Critical failure can be tuned by material rheology:

A model ...

J. Baró\textsuperscript{a,b,c}, J. Davidsen\textsuperscript{a}

... and a case study

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Deformation Avalanches and Critical Failure

- Non-linear Deformation $\rightarrow$ Avalanches

(Crystalline nano-pillars under compression)

$[N.~Friedman~et~al.,~PRL~(2012)]$
Deformation Avalanches and Critical Failure

- Non-linear Deformation → Avalanches
- Macro-Failure ≈ Phase Transition?
- Is failure predictable?
  - ‘Canonical’ State
  - Statistics of Avalanches: slips, acoustic emission, etc.

(Crystalline nano-pillars under compression)

[Crystalline nano-pillars under compression]

[N. Friedman et al., PRL (2012)]
Hypothesis: Failure is a Critical Point

Distribution of Sizes ($\Delta$)

$$D(\Delta; |t_c - t|) d\Delta = \Delta^{-\kappa} D(\Delta|t_c - t|^{1/\sigma}) d\Delta$$

(in mean field models) \cite{Dahmen2011}

Accelerated Energy Release

$$d\Delta/dt \propto \langle \Delta \rangle \sim |t_c - t|^{\kappa - 2/\sigma}$$

(Crystalline nano-pillars under compression)

Salje EKH, Dahmen KA. 2014.
An experimental case: Soft uniaxial compression of SiO$_2$ porous materials

**Agree with models of critical failure**
- Brittle failure
- Accelerated $E$ release: $dE/dt \sim (t - t_f)^{-m}$

**Disagree with models of critical failure**
- Stationary energies:
  - $P(E_{AE}|t - t_f) dE \sim E_{AE}^{-\epsilon} dE_{AE}$
  - constant $\langle E_{AE}(t) \rangle$

$E \sim \int |\text{Signal}(t)|^2 dt$  [Baró et al. PRL (2013), Baró et al. PRL (2018)]
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- Variations in activity rate $dn/dt$:
  - Precursors (foreshocks)
  - Aftershock sequences

Could rheology explain it all?

$E \sim \int |\text{Signal}(t)|^2 dt$  [Baró et al. PRL (2013), Baró et al. PRL (2018)]
Viscoelasticity in a prototype model

Standard (democratic fiber bundle) Model

- Elastic Fibers

Brittle fibers of stochastic strengths $S_i$:

Mean field interactions:

$$\sigma_i = \begin{cases} \sigma/N & (\varepsilon < S_i) \\ 0 & (\varepsilon \geq S_i) \end{cases}$$

Avalanches from strength-distribution:

$$N(\varepsilon) \sim N(S_i < \varepsilon)$$

determines stability strain-stress: $\sigma(\varepsilon)$:

micro. avalanches + macro. failure
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- Constitutive Equation
  
  \[ \sigma(\varepsilon) = \varepsilon N(S_i < \varepsilon) \]  
  (analytic)

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**Viscoelastic Fibers**

\[ \sigma = \sigma(\varepsilon) + H(\varepsilon, t; h) \]

(also analytic !)

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$\xi(\Delta_i) > B(S_i|h)\Delta_i$

Avalanche Sizes $\rightarrow$ Hitting Times

- $\xi(\Delta)$: Poisson counting process of $\Delta$ trials.
- Against a boundary proportional to $\Delta$

[Baró & Davidsen, PRE (2018)]
Avalanche Sizes at Failure

- From const. eq. an avalanche starts at $S_i$, stops at:

$$\sigma(\varepsilon, t; h) \geq \sigma(S_i)$$

$$E_{\sigma} S_i + \Delta \sigma(E_{\sigma})$$

- Driving $\xi(\Delta)$:

Avalanche Sizes $\rightarrow$ Hitting Times

- $\xi(\Delta)$: Poisson counting process of $\Delta$ trials.

