# Matching the Quasi Parton Distribution in a Momentum Subtraction Scheme

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Lattice PDF Workshop
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I. Stewart and Y.Z., PRD 2018, arXiv:1709.04933

# Outline

- PDF from lattice QCD through LaMET
- Nonperturbative renormalization of the quasi PDF
- Match quasi RI/MOM PDF to MSbar PDF
- Numerical results

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# Parton Distribution Function

- Definition of PDFs in QCD factorization theorems:

$$q(x,\mu)=\int\frac{d\xi^-}{4\pi}e^{-ixP^+\xi^-}\left\langle P\middle|\bar{\psi}(\xi^-)\gamma^+U(\xi^-,0)\psi(0)\middle|P\right\rangle$$

$$\sigma=\sum_{a,b}f_a(x_1)\otimes f_b(x_2)\otimes\sigma_{ab}$$

$$\xi^\pm=(t\pm z)/\sqrt{2}\quad U(\xi^-,0)=P\exp\left[-ig\int_0^{\xi^-}d\eta^-A^+(\eta^-)\right]$$

- Gauge-invariant and boost-invariant light-cone correlation;
- In the light-cone gauge $A^+=0$, has a clear interpretation as parton number density;
- Not directly calculable from lattice QCD due to real-time dependence of the light-cone.

# Large momentum effective theory

- Quasi-PDF:

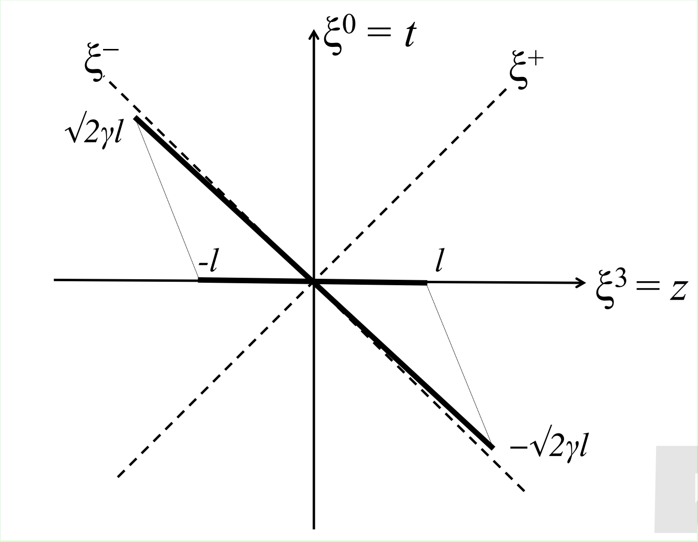
X. Ji, PRL 2013; Sci.China Phys.Mech.Astron. 2014.

$$\tilde{q}(x,P^z,\tilde{\mu}) = \int \frac{dz}{4\pi} e^{ixP^z z} \left\langle P \left| \bar{\psi}(z)\gamma^z U(z,0)\psi(0) \right| P \right\rangle$$

$$z^\mu = (0,0,0,z)$$

$$U(z,0) = P\exp\left[-ig\int_0^z dz' A^z(z')\right]$$

- Equal-time correlation along the *z* direction, calculable in lattice QCD when $P^z << a^{-1}$, dependent of $P^z$;
- Under an infinite Lorentz boost along the *z* direction, the spatial gauge link approaches the light-cone direction, and the quasi-PDF reduces to the (light-cone) PDF.



# Large momentum effective theory

- Taking the $P^z \to \infty$ limit of the quasi-PDF is ill-defined due to the latter's nontrivial dependence of $P^z$,
- The (renormalized) quasi PDF is related to the PDF through a factorization formula:

$$\tilde{q}_i^X(x, P^z, \tilde{\mu}) = \int_{-1}^{+1} \frac{dy}{|y|} C_{ij}^X\left(\frac{x}{y}, \frac{\tilde{\mu}}{P^z}, \frac{\mu}{|y|P^z}\right) q_j(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}\right), \quad (8)$$

  - They have the same IR divergences;
  - $C$ factor matches their UV difference, and can be calculated in perturbative QCD;
  - Higher-twist corrections suppressed by powers of $P^z$.

