Critical failure can be tuned by material rheology:

A model ...

J. Baró$^{a,b,c}$, J. Davidsen$^a$

... and a case study

... K.A. Dahmen$^b$, G. Nataf$^{c,d}$, P. O. Castillo-Villa$^{c,e}$, E.H.K. Salje$^f$, A. Planes$^c$, E. Vives$^c$

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Deformation as Avalanche Phenomena

- Non-linear Deformation $\rightarrow$ Avalanche Dynamics:
  - Low Temperature
  - Quenched Disorder
  - Interactions

Crystalline nano-pillars

Tectonic Gouges

- [N. Friedman et al., PRL (2012)]
- [T. Hatano, C. Narteau, P. Schebalin, SREP (2015)]
Deformation as Avalanche Phenomena

- Non-linear Deformation → Avalanche Dynamics:

- Failure: Avalanche ≈ Phase Transition

\[ \text{order} \quad \text{disorder} \]

\[ \text{control} \quad \text{disorder} \]

- Low Temperature
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Deformation as Avalanche Phenomena

- Non-linear Deformation $\rightarrow$ Avalanche Dynamics:
- Failure: Avalanche $\approx$ Phase Transition
- Order $\rightarrow$ Disorder
- Failure Prediction?
  - Properties / State: (tomography, seismography)
  - Statistics of Avalanches: Acoustic Emission

Low Temperature
Quenched Disorder
Interactions

Crystalline nano-pillars

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Critical Failure & Rheology

2018 CAP Congress June 13, 2018
Power Laws everywhere! (scale-invariance)

[A. De-Santis et al., BSSA (2011)]  [S. Goodfellow & P. Young, GRL (2014)]

Acceleration of Activity / Energy Released

[D. Amitrano, JGR (2003)]
Power Laws everywhere! (scale-invariance)

- Day w.r.t. main shock (6 Apr. 2009)
  - $b = 0.89 \pm 0.03$

- Distribution of Sizes
  \[ D(S; |t_c - t|)dS = S^{-\kappa}D_S(S|t_c - t|^{1/\sigma})dS \]
  (reproduced by most models) \[ \text{[K. Dahmen, et al. (2011)\]} \]

Failure is a Critical Point (RG)

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Failure is a Critical Point (RG)

Distribution of Sizes

\[ \frac{dS}{dt} \propto \langle S \rangle \sim |t_c - t|^{\frac{\kappa - 2}{\sigma}} \]
An experimental case: Uniaxial compression of SiO$_2$ porous materials

- Porous SiO$_2$ ($\Phi \sim 10\% - 40\%$)
- Soft Uniaxial Compression
  - stress control ($\sim 1$ kPa/s)
  - no lateral confinement
- Strain Monitoring ($\sim \mu m$)
- Acoustic Emission Recording ($\sim MHz$)
- Fractures & Crackling Noise (AE)
  (10$k$ – 30$k$ events)

$E \sim \int |\text{Signal}(t)|^2 dt$
Critical Failure?

Data Agree

- Ultimate brittle event ($P^5_c$)

Data Disagree

$E \sim \int |\text{Signal}(t)|^2 dt$
An experimental case: Uniaxial compression of SiO$_2$ porous materials

Critical Failure ?

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Data Disagree

- Stationary energies:
  
  $P(E|t-t_f) dE \sim E^{-\epsilon} dE$.

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An experimental case: Uniaxial compression of SiO$_2$ porous materials

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- Brittle precursors (not SOC).

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Critical Failure & Rheology

2018 CAP Congress June 13, 2018 4 / 8

An experimental case: Uniaxial compression of SiO$_2$ porous materials

Critical Failure?

Data Agree
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An experimental case: Uniaxial compression of SiO\textsubscript{2} porous materials

Critical Failure?

**Data Agree**

- Ultimate brittle event ($P_c^5$) not random
- Accelerated $E$ release: $\frac{dE}{dt} \sim (t - t_f)^{-m}$

**Data Disagree**

- Stationary energies: $P(E|t - t_f)\,dE \sim E^{-\epsilon} \, dE$.
- Brittle precursors (not SOC).
- Variations in activity rate $dn/dt$.

$E \sim \int |\text{Signal}(t)|^2 \, dt$
An experimental case: Uniaxial compression of SiO$_2$ porous materials

Critical Failure?

