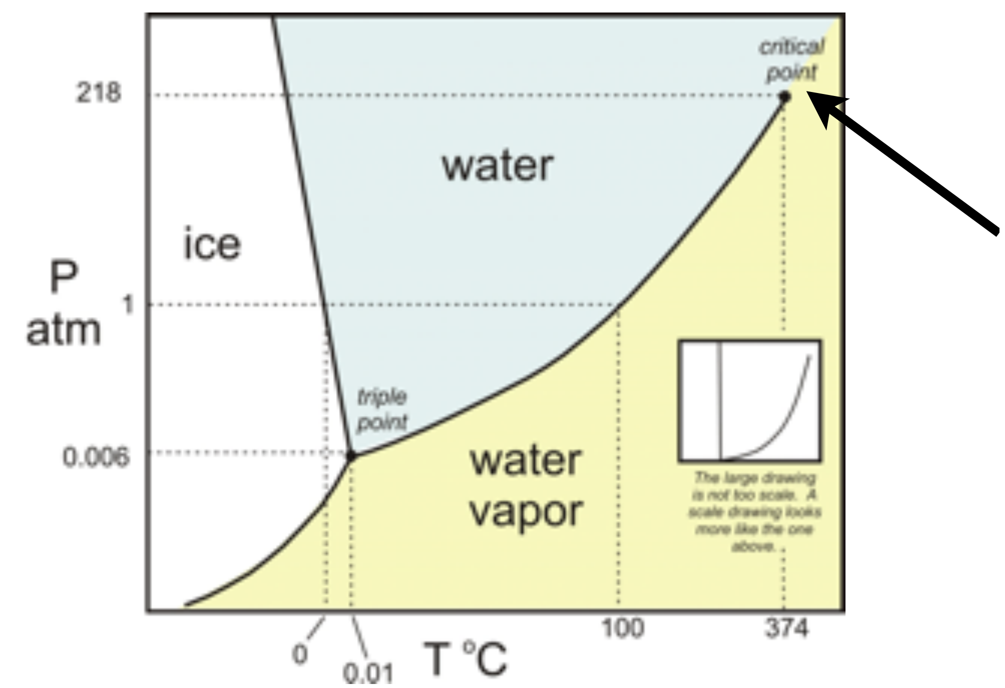
[SCH '17]

Application to 3D Ising

3D Ising Model: IR fixed point of Z2-symmetric scalar field theory

$$L = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4 \; \left[+ \kappa\phi^6 \ldots \right]$$

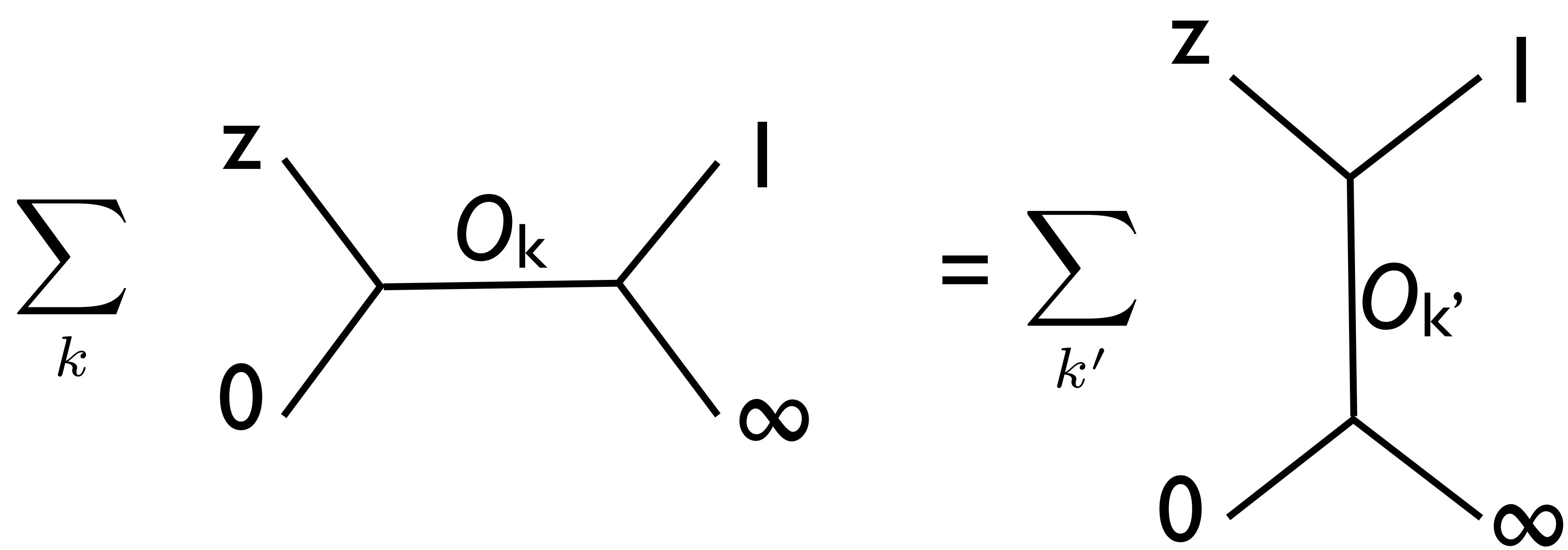




[slide from Rychkov]

Lightest operators:
$$\Delta_\sigma = 0.5181489(10), \quad Z_2 \text{ odd}$$
$$\Delta_\epsilon = 1.412625(10), \quad Z_2 \text{ even}$$

Lots of data available from *numerical bootstrap*.
(Input: series expansions must match)



spin & $\mathbb{Z}_2$	name	Δ	OPE coefficient
$\ell = 0,\ \mathbb{Z}_2 = -$	σ	$0.518154(15)$	
$\ell = 0,\ \mathbb{Z}_2 = +$	ϵ	$1.41267(13)$	$f^2_{\sigma\sigma\epsilon} = 1.10636(9)$
	ϵ'	$3.8303(18)$	$f^2_{\sigma\sigma\epsilon'} = 0.002810(6)$
$\ell = 2,\ \mathbb{Z}_2 = +$	T	3	$c/c_{\text{free}} = 0.946534(11)$
	T'	$5.500(15)$	$f^2_{\sigma\sigma T'} = 2.97(2) \times 10^{-4}$