# Procedure of Systematic Calculation

1. Simulation of the quasi PDF in lattice QCD

3. Subtraction of higher twist corrections

$$\tilde{q}_i(x, P^z, \tilde{\mu}) = \int_{-1}^{+1} \frac{dy}{|y|}\, C_{ij}\left(\frac{x}{y}, \frac{\tilde{\mu}}{P^z}, \frac{\mu}{|y|P^z}\right) q_j(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}\right) ,$$

2. Renormalization of the lattice quasi PDF, and then taking the continuum limit

4. Matching to the MSbar PDF.

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# Renormalization

- The gauge-invariant quark Wilson line operator can be renormalized multiplicatively in the coordinate space:

$$\tilde{O}_\Gamma(z) = \bar{\psi}(z)\Gamma W(z,0)\psi(0) = Z_{\psi,z} e^{-\delta m|z|} \left(\bar{\psi}(z)\Gamma W(z,0)\psi(0)\right)^R$$

X. Ji, J.-H. Zhang, and Y.Z., 2017; J. Green, K. Jansen, and F. Steffens, 2017; T. Ishikawa, Y.-Q. Ma, J. Qiu, S. Yoshida, 2017.

- Different renormalization schemes can be converted to each other in coordinate space;

$$\begin{aligned}\tilde{Q}^X(\zeta, z^2\mu_R^2) &= \frac{Z_{\overline{\rm MS}}(\epsilon,\mu)}{Z_X(z^2\mu_R^2,\epsilon)}\, \tilde{Q}^{\overline{\rm MS}}(\zeta, z^2\mu^2) \\ &= Z'_X(z^2\mu_R^2, \mu_R^2/\mu^2)\, \tilde{Q}^{\overline{\rm MS}}(\zeta, z^2\mu^2)\,,\end{aligned}$$

- We can implement a nonperturbative renormalization scheme on the lattice.

# Regulator independence

- If we apply the same renormalization scheme in both lattice and continuum theories,

$$\begin{aligned}\tilde{O}_\Gamma^R(z,\mu) &= Z_X^{-1}(z,\varepsilon,\mu)\tilde{O}_\Gamma(z,\varepsilon) \\ &= \lim_{a\to 0} Z_X^{-1}(z,a^{-1},\mu)\tilde{O}_\Gamma(z,a^{-1})\end{aligned}$$

- This should apply to all renormalization schemes;
- After renormalization, we can just calculate the matching coefficient in DimReg;
- However, not all schemes can be implemented nonperturbatively on the lattice.

# A momentum subtraction scheme

Martinelli et al., 1994

- Regulator-independent momentum subtraction scheme (RI/MOM):

$$Z_{OM}^{-1}(z,a^{-1},p_R^z,\mu_R)\left\langle p\middle|\tilde{O}_\Gamma(z,a^{-1})\middle|p\right\rangle\Big|_{\substack{p^2=\mu_R^2\\ p^z=p_R^z}}=\left\langle p\middle|\tilde{O}_\Gamma(z)\middle|p\right\rangle_{\text{tree}}$$

$$Z_{OM}(z,a^{-1},p_R^z,\mu_R)=\frac{\left\langle p\middle|\tilde{O}_\Gamma(z,a^{-1})\middle|p\right\rangle\Big|_{\substack{p^2=\mu_R^2\\ p^z=p_R^z}}}{\left\langle p\middle|\tilde{O}_\Gamma(z)\middle|p\right\rangle_{\text{tree}}}=\frac{\left\langle p\middle|\tilde{O}_\Gamma(z)\middle|p\right\rangle\Big|_{\substack{p^2=\mu_R^2\\ p^z=p_R^z}}}{(4p_R^\Gamma\zeta)e^{-ip_R^z*z}}$$

- Can be implemented nonperturbatively on the lattice.
- Scales in renormalization: $\mu_R$, $p_R{}^z$