Data Agree

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- Accelerated $E$ release: $dE/dt \sim (t - t_f)^{-m}$

Data Disagree

- Stationary energies: $P(E|t - t_f)dE \sim E^{-\epsilon}dE$.
- Brittle precursors (not SOC).
- Variations in activity rate $dn/dt$.
- Aftershock sequences.

$E \sim \int |\text{Signal}(t)|^2 dt$

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Critical Failure & Rheology

2018 CAP Congress June 13, 2018 4/8
Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

**Standard (Democratic Fiber Bundle) Model**

- **Micromechanics:**
  \[ \sigma_l = \begin{cases} 
  E\varepsilon & (E\varepsilon < S_i) \\
  0 & (E\varepsilon \geq S_i) 
  \end{cases} \]

- **Mean Field:**
  \[ \sigma_l = \frac{\sigma}{N_{fibers}} \]
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  \[ \sigma(E\varepsilon) = (1 - F(S = E\varepsilon))E\varepsilon \]
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Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

(generalized) Viscoelastic (democratic fiber bundle) Model

- **Micromechanics:**
  \[ E \Delta \varepsilon(t) = (1 - H_{\alpha}(t/\tau)) \Delta \sigma_l \]

Transient: \[ \begin{align*}
  H_{\alpha}(0) & \rightarrow h \\
  H_{\alpha}(\infty) & \rightarrow 0
\end{align*} \]
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- Micromechanics:
  \[ E \Delta \varepsilon(t) = (1 - H_\alpha(t/\tau)) \Delta \sigma_1 \]
  Transient: \[
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  \]

- Macroscopic constitutive equation:
  \[
  \sigma(E\varepsilon, t) = E\varepsilon(t) \left(\frac{1}{1 - F(E\varepsilon)} - \sum_{S_j < E\varepsilon} \phi_j(t - t_j)\right)^{-1}
  \]
Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

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Universal Avalanche Statistics for Fibrous Models

- Avalanche start at $S_i$, stops when: $\sigma(E\varepsilon) \geq \sigma(S_i)$

Universal Avalanche Condition:

$$\xi(\Delta_i) > B(S_i|h)\Delta_i$$

- $\xi(\Delta)$: Poisson counting process of $\Delta$ steps.
Avalanche start at $S_i$, stops when: \[ \sigma(E\varepsilon) \geq \sigma(S_i) \]

**Universal Avalanche Condition:**
\[ \xi(\Delta_i) > B(S_i|h)\Delta_i \]

- $\xi(\Delta)$: Poisson counting process of $\Delta$ steps.
  - $B > 1$: Prob. $\Delta \to \infty$.
  - $B < 1$: Size distribution: $D(\Delta; B)d\Delta = \Delta^{-3/2} D(\Delta|1-B)|d\Delta$.
  - $B = 1$: Critical.

[Baró & Davidsen, PRE (2018)]
Avalanche start at $S_i$, stops when:

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- Slope $B$ is function of state:

$$B(S_i|h) = \frac{S_i \text{pdf}(S_i)}{1 - F(S_i)}(1 - h)$$

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\]

- Slope \( B \) is function of state:

\[
B(S_i|h) = \frac{S_i \, pdf(S_i)}{1 - F(S_i)} (1 - h)
\]

- At failure: \( d\sigma/d\varepsilon|_{\sigma_f} = 0 \):

\[
B(E\varepsilon_f|h) = (1 - h)
\]

- \( \xi(\Delta) \): Poisson counting process of \( \Delta \) steps.
  - \( B > 1 \): Prob. \( \Delta \to \infty \).
  - \( B < 1 \): Size distribution: \( D(\Delta;B)d\Delta = \Delta^{-3/2} \, D(\Delta|1 - B|) \, d\Delta \).
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**Universal Avalanche Condition:**

$$\xi(\Delta_i) > B(S_i|h)\Delta_i$$

- Slope $B$ is function of state:
  $$B(S_i|h) = \frac{S_i}{1 - F(S_i)} \left(1 - h\right)$$

- At failure: $d\sigma/d\varepsilon|_{\sigma_f} = 0$:
  $$B(E\varepsilon_f|h) = (1 - h)$$

- **Critical failure for** $h = 0$.
- **Subcrit. failure for** $h > 0$.