[El-Showk,Paulos,Poland,
Rychkov,Simmons-Duffin&Vichi '14]

Large-spin expansion

Organizing principle for CFT spectrum: analyticity in spin

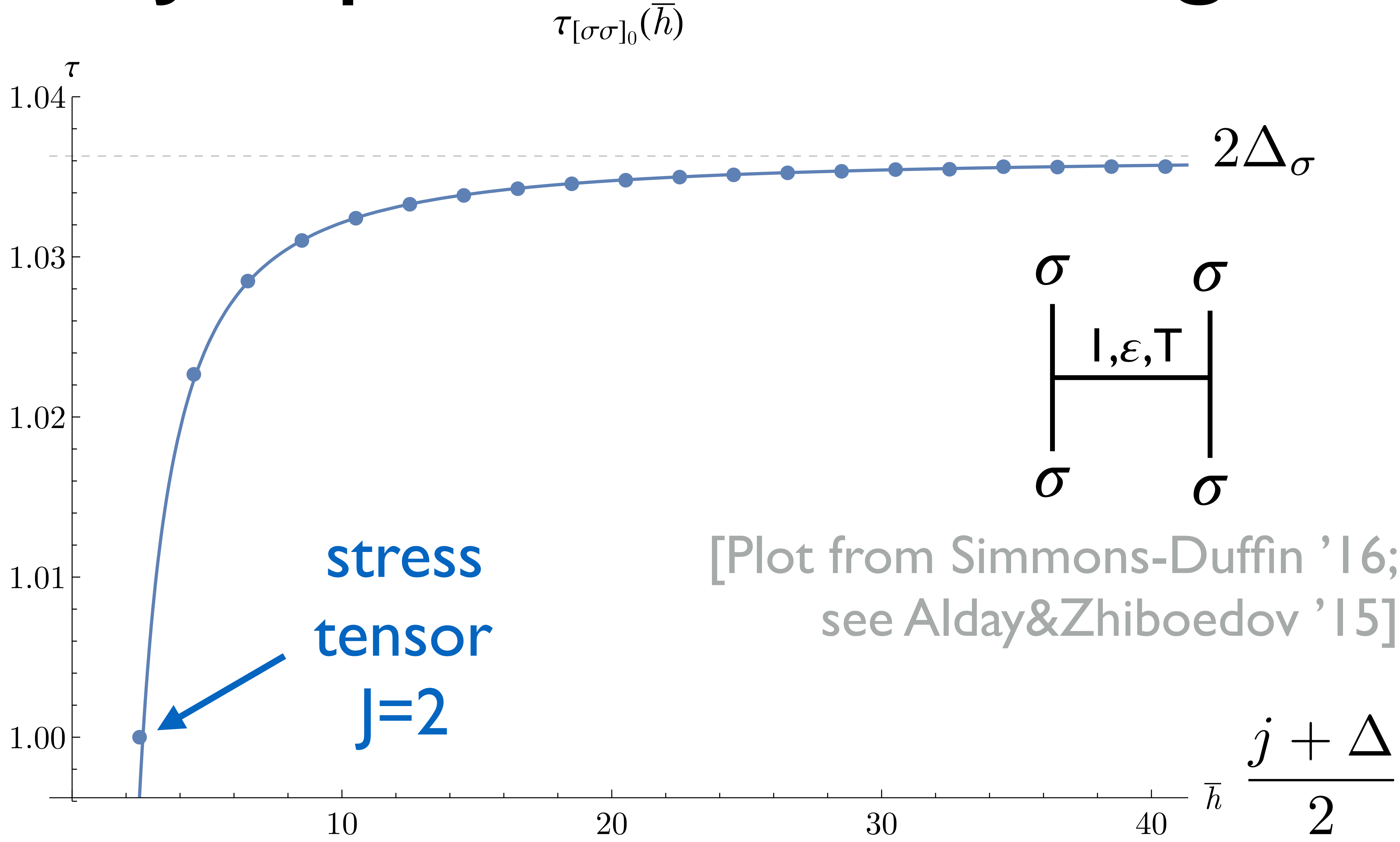
Easiest at large-J: integral pushed to corner $(z,\bar{z}) \to (0,1)$

large spin in s-channel $\leftrightarrow$ low twist in t-channel

$\Rightarrow$ Solve crossing in asymptotic series in 1/J

[Komargodski&Zhiboedov,
Fitzpatrick,Kaplan,Poland&Simmons-Duffin,
Alday&Bissi&…,
Kaviraj,Sen,Sinha&…,
Alday,Bissi,Perlmutter&Aharony,…]