- Against a boundary proportional to $\Delta$
  - $B > 1$: Prob. $\Delta \rightarrow \infty$
  - $B < 1$: Size distribution: $D(\Delta; B)d\Delta = \Delta^{-3/2} \mathcal{D}(\Delta|1-B|) d\Delta$
  - $B = 1$: Critical

\[\xi(\Delta_i) > B(S_i|h)\Delta_i\]
Avalanche Sizes at Failure

From const. eq. an avalanche starts at $S_i$, stops at:

$$\sigma(\varepsilon, t; h) \geq \sigma(S_i)$$

- At failure: $B(\varepsilon_f | h) = (1 - h)$

Avalanche Sizes → Hitting Times

- $\xi(\Delta_i) > B(S_i | h)\Delta_i$

- $\xi(\Delta)$: Poisson counting process of $\Delta$ trials.

- Against a boundary proportional to $\Delta$
  - $B > 1$: Prob. $\Delta \to \infty$
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  - $B = 1$: Critical

$$\Gamma(1/2)$$

[Baró & Davidsen, PRE (2018)]
Avalanche Sizes at Failure

- From const. eq. an avalanche starts at \( S_i \), stops at:

\[
\sigma(\varepsilon, t; h) \geq \sigma(S_i)
\]

- At failure: \( B(\varepsilon_f | h) = (1 - h) \)
- Critical failure for \( h = 0 \).
- Subcrit. failure for \( h > 0 \).

\[\xi(\Delta_i) > B(S_i|h)\Delta_i\]

- Avalanche Sizes \( \rightarrow \) Hitting Times
  - \( \xi(\Delta) \): Poisson counting process of \( \Delta \) trials.
  - Against a boundary proportional to \( \Delta \)
    - \( B > 1 \): Prob. \( \Delta \rightarrow \infty \)
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    - \( B = 1 \): Critical

[Baró & Davidsen, PRE (2018) ]
Tuning Acceleration: Critical Failure or Foreshocks?

**Standard Model \((h = 0)\): Critical Failure**

\[
\langle \Delta \rangle \sim f^\beta (\tau - 2)
\]

**Viscoelastic Model \((0 < h < 1)\): Foreshocks**

\[
\langle \Delta \rangle \sim \text{cnt.}
\]
Tuning Acceleration: Critical Failure or Foreshocks?

### Standard Model ($h = 0$): Critical Failure

- Critical Failure: $\langle \Delta \rangle \sim f^\beta (\tau^{-2})$
- Stationary activity: $\frac{dn}{dt} \sim \text{cnt.}$

### Viscoelastic Model ($0 < h < 1$): Foreshocks

- Sub-critical Failure: $\langle \Delta \rangle \sim \text{cnt.}$
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Tuning Acceleration: Critical Failure or Foreshocks?

Standard Model \((h = 0)\): Critical Failure

- Critical Failure: \(\langle \Delta \rangle \sim f^\beta(\tau - 2)\)
- Stationary activity: \(\frac{dn}{dt} \sim \text{cnt.}\)
- Acc. Energy: \(\frac{d\Delta}{dt} = \langle \Delta \rangle \times \frac{dn}{dt} \sim f^\beta(\tau - 2)\)

Viscoelastic Model \((0 < h < 1)\): Foreshocks

- Sub-critical Failure: \(\langle \Delta \rangle \sim \text{cnt.}\)
- Precursory activity: \(\frac{dn}{dt} \sim f^\beta(\tau - 2)\)
- Acc. Energy: \(\frac{d\Delta}{dt} = \langle \Delta \rangle \times \frac{dn}{dt} \sim f^\beta(\tau - 2)\)
Critical failure can be tuned by material rheology:

- Models of critical failure do not reproduce some experimental observations:
  - Accelerated energy release without divergence of energy scales.
  - Variations in activity: aftershocks, foreshocks.

- Addition of rheology to models of critical failure can:
  - Generate event correlations (aftershocks) and preceding activity (foreshocks).
  - Prevent criticality at failure, preserving acceleration.

- Statistics of avalanches in the viscoelastic democratic fiber bundle model are universal and only depend on the distance to criticality (not to failure).

a case of study ...