**$\xi(\Delta)$:** Poisson counting process of $\Delta$ steps.
- **$B > 1$:** Prob. $\Delta \to \infty$.
- **$B < 1$:** Size distribution:
  $$D(\Delta; B)d\Delta = \Delta^{-3/2} D(\Delta|1 - B) d\Delta.$$ 
- **$B = 1$:** Critical.

[Baró & Davidsen, PRE (2018)]
Tuning Acceleration: Critical Failure or Foreshocks?

Standard Model ($h = 0$): **Critical Failure**

- Critical Failure: $\langle \Delta \rangle \sim f^\beta (\tau - 2)$

Viscoelastic Model ($0 < h < 1$): **Foreshocks**

- Sub-critical Failure: $\langle \Delta \rangle \sim cnt.$
Tuning Acceleration: Critical Failure or Foreshocks?

**Standard Model ($h = 0$): Critical Failure**

- Critical Failure: $\langle \Delta \rangle \sim f^\beta(\tau^{-2})$
- Stationary activity: $\frac{dn}{dt} \sim cnt.$

**Viscoelastic Model ($0 < h < 1$): Foreshocks**

- Sub-critical Failure: $\langle \Delta \rangle \sim cnt.$
- Precursory activity: $\frac{dn}{dt} \sim f^\beta(\tau^{-2})$
Standard Model ($h = 0$): Critical Failure

- Critical Failure: $\langle \Delta \rangle \sim f^{\beta(\tau - 2)}$
- Stationary activity: $\frac{dn}{dt} \sim \text{cnt.}$
- Acc. Energy: $\frac{d\Delta}{dt} \frac{dn}{dt} \sim f^{\beta(\tau - 2)}$

Viscoelastic Model ($0 < h < 1$): Foreshocks

- Sub-critical Failure: $\langle \Delta \rangle \sim \text{cnt.}$
- Precursory activity: $\frac{dn}{dt} \sim f^{\beta(\tau - 2)}$
- Acc. Energy: $\frac{d\Delta}{dt} \frac{dn}{dt} \sim f^{\beta(\tau - 2)}$

\[ (\tau_{-2})^{\beta_\sigma} = -0.5 \]
Critical failure can be tuned by material rheology:

- Models of **critical failure** do not reproduce some experimental observations:
  - Accelerated energy release without divergence of energy scales.
  - Temporal correlations: aftershocks, foreshocks.

- Addition of **transient hardening** to models of **critical failure**:
  - Generate triggering (aftershocks) and preceding activity (foreshocks).
  - Prevent criticality at failure, preserving acceleration.

- Statistics of avalanches in the **viscoelastic democratic fiber bundle model** are **universal** and only depend on the **distance to criticality** (not to failure).

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**a case of study ...**


⇒ Experimental evidence of accelerated seismic release without critical failure in acoustic emissions of compressed nanoporous materials,


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**... and a model**

J. Baró, J. Davidsen,

⇒ Universal avalanche statistics and triggering close to failure in a mean field model of rheological fracture,


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**Critical Failure & Rheology**

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Table: First three top rows: fitted exponents in experimental data, compared to the MF exponents for slip and fracture MF models. Bottom rows: fundamental exponents estimated from MF theory. Superscripts $a$ and $b$ denote two different interpretations of ASR in terms of MF theory.
<table>
<thead>
<tr>
<th></th>
<th>area $A$ (mm$^2$)</th>
<th>height $h$ (mm)</th>
<th>driving rate $dP/\text{dt}$ (kPa/s)</th>
<th>$Th$ (dB)</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vycor (V32)</td>
<td>17.0</td>
<td>5.65</td>
<td>5.7</td>
<td>23</td>
<td>34138</td>
</tr>
<tr>
<td>Gelsil (G26)</td>
<td>46.7</td>
<td>6.2</td>
<td>0.7</td>
<td>26</td>
<td>5412</td>
</tr>
<tr>
<td>Sands. (SR2)</td>
<td>17.0</td>
<td>4.3</td>
<td>2.4</td>
<td>23</td>
<td>27271</td>
</tr>
</tbody>
</table>

Table: Sample details: cross-sectional area $A$; height $h$; compression rate $dP/\text{dt}$; number $N$ of recorded signals above threshold $Th$. 
Filling the Checklist: Minimal Structured Model

- Model with minimal Structure (non-MF)