1/J expansion in 3D Ising



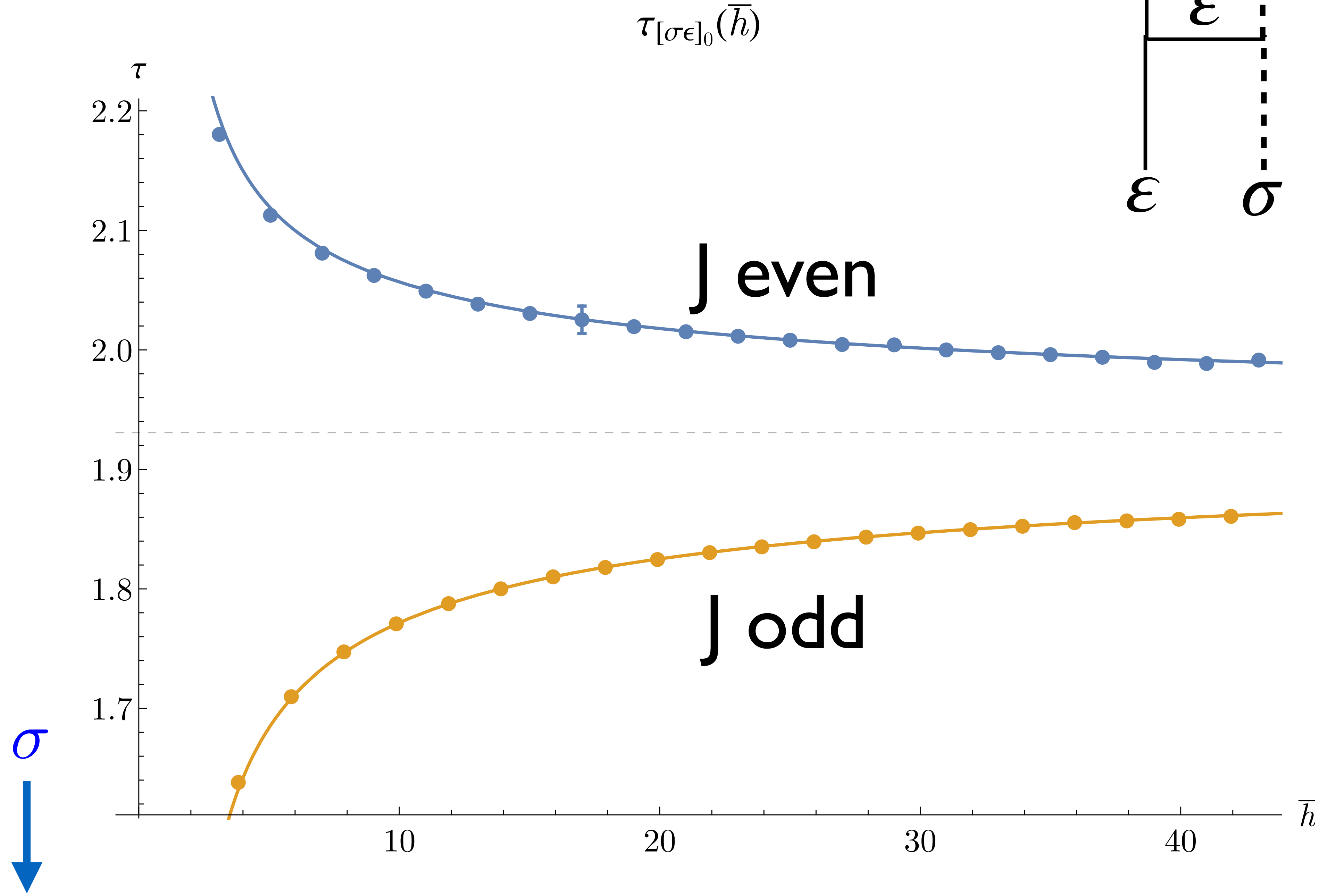
new: -formula shows there can't be outliers

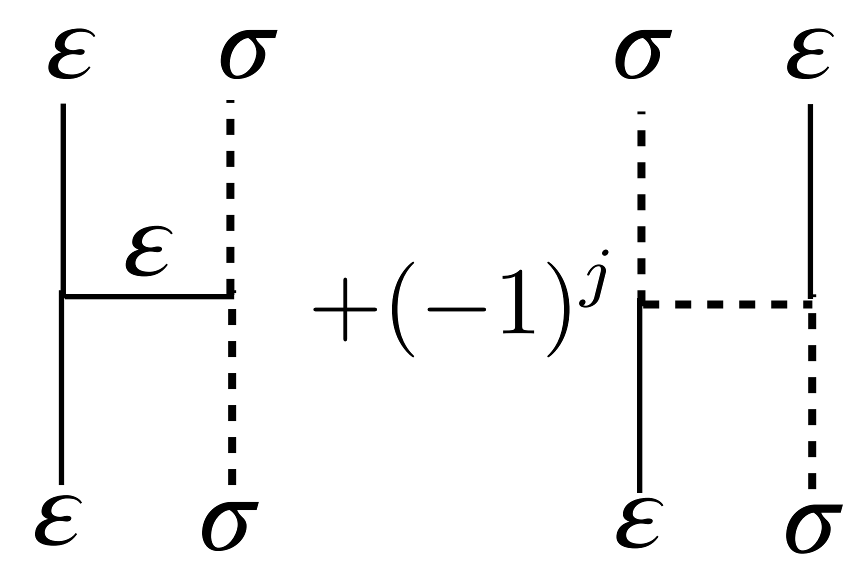
-all states (at least with J>1) must lie on trajectory

Best Before J> 1

What about J=0?