# Nonperturbative renormalization on the lattice

- For $\Gamma=\gamma^z$, we have to choose $p_R^z \neq 0$; for $\Gamma=\gamma^t$, we can choose $p_R^z = 0$ while $|p^2| >> \Lambda_{QCD}$;

$$Z_{OM}(z, a^{-1}, p_z^R, \mu_R) = \langle p | \tilde{O}_\Gamma(z) | p \rangle \Big|_{\substack{p^2=\mu_R^2 \\ p_z=p_z^R}} / (4 p_R^\Gamma \zeta e^{-ip_R^z * z})$$

- For nonzero $p_R^z$, $Z_{OM}$ is a complex number, real part symmetric and imaginary part anti-symmetric;
- Operator mixing on the lattice between $O_\Gamma$ and $O_1$ at $O(a^0)$ (for $\gamma^z$) and $O(a^1)$ (for $\gamma^t$) due to broken chiral symmetry.

M. Constantinou and H. Panagopoulos, 2017;
T. Ishikawa et al. (LP3), 2017.

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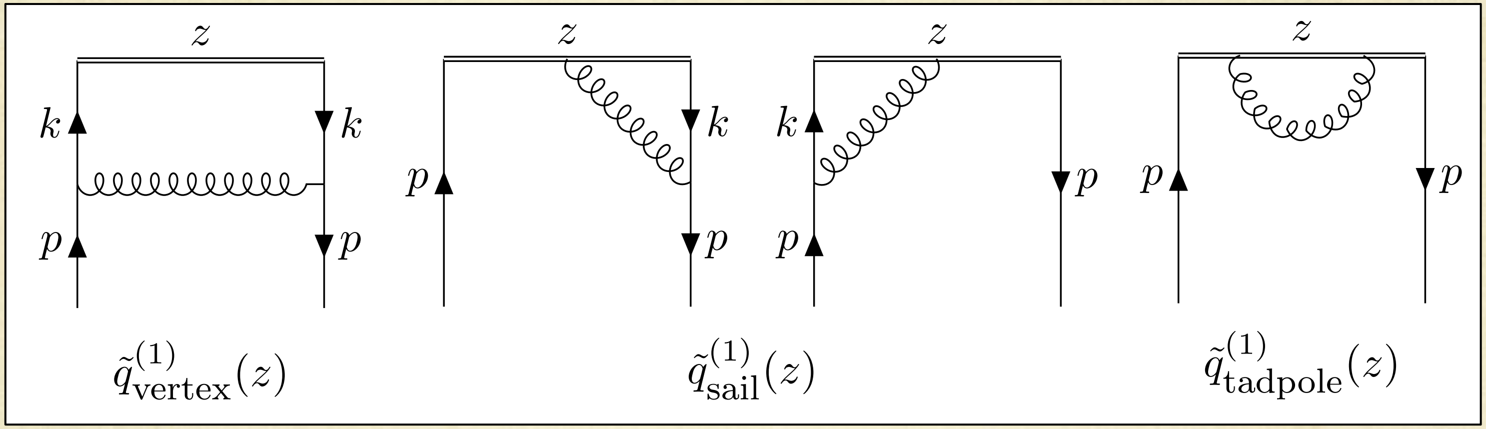
# Matching coefficient

**Strategy:**

- Extracting matching coefficient by comparing the quasi-PDF and light-cone PDF in an off-shell quark state;
- Quark off-shellness $p^2<0$ regulates the infrared (IR) and collinear divergences;

# One-loop Feynman diagrams




- Dimensional regularization $d=4-2\varepsilon$;
- $\Gamma=\gamma^z$ for discussion in this talk. External momentum $p^\mu=(p^0,0,0,p^z)$ and $p^2<0$;
- Fourier transform to the momentum space to obtain the quasi-PDF;

# One-loop Feynman diagrams

- Feynman rules:
  - e.g.