⇒ Experimental evidence of accelerated seismic release without critical failure in acoustic emissions of compressed nanoporous materials,


... and a model

J. Baró, J. Davidsen,

⇒ Universal avalanche statistics and triggering close to failure in a mean field model of rheological fracture,

<table>
<thead>
<tr>
<th></th>
<th>V32</th>
<th>G26</th>
<th>SR2</th>
<th>slip MF</th>
<th>fracture MF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>3.0 (4)</td>
<td>3.4 (4)</td>
<td>3.2 (4)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1.40 (5)</td>
<td>1.40 (5)</td>
<td>1.50 (5)</td>
<td>4/3</td>
<td>4/3</td>
</tr>
<tr>
<td>$m$</td>
<td>1.02 (13)</td>
<td>1.11 (20)</td>
<td>0.99 (8)</td>
<td>1$^a$</td>
<td>2$^b$</td>
</tr>
<tr>
<td>$\sigma_{\nu z}$</td>
<td>0.50 (6)</td>
<td>0.45 (6)</td>
<td>0.48 (5)</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1.60 (8)</td>
<td>1.62 (8)</td>
<td>1.76 (8)</td>
<td>3/2</td>
<td>3/2</td>
</tr>
<tr>
<td>$\sigma^a$</td>
<td>0.40 (9)</td>
<td>0.34 (9)</td>
<td>0.24 (8)</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma^b$</td>
<td>0.88 (12)</td>
<td>0.80 (16)</td>
<td>0.76 (7)</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>$\beta^a$</td>
<td>3.7 ± 0.8</td>
<td>4.6 ± 1.2</td>
<td>6.3 ± 2.1</td>
<td>3</td>
<td>3/2</td>
</tr>
<tr>
<td>$\beta^b$</td>
<td>1.67 (24)</td>
<td>1.83 (37)</td>
<td>2.00 (25)</td>
<td>3</td>
<td>3/2</td>
</tr>
</tbody>
</table>

Table: First three top rows: fitted exponents in experimental data, compared to the MF exponents for slip and fracture MF models. Bottom rows: fundamental exponents estimated from MF theory. Superscripts $a$ and $b$ denote two different interpretations of ASR in terms of MF theory.
<table>
<thead>
<tr>
<th></th>
<th>area $A$ (mm$^2$)</th>
<th>height $h$ (mm)</th>
<th>driving rate $dP/dt$ (kPa/s)</th>
<th>$Th$ (dB)</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vycor (V32)</td>
<td>17.0</td>
<td>5.65</td>
<td>5.7</td>
<td>23</td>
<td>34138</td>
</tr>
<tr>
<td>Gelsil (G26)</td>
<td>46.7</td>
<td>6.2</td>
<td>0.7</td>
<td>26</td>
<td>5412</td>
</tr>
<tr>
<td>Sands. (SR2)</td>
<td>17.0</td>
<td>4.3</td>
<td>2.4</td>
<td>23</td>
<td>27271</td>
</tr>
</tbody>
</table>

Table: Sample details: crossectional area $A$; height $h$; compression rate $dP/dt$; number $N$ of recorded signals above threshold $Th$. 
Filling the Checklist: Minimal Structured Model

- Model with minimal Structure (non-MF)

\begin{align*}
\text{Experiments in SiO}_2: & \\
\text{Vycor (SiO}_2) & \\
\text{Sandstone} & \\
\end{align*}
The Generalized Zener Element

- Adds **dissipation**, **temporal scales** and **power-law memory**.
- Reproduces the response of certain bulk amorphous materials

\[
\sigma_m = E_m \varepsilon_m
\]
\[
\sigma_X = \sigma_l = E \varepsilon
\]
\[
\sigma_l = X \frac{d^\alpha}{dt^\alpha} \varepsilon_X
\]