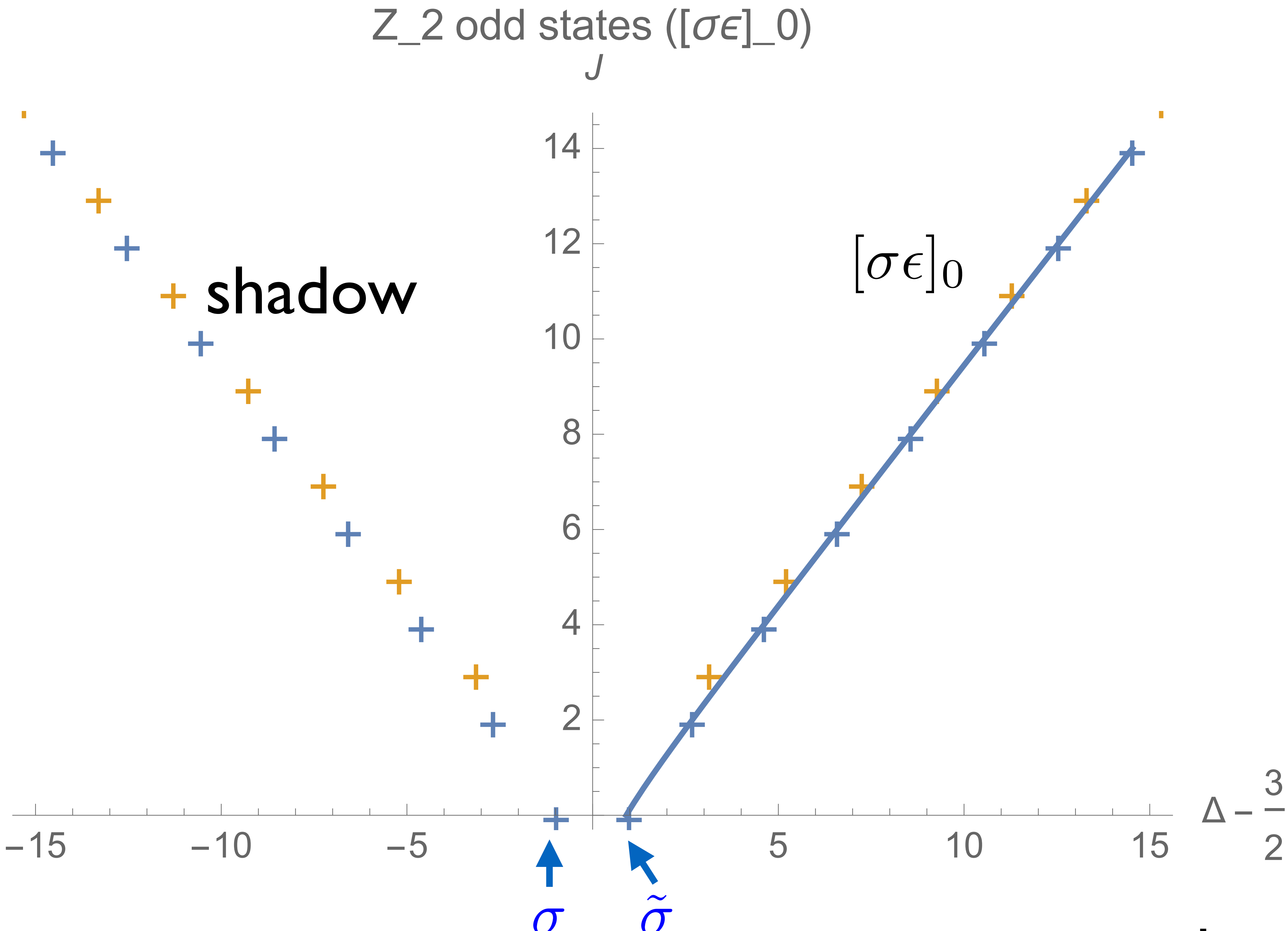
Z_2-odd operators





Works great for J>1, but $\Delta_\sigma = 0.51$ seems not even close!

What's analytic in spin is shadow symmetric $(\Delta, d - \Delta)$



more seems to be true!

[SCH, Gobeil, Maloney& Zahraee, in progress]

Application to AdS/CFT

Theories with AdS gravity duals have:

- Large-N expansion (small $\hbar$ in AdS)
- Few light single-traces, all with small spin ≤ 2 (up to a very high dimension $\Delta_{\text{gap}} \gg 1$) [HPPS '09]

simple statement for dDisc:

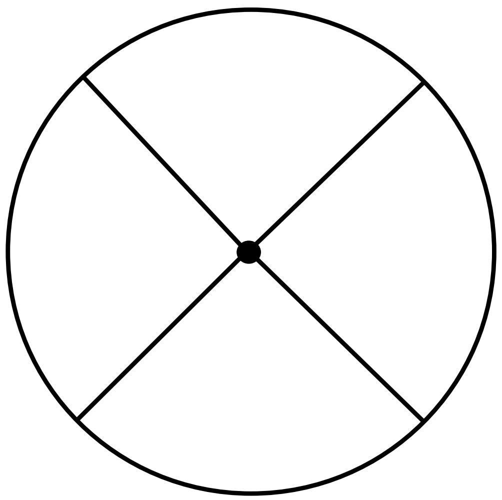
$$\mathrm{dDisc}\, G = \sum_{J',\Delta'} \sin^2\!\left(\tfrac{\pi}{2}(\Delta' - 2\Delta)\right) \left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{\Delta'+J'} \left(\frac{1-\sqrt{\bar\rho}}{1+\sqrt{\bar\rho}}\right)^{\Delta'-J'}$$

kills double-traces

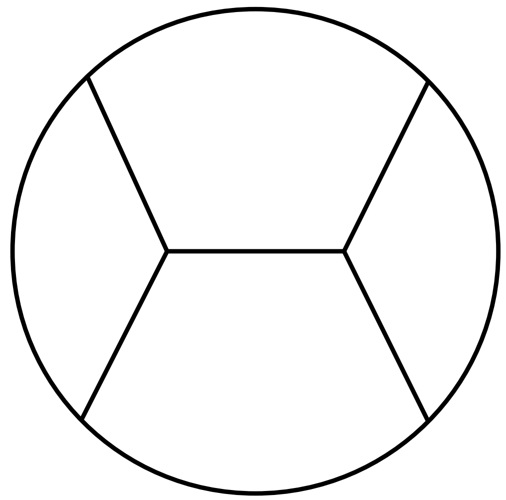
kills heavy

theories with local AdS dual $\Leftrightarrow$ dDisc saturated by few light primaries

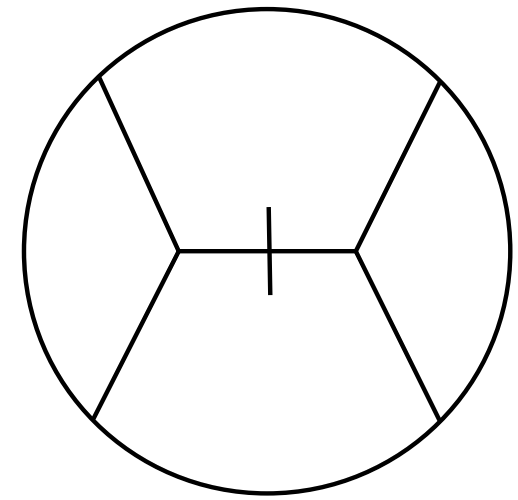
S-matrix unitarity for Witten diagrams:



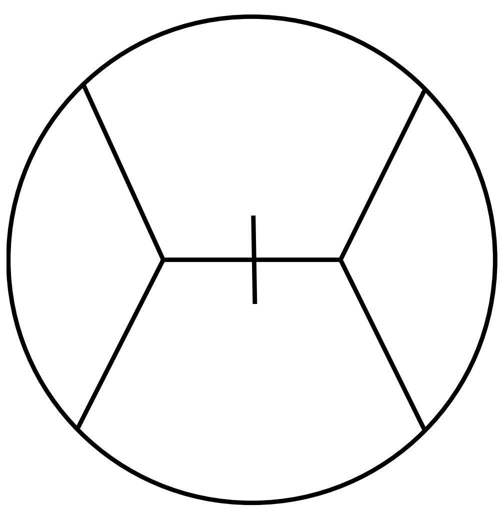
$+$



$+ \text{ crossed} = \int$



+ contact terms with finite J



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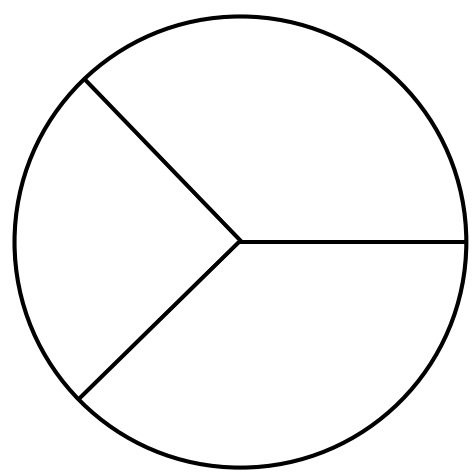
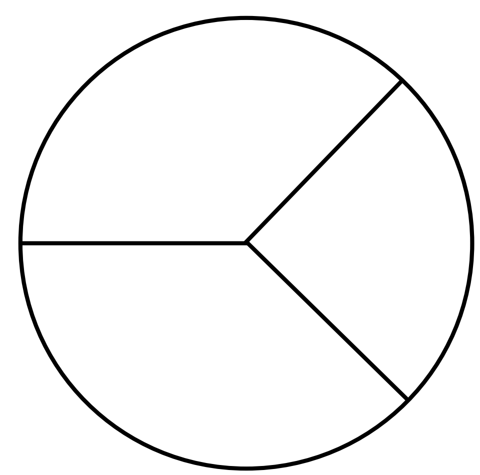
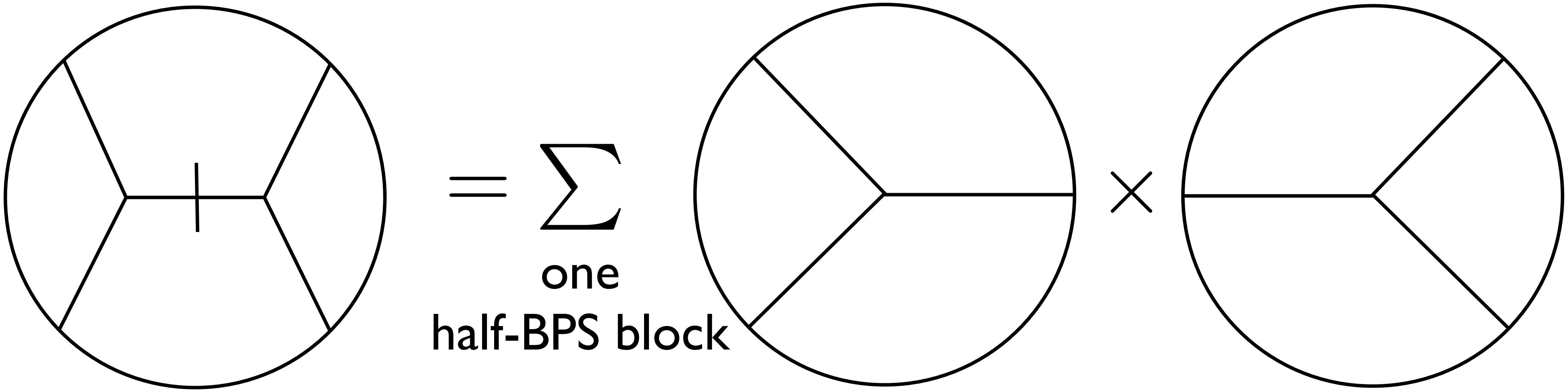


$\times$



= single conformal block (=known)

Correlator of Tr[Z^2] N=4 SYM:



[Alday& SCH '17]

dDisc is just the polar part as v->0 of one block:

$$\mathcal{G}(u,v) = \frac{1}{v^2} + \frac{2u^2 \log u - 3u^2 + 4u - 1}{v(u-1)^3}\,\frac{1}{c}$$

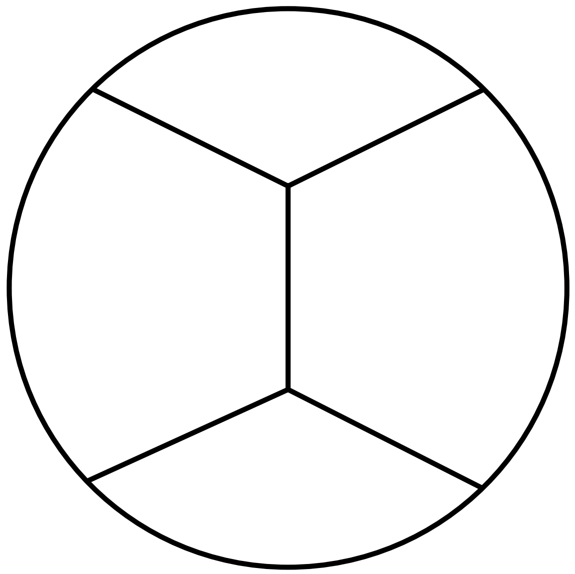
Plug into inversion integral gives all OPE data!!

$$\langle a^{(0)} \rangle_{n,\ell} = 2(\ell+1)(6+\ell+2n)$$

$$\langle a^{(0)}\gamma^{(1)} \rangle_{n,\ell} = -(n+1)(n+2)(n+3)(n+4)$$

$$\langle a^{(1)} \rangle_{n,\ell} = \frac{1}{2}\partial_n \langle a^{(0)}\gamma^{(1)} \rangle_{n,\ell}$$

Result matches perfectly supergravity Witten diagrams



$$\mathcal{G} = 1 + \frac{1}{v^2} + \frac{1}{c}\left(\frac{1}{v} - u^2 \bar{D}_{2,4,2,2}(z,\overline{z})\right) + O(1/c^2)$$