$$\tilde{q}^{(1)}_{\rm w.fn.}(z,p^z,\epsilon,-p^2) \quad = \quad \delta Z_\psi \;\; \tilde{q}^{(0)}(z,p^z)$$

$$\tilde{q}^{(1)}_{\rm vertex}(z,p^z,\epsilon,-p^2)+\tilde{q}^{(1)}_{\rm w.fn.}(z,p^z,\epsilon,-p^2)=\zeta\, p^z\int_{-\infty}^{\infty}dx\, e^{-ixp^z z}\,{\rm Tr}\left[\not{p}\int\frac{d^dk}{(2\pi)^d}(-igT^a\gamma^\mu)\frac{i}{\not{k}}\gamma^z\frac{i}{\not{k}}(-igT^a\gamma^\nu)\frac{-ig_{\mu\nu}}{(p-k)^2}\right]$$
$$\times\left[\delta(k^z-xp^z)-\delta(p^z-xp^z)\right], \qquad (22)$$

Three dimensional integration;
Ultraviolet (UV) convergent.

Four dimensional integration;
UV divergent, regularized by $\varepsilon$.

At bare level, which means keeping $\varepsilon$ finite, satisfies vector current conservation (V.C.C.);

$$\int dx\; \tilde{q}^{(1)}(x,p^z,\varepsilon,-p^2)=0$$

In the MSbar renormalization, a careful $\varepsilon$ expansion is needed;

Izubuchi, Ji, Jin, Stewart and Y.Z., 2018

In the first calculation with transverse momentum cutoff (Xiong, Ji, Zhang and Y.Z., 2014), we took $\varepsilon=0$ and write quasi-PDF as a plus function to enforce V.C.C..

$$\tilde{q}^{(1)}(x,p^z)=h(x,p^z)-\delta(1-x)\int dx'\,h(x',p^z)$$

# One-loop results

I. Stewart and YZ, PRD 2018, arXiv:1709.04933

- One-loop **bare** matrix element (with V.C.C.):

$$\tilde{q}^{(1)}(z,p^z,0,-p^2)=\frac{\alpha_s C_F}{2\pi}(4p^z\zeta)\int_{-\infty}^{\infty}dx\,\left(e^{-ixp^z z}-e^{-ip^z z}\right)h(x,\rho)$$

$$\rho\equiv\frac{(-p^2-i\varepsilon)}{p_z^2},$$

$$h(x,\rho)\equiv\begin{cases}\frac{1}{\sqrt{1-\rho}}\left[\frac{1+x^2}{1-x}-\frac{\rho}{2(1-x)}\right]\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}}-\frac{\rho}{4x(x-1)+\rho}+1 & x>1\\ \frac{1}{\sqrt{1-\rho}}\left[\frac{1+x^2}{1-x}-\frac{\rho}{2(1-x)}\right]\ln\frac{1+\sqrt{1-\rho}}{1-\sqrt{1-\rho}}-\frac{2x}{1-x} & 0<x<1\\ \frac{1}{\sqrt{1-\rho}}\left[\frac{1+x^2}{1-x}-\frac{\rho}{2(1-x)}\right]\ln\frac{2x-1-\sqrt{1-\rho}}{2x-1+\sqrt{1-\rho}}+\frac{\rho}{4x(x-1)+\rho}-1 & x<0\end{cases},$$

- Potential problem:

$$\lim_{|x|\to\infty}h(x,\rho)\sim-\frac{3}{2|x|},\quad \int_{-\infty}^{\infty}dx\;h(x,\rho)\text{ is logarithmically divergent needs }\varepsilon\text{ to be regularized!}$$

  - This logarithmic divergence is what needs to be treated carefully for the MSbar scheme;
  - **Not a problem for the RI/MOM scheme!**

# RI/MOM renormalization

- Renormalization in coordinate space: $\tilde{q}^{(1)}_{\rm CT}(z,p^z,p^z_R,\mu_R) = -Z^{(1)}_{\rm OM}(z,p^z_R,0,\mu_R)\,\tilde{q}^{(0)}(z,p^z)$.