Constitutive equation for the generalized Zener Element:
\[
\left[1 + \frac{X}{E_m} \frac{d^\alpha}{dt^\alpha}\right] \sigma_l = \left[1 + \frac{X(E_m + E)}{E_m E} \frac{d^\alpha}{dt^\alpha}\right] E \varepsilon.
\]

\[
H_\alpha(t/\tau) := \frac{E_m}{E_m + E} E_\alpha \left(-\left(\frac{t}{\tau}\right)^\alpha\right) \begin{cases} H_{\alpha}(0) &=\frac{E_m}{E_m + E} := h \\ H_{\alpha}(t \gg \tau) &\rightarrow 0 \end{cases}
\]

\[
J_{GZ}(t) = \frac{1}{E} \left(1 - H_{\alpha}\left(\frac{t}{\tau}\right)\right)
\]
Magnitude Relations:

\[
D_{AE} = t - t_i | V < V_{th}
\]

- AE magn.

\[
A_{AE} = \max(V(t))
\]

\[
E_{AE} = \int_{t_i}^{t_i + D_{AE}} |V(t)|^2 dt
\]

- Signal Hypothesis:

\[
V(t) = G \int_{-\infty}^{t} v(t) e^{i\omega_0 t - \frac{t-t'}{\tau}} dt'
\]

- Parabolic shape:

\[
\tilde{v}(t/T) = 4 \left( \frac{t}{T} - \left( \frac{t}{T} \right)^2 \right)
\]
- Acceleration and energy exponent before failure:

![Graphs and plots showing data points and curves for different materials and parameters.](Figures/PRL2018)

- Critical parameters:
  - $f_k = 1 - P/P_c^k$
  - $m = 1.02(12)$ for V32
  - $m = 1.11(20)$ for G26
  - $m = 0.99(8)$ for SR2

Contact: jordi.barourbea@ucalgary.ca (UCalgary)
Deceleration and energy exponent after failure:

\[ \frac{dE_{AE}}{dt} (\text{aJ/s}) \]

\[ f_k^* = \frac{P}{P_c k^{-1}} \]

- **Figures PRL2018**

- **V32**
- **G26**
- **SR2**

- **a)**
- **b)**
- **c)**

- **d)**
- **e)**
- **f)**

- **g)**
- **h)**
- **i)**

- **m^* = 1.11(23)**
- **m^* = 1.13(50)**
- **m^* = 1.53(25)**

J. Baró: jordi.barourbea@ucalgary.ca (UCalgary)

Critical Failure & Rheology

2018 CAP Congress June 13, 2018
Complementary Cumulative Distribution of Energies:

V32: $k=2$

$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

G26: $k=3$

$2.5e-07 < f < 1.1e-03$

$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

SR2: $k=1$

$2.2e-07 < f < 7.4e-05$

$7.4e-05 < f < 2.3e-03$

$2.3e-03 < f < 1.2e-02$

$1.2e-02 < f < 2.6e-02$

V32: $k=3$

$2.5e-07 < f < 1.1e-03$

$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

G26: $k=4$

$2.5e-07 < f < 1.1e-03$

$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

SR2: $k=2$

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$1.3e-02 < f < 1.6e-01$

V32: $k=4$

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$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

G26: $k=5$

$2.5e-07 < f < 1.1e-03$

$1.1e-03 < f < 1.2e-03$

$1.2e-03 < f < 1.3e-02$

$1.3e-02 < f < 1.6e-01$

SR2: $k=3$

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$1.1e-03 < f < 1.2e-03$

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An experimental case: Uniaxial compression of SiO$_2$ porous materials

- Porous SiO$_2$ ($\Phi \sim 10\%-40\%$)
- Soft Uniaxial Compression
  - stress control ($\sim 1$ kPa/s)
  - no lateral confinement

- Strain Monitoring ($\sim \mu$m)
- Acoustic Emission Recording ($\sim MHz$)
- Fractures & Crackling Noise (AE) (10$k - 30$k events)

$$E \sim \int |\text{Signal}(t)|^2 dt$$
An experimental case: Uniaxial compression of SiO$_2$ porous materials

Critical Failure?

Data Agree

- Ultimate brittle event ($P_c^5$)

Data Disagree

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- Aftershock sequences.

Rheology?