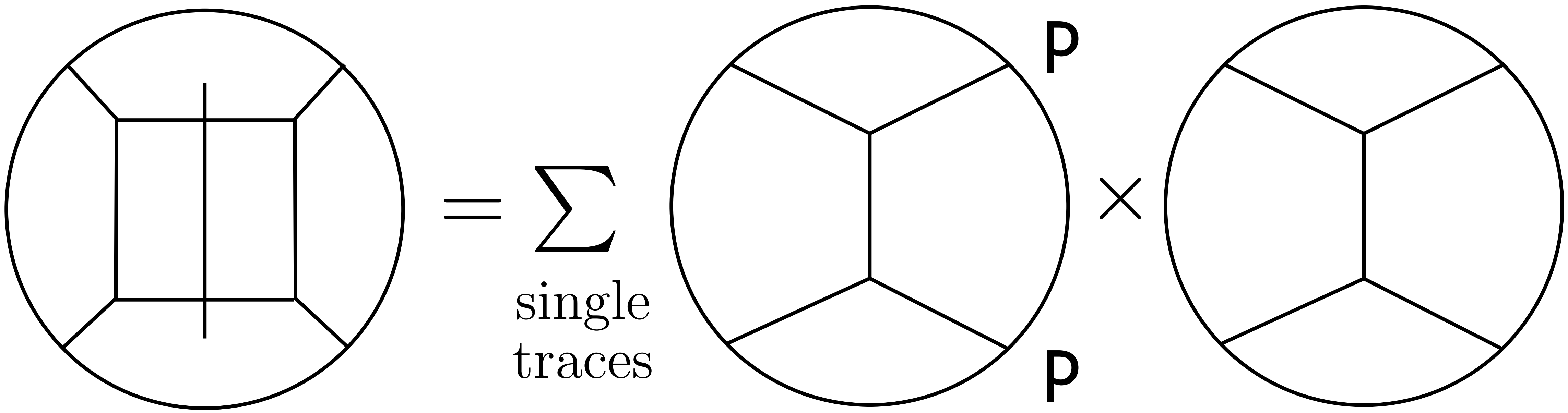
[D'Hoker, Freedman, Mathur, Matusis& Rastelli, ~99]

- New: control over contact ambiguities ('cR⁴') usingCFT only
- In progress: general S_5 Kaluza-Klein modes. Nice 10D structure…

[SCH & Anh-Khoi Trinh, …]

see also: [Rastelli&Zhou 16, Drummond et al. 17]

One-loop supergravity (from 1/N CFT correlator):



Product of trees: $\langle O_2 O_2 O_p O_p \rangle \times \langle O_p O_p O_2 O_2 \rangle$

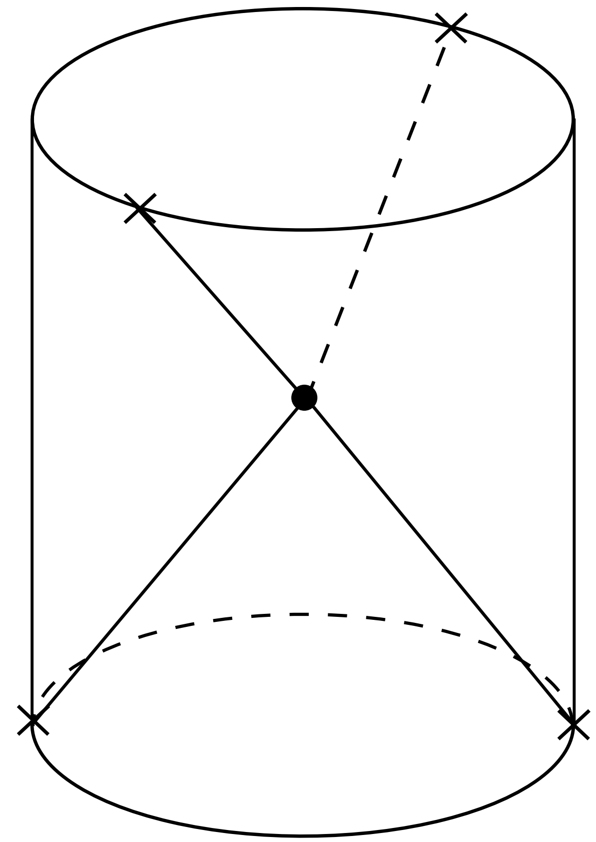
Accounts for mixing between $O_2 \Box^n O_2$ and $O_p \Box^{n'} O_p$

Ongoing object of study by several groups

[Alday&Bissi, Aprile,Drummond,Heslop&Paul]

We obtained the full 1-loop OPE data, and studied 'bulk point' limit

[Alday,SCH '17]



large-Δ OPE data ⇔ flat-space partial waves:

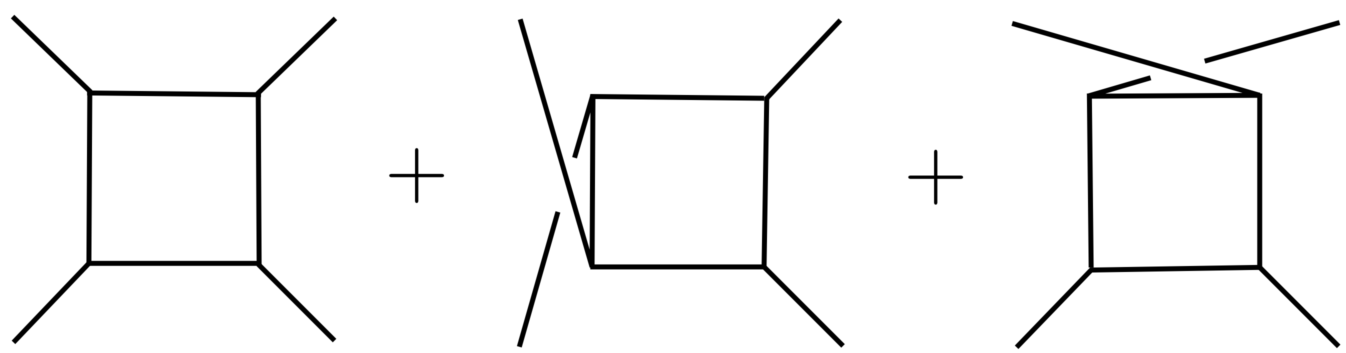
$$\lim_{n\to\infty} \frac{\left\langle a e^{-i\pi\gamma}\right\rangle_{n,\ell}}{\left\langle a^{(0)}\right\rangle_{n,\ell}} = b_\ell(s) \qquad \sqrt{s} = 2n/L$$

[HPPS]

Flat space limit is 10D type-IIB supergravity

one-loop IIB amplitude is simple:

$$A_{10}^{sugra}(s,t) = 8\pi G_N \frac{s^3}{tu} + \frac{(8\pi G_N)^2}{(4\pi)^5}\left(I_{box}(s,t) + I_{box}(s,u) + I_{box}(t,u)\right)$$