$$\tilde{q}^{(1)}_{\rm OM}(z,p^z,p^z_R,-p^2,\mu_R) = \tilde{q}^{(1)}(z,p^z,0,-p^2) + \tilde{q}^{(1)}_{\rm CT}(z,p^z,p^z_R,\mu_R)$$

$$\tilde{q}^{(1)}(z,p^z,0,-p^2) = \frac{\alpha_s C_F}{2\pi}(4p^z\zeta)\int_{-\infty}^{\infty}dx\,\left(e^{-ixp^z z} - e^{-ip^z z}\right)h(x,\rho) \qquad \rho = \frac{-p^2}{p_z^2} = \frac{p_z^2 - p_0^2}{p_z^2} < 1$$

$$\tilde{q}^{(1)}_{\rm CT}(z,p^z,p^z_R,\mu_R) = -\frac{\alpha_s C_F}{2\pi}(4p^z\zeta)\int_{-\infty}^{\infty}dx\,\left(e^{i(1-x)p^z_R z - ip^z z} - e^{-ip^z z}\right)h(x,r_R) \qquad r_R = \frac{\mu_R^2}{(p_R^z)^2} = \frac{(p_R^4)^2 + (p_R^z)^2}{(p_R^z)^2} > 1 \text{ for lattice momentum,}$$

analytical continuuation from $\rho < 1$!

- Identify the collinear divergence: onshell limit!

$$\tilde{q}^{(1)}_{\rm OM}(z,p^z,p^z_R,-p^2 << p_z^2,\mu_R) = \tilde{q}^{(1)}(z,p^z,0,-p^2 << p_z^2) + \tilde{q}^{(1)}_{\rm CT}(z,p^z,p^z_R,\mu_R)$$

$$\tilde{q}^{(1)}(z,p^z,0,-p^2 \ll p_z^2) = \frac{\alpha_s C_F}{2\pi}(4p^z\zeta)\int_{-\infty}^{\infty}dx\,\left(e^{-ixp^z z} - e^{-ip^z z}\right)h_0(x,\rho),$$

$$h_0(x,\rho) \equiv \begin{cases} \dfrac{1+x^2}{1-x}\ln\dfrac{x}{x-1} + 1 & x > 1 \\ \dfrac{1+x^2}{1-x}\ln\dfrac{4}{\rho} - \dfrac{2x}{1-x} & 0 < x < 1 \\ \dfrac{1+x^2}{1-x}\ln\dfrac{x-1}{x} - 1 & x < 0 \end{cases},$$

# RI/MOM renormalization

- Fourier transform to obtain the *x*-dependent quasi-PDF:

$$\tilde{q}^{(1)}_{\rm OM}(x,p^z,p^z_R,\mu_R) = \int \frac{dz}{2\pi}\, e^{ixzp^z}\, \tilde{q}^{(1)}_{\rm OM}(z,p^z,p^z_R,\mu_R)$$
$$= \frac{\alpha_s C_F}{2\pi}(4\zeta)\left\{\int dy\,\left[\delta(y-x)-\delta(1-x)\right]\left[h_0(y,\rho)-h(y,r_R)\right] + h(x,r_R) - |\eta|\, h\big(1+\eta(x-1),r_R\big)\right\},$$

$$\eta \equiv \frac{p^z}{p^z_R}$$

One can explicitly check that the RI/MOM quasi-PDF satisfies vector current conservation:

$$\int_{-\infty}^{\infty} dx\; \tilde{q}^{(1)}_{\rm OM}(x,p^z,p^z_R,-p^2,\mu_R) = \frac{\alpha_S C_F}{2\pi}(4\zeta)\left[\int_{-\infty}^{\infty} dx\; h(x,r_R) - \int_{-\infty}^{\infty} dx\; |\eta| h(1+|\eta|(x-1),r_R)\right] = 0$$

# RI/MOM renormalization

- Full result of RI/MOM quasi-PDF: Plus functions with $\delta$-function at $x$=1

$$\tilde{q}^{(1)}_{\rm OM}(x,p^z,p^z_R,\mu_R) \tag