$E \sim \int |\text{Signal}(t)|^2 dt$  [Baró et al. PRL (2013), Baró et al. PRL (2018)]
An experimental case: Uniaxial compression of SiO$_2$ porous materials

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$E \sim \int |\text{Signal}(t)|^2 dt$ [Baró et al. PRL (2013), Baró et al. PRL (2018)]
Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

Standard (democratic fiber bundle) Model

Micromechanics:

\[
\sigma_l = \begin{cases} 
E \varepsilon & (E \varepsilon < S_i) \\
0 & (E \varepsilon \geq S_i)
\end{cases}
\]

Mean Field:

\[
\sigma_l = \frac{\sigma}{N_{\text{fibers}}}
\]
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- **Macroscopic constitutive equation:**
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Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

(generalized) Viscoelastic (democratic fiber bundle) Model

Micromechanics:

\[ E \Delta \varepsilon(t) = (1 - H_\alpha(t/\tau)) \Delta \sigma_i \]

Transient:

\[
\begin{align*}
H_\alpha(0) & \rightarrow h \\
H_\alpha(\infty) & \rightarrow 0
\end{align*}
\]
Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

(generalized) Viscoelastic (democratic fiber bundle) Model

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  \[ E \Delta \varepsilon(t) = (1 - H_\alpha(t/\tau)) \Delta \sigma_i \]
  
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  \[ \begin{cases} 
    H_\alpha(0) & \rightarrow h \\
    H_\alpha(\infty) & \rightarrow 0 
  \end{cases} \]

- Macroscopic constitutive equation:
  \[ \sigma(E\varepsilon, t) = E\varepsilon \left( \frac{1}{1 - F(E\varepsilon)} - \sum_{s_j < E\varepsilon} \phi_j(t - t_j) \right)^{-1} \]
Viscoelasticity in the Democratic (mean field) Fiber Bundle Model

(generalized) Viscoelastic (democratic fiber bundle) Model

- Micromechanics:
  \[ E \Delta \varepsilon(t) = (1 - H_\alpha(t/\tau)) \Delta \sigma_i \]

Transient:
\[
\begin{align*}
H_\alpha(0) & \rightarrow h \\
H_\alpha(\infty) & \rightarrow 0
\end{align*}
\]

- Macroscopic constitutive equation:
  \[
  \sigma(E\varepsilon, t) = E\varepsilon \left( \frac{1}{1 - F(E\varepsilon)} - \sum_{j} \phi_j (t - t_j) \right)^{-1}
  \]

\[ \sigma(E\varepsilon|h = 0) \quad \sigma(E\varepsilon, t|h) \]

Driving

\[ S_i \quad E\varepsilon \quad \beta \quad E\varepsilon \quad S_i \]

\[ 2H(0) \quad 3 \]

\[ S_i \quad S_{i+\Delta t} \quad S_i \quad S_{i+\Delta t} \quad S_i \quad S_{i+\Delta t} \quad S_{i+\Delta t+\Delta t'} \]

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2018 CAP Congress June 13, 2018 7/9
From const. eq.: An avalanche start at $S_i$, stops when: $\sigma(\varepsilon) \geq \sigma(S_i)$

**Universal Avalanche Definition:**

$$\xi(\Delta_i) > B(S_i|h)\Delta_i$$

(hitting times problem)

$\xi(\Delta)$ : Poisson counting process of $\Delta$ steps.

[Baró & Davidsen, PRE (2018)]
From const. eq.: An avalanche start at \( S_i \), stops when: 
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\[ \xi(\Delta_i) > B(S_i|h)\Delta_i \]  
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\( \xi(\Delta) \) : Poisson counting process of \( \Delta \) steps.
- \( B > 1 \): Prob. \( \Delta \rightarrow \infty \).
- \( B < 1 \): Size distribution: 
  \[ D(\Delta;B)d\Delta = \Delta^{-3/2}D(\Delta|1-B)d\Delta. \]
- \( B = 1 \): Critical.

[Baró & Davidsen, PRE (2018)]
Universal Avalanche Statistics for Fibrous Models

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- Slope $B$ is function of state:

$$B(S_i|h) = \frac{S_i \text{ pdf}(S_i)}{1 - F(S_i)} (1 - h)$$

- $\xi(\Delta)$: Poisson counting process of $\Delta$ steps.
  - $B > 1$: Prob. $\Delta \to \infty$.
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- At failure ($d\sigma/d\epsilon|_{\sigma_f} = 0$):
  $$B(E\epsilon_f|h) = (1-h)$$

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- At failure ($d\sigma/d\varepsilon|_{\sigma_f} = 0$):
  $$B(E\varepsilon_f|h) = (1 - h)$$

- Critical failure for $h = 0$.
- Subcrit. failure for $h > 0$.