We expand it over 5D partial waves: $A_5 = A_{10}/\mathrm{vol}\ \mathrm{S}_5$

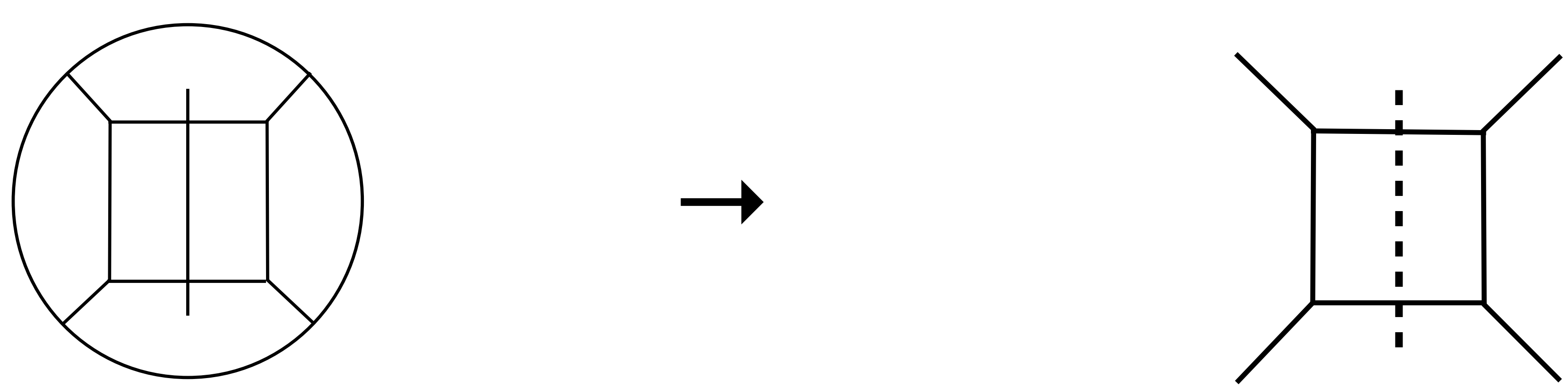
$$A_5(s,t) = \frac{128\pi}{\sqrt{s}} \sum_{\ell\ \mathrm{even}} (\ell+1)^2 b_\ell(s)\, P_\ell(\cos\theta)$$

Perfect match!

$$\lim_{n\to\infty} \frac{\left\langle a e^{-i\pi\gamma} \right\rangle_{n,\ell}}{\left\langle a^{(0)} \right\rangle_{n,\ell}} = b_\ell(s)$$

CFT (N=4 SYM) $\Rightarrow$ IIB supergravity at one-loop

$$\begin{tikzcd}
\text{correlator } \mathcal{G}(z,\bar{z}) \arrow[rr, "\text{flat space}"'] & & \text{amplitude } A_5(s,t) \\
\text{double-disc. } \mathrm{dDisc}\left[\mathcal{G}\right] \arrow[u, Rightarrow, "\text{inversion integral}"] \arrow[rr, "\text{flat space}"'] & & \text{discontinuity } \mathrm{Disc}\left[A_5\right] \arrow[u, Rightarrow, "\text{dispersion relation}"']
\end{tikzcd}$$



One-loop UV divergence: low-spins ill-defined

Dispersion relation is positive definite after UV-completion, so divergence can't be fully canceled:

$$C_{R^4} \geq \Lambda_{\mathrm{UV}}^2 \equiv (\Delta_{\mathrm{gap}}/L_{\mathrm{AdS}})^2$$

Minimal subtraction is in the swampland

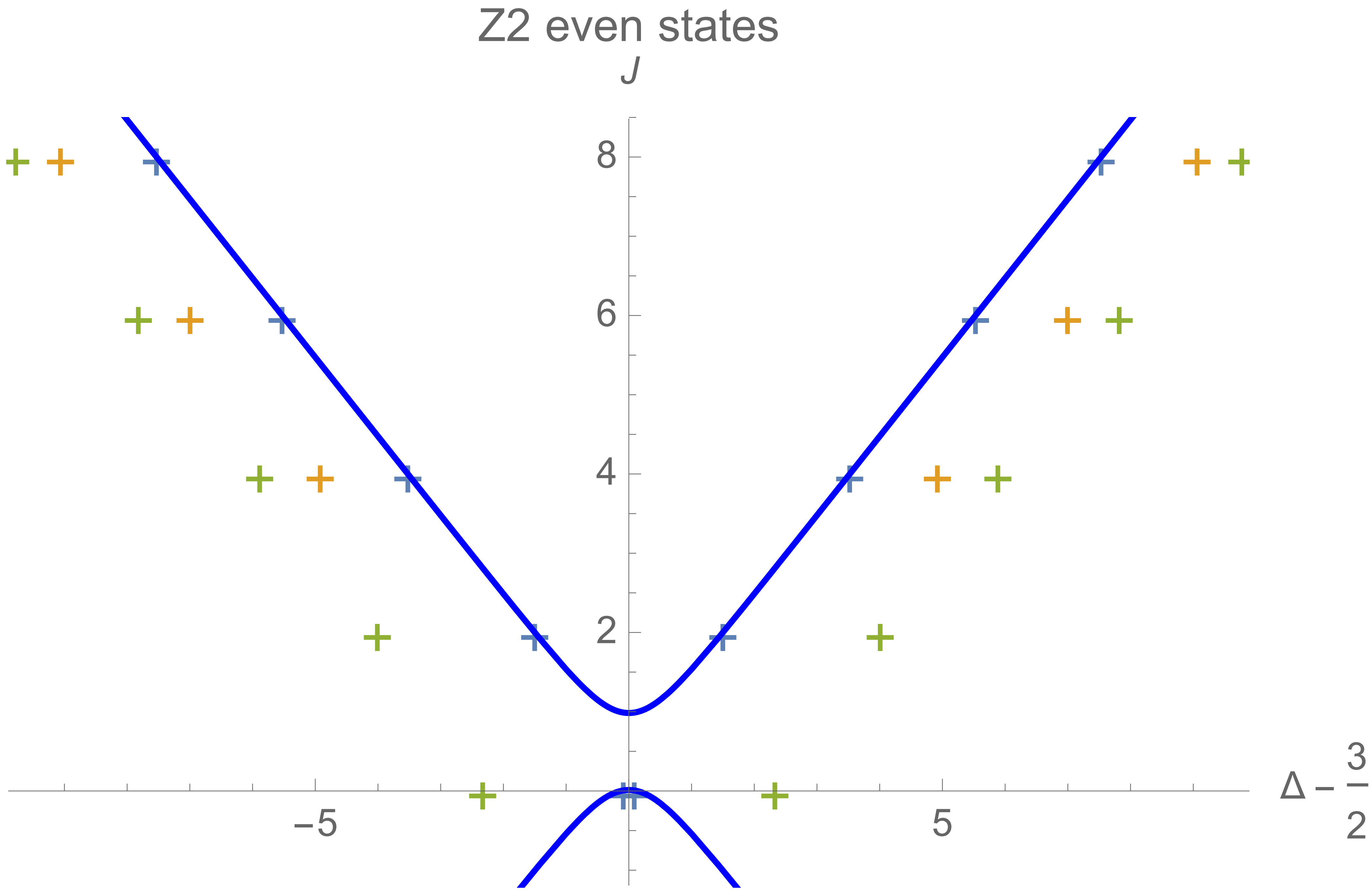
Summary

- Dispersion relation for OPE coefficients:

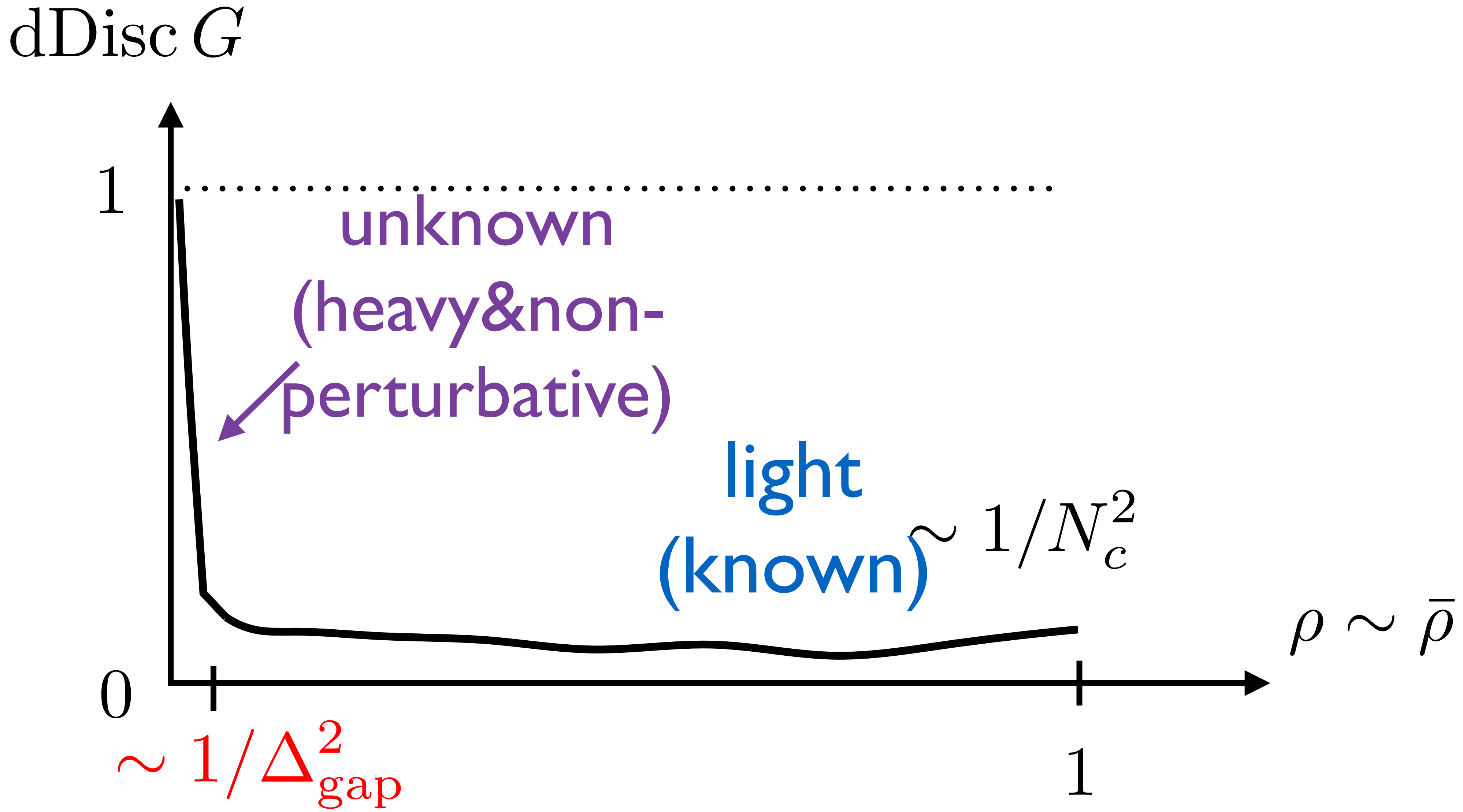
$$\underbrace{c(j,\Delta)}_{\text{s-channel}} \equiv \int_0^1 d\rho d\bar{\rho}\, g_{\Delta,j}\, \underbrace{\mathrm{dDisc}\, G}_{\text{cross-channels}}$$

- -Organizes spectrum into analytic families
  -Efficient cutting rules for AdS/CFT

- *Open directions:*
  - interplay with numerical bootstrap?
  - why/when does it work for J≤1?
  - study non-AdS / non-CFT?
  - heavy-light correlators and black holes?



(fit accounts for possible square-root branch point)



$$c_{j,\Delta} = \int F_{j,\Delta}\, \mathrm{dDisc}\, G \quad = c_{j,\Delta}\Big|_{\text{light}} + c_{j,\Delta}\Big|_{\text{heavy}}$$

'minimal solution'

correction small for j>2

[see also: Alday, Bissi&Perlmutter; Li, Meltzer&Poland]

'Heavy' part depends on nonperturbative UV completion.

It's weighed by $\sim (\rho\bar{\rho})^{J/2}$. Use positivity + boundedness:

$$\boxed{\left|c(j, \tfrac{d}{2}+i\nu)_{\text{heavy}}\right| \leq \frac{1}{c_T}\frac{\#}{(\Delta_{\text{gap}}^2)^{j-2}}}$$

This establishes, from CFT, an EFT power-counting in AdS.