- Experiments in SiO$_2$:
The Generalized Zener Element

- Adds **dissipation**, **temporal scales** and **power-law memory**.
- Reproduces the response of certain bulk amorphous materials

\[
\sigma_m = E_m \varepsilon_m
\]
\[
\sigma_X = X \frac{d^\alpha}{dt^\alpha} \varepsilon_X
\]
\[
\sigma_e = E \varepsilon
\]

Constitutive equation for the generalized Zener Element:

\[
\left[ 1 + \frac{X}{E_m} \frac{d^\alpha}{dt^\alpha} \right] \sigma_I = \left[ 1 + \frac{X(E_m + E)}{E_mE} \frac{d^\alpha}{dt^\alpha} \right] E \varepsilon.
\]

\[
H_\alpha(t/\tau) := \frac{E_m}{E_m + E} E_\alpha \left( - \left( \frac{t}{\tau} \right)^\alpha \right)
\]

\[
\begin{align*}
H_\alpha(0) &= \frac{E_m}{E_m + E} := h \\
H_\alpha(t \gg \tau) &\to 0
\end{align*}
\]

**Creep compliance**:

\[
J_{GZ}(t) = \frac{1}{E} \left( 1 - H_\alpha \left( \frac{t}{\tau} \right) \right)
\]
Magnitude Relations:

\[ D_{AE} = t - t_i \mid V < V_{th} \]

\[ A_{AE} = \max(V(t)) \]

\[ E_{AE} = \int_{t_i}^{t_i + D_{AE}} |V(t)|^2 dt \]

Signal Hypothesis:

\[ V(t) = G \int_{-\infty}^{t} v(t)e^{i\omega_0 t - \frac{t-t'}{\tau}} dt' \]

Parabolic shape:

\[ \tilde{v}(t/T) = 4 \left( t/T - (t/T)^2 \right) \]
- Acceleration and energy exponent before failure:

\[ \frac{dE}{dt} (\text{aJ/s}) \]

\[ f_k = 1 - \frac{P}{P_c} \]

- Critical Failure & Rheology

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Deceleration and energy exponent after failure:

\[ \frac{dE}{dt} (\text{aJ/s}) \]

\[ f_k^* = \frac{P}{P_c k^{-1}} \]

\[ m^* = 1.11(23) \]

\[ m^* = 1.13(50) \]

\[ m^* = 1.53(25) \]
Complementary Cumulative Distribution of Energies:

V32: $k=2$

- $1.1e-03 < f < 1.2e-03$
- $1.6e-03 < f < 1.4e-02$

V32: $k=3$

- $8.3e-07 < f < 1.1e-03$
- $1.1e-03 < f < 1.2e-03$

V32: $k=4$

- $2.6e-07 < f < 1.8e-05$
- $1.8e-05 < f < 1.6e-04$

V32: $k=5$

- $2.8e-07 < f < 1.8e-05$
- $1.5e-06 < f < 2.1e-04$

G26: $k=3$

- $2.8e-07 < f < 4.4e-03$
- $4.1e-03 < f < 4.8e-03$

G26: $k=4$

- $2.6e-07 < f < 5.6e-03$
- $5.4e-02 < f < 4.5e-02$

G26: $k=5$

- $2.2e-07 < f < 1.8e-05$
- $7.4e-05 < f < 1.2e-03$

SR2: $k=1$

- $2.2e-07 < f < 7.4e-05$
- $7.4e-05 < f < 2.3e-03$

SR2: $k=2$

- $1.0e-06 < f < 3.5e-05$
- $7.4e-05 < f < 2.3e-03$

SR2: $k=3$

- $1.2e-06 < f < 3.2e-05$
- $3.2e-05 < f < 7.7e-05$

SR2: $k=4$

- $4.9e-08 < f < 2.2e-05$
- $2.2e-05 < f < 4.2e-05$

SR2: $k=5$

- $5.9e-07 < f < 3.6e-03$
- $5.0e-03 < f < 2.0e-02$

E (eJ)

N(E_{max})
Triggering in experiments and model

**Triggering in lab:** [Baró et al., PRL (2013)]

![Graph showing aftershock rate](image)

**Triggering in viscoelastic model:** [Baró & Davidsen, PRE (2018)]

![Graph showing energy release and avalanche sizes](image)

Under slow $\sigma$ driving:

- **Accelerated energy release?**
  \[
  \frac{d(Energy)}{dt} \propto \frac{dF(E\varepsilon)}{d\sigma} \propto \frac{d(E\varepsilon)}{d\sigma}
  \]

- **Avalanche sizes?**