- $\xi(\Delta)$: Poisson counting process of $\Delta$ steps.
  - $B > 1$: Prob. $\Delta \to \infty$.
  - $B < 1$: Size distribution: $D(\Delta; B)d\Delta = \Delta^{-3/2} D(\Delta|1 - B)) d\Delta$.
  - $B = 1$: Critical.

[Baró & Davidsen, PRE (2018)]
Triggering in experiments and model

**Triggering in lab:** [Baró et al., PRL (2013)]

**Triggering in viscoelastic model:** [Baró & Davidsen, PRE (2018)]

Under slow $\sigma$ driving:

- Accelerated energy release?
  \[
  \frac{d\text{Energy}}{dt} \propto \frac{dF(E\varepsilon)}{d\sigma} \propto \frac{d(E\varepsilon)}{d\sigma}
  \]

- Avalanche sizes?

\[
\sigma_f
\]
Deformation as Avalanche Phenomena

- Non-linear Deformation $\rightarrow$ Avalanche Dynamics:

  - Low Temperature
  - Quenched Disorder
  - Interactions

Crystalline nano-pillars

Tectonic Gouges

[N. Friedman et al., PRL (2012)]
[T. Hatano, C. Narteau, P. Schebalin, SREP (2015)]
Deformation as Avalanche Phenomena

- Non-linear Deformation → Avalanche Dynamics:

- Failure: Avalanche ≈ Phase Transition

\[ \text{order} \quad \text{disorder} \]

\[ \text{FOPT} \quad \text{CPT} \]

Crackling Snapping Popping

\[ \begin{align*}
\text{Low Temperature} \\
\text{Quenched Disorder} \\
\text{Interactions}
\end{align*} \]

Crystalline nano-pillars

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Deformation as Avalanche Phenomena

- Non-linear Deformation → Avalanche Dynamics:
- Failure: Avalanche ≈ Phase Transition

Low Temperature
- Quenched Disorder
- Interactions

Crystalline nano-pillars

- Failure Prediction?
  - Properties / State: (tomography, seismography)
  - Statistics of Avalanches: Acoustic Emission

[9. Friedman et al., PRL (2012)]

[T. Hatano, C. Narteau, P. Schebalin, SREP (2015)]

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Hypothesis: Failure is Critical

- Power Laws everywhere! (scale-invariance)

[A. De-Santis et al., BSSA (2011)]
[S. Goodfellow & P. Young, GRL (2014)]

- Acceleration of Activity / Energy Released

[D. Amitrano, JGR (2003)]
Hypothesis: Failure is Critical

- Power Laws everywhere! (scale-invariance)
  - Day w.r.t. main shock (6 Apr. 2009)
  - [A. De-Santis et al., BSSA (2011)]
  - [S. Goodfellow & P. Young, GRL (2014)]

- Failure is a Critical Point
  - Distribution of Sizes ($\Delta$)
    \[ D(\Delta; |t_c - t|)d\Delta = \Delta^{-\kappa}D(\Delta|t_c - t|^{1/\sigma})d\Delta \]
    (reproduced by most models) [K. Dahmen, et al. (2011)]

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\[ D(\Delta; |t_c - t|)d\Delta = \Delta^{-\kappa} D(\Delta|t_c - t|^{1/\sigma})d\Delta \]

(reproduced by most models) \[K.\ Dahmen, et al. (2011)]

- Acceleration of Activity / Energy Released

\[ d\Delta/dt \propto \langle \Delta \rangle \sim |t_c - t|^{-\kappa - 2/\sigma} \]

[D. Amitrano, JGR (2003)]

[A. De-Santis et al., BSSA (2011)]

[S. Goodfellow & P. Young, GRL (2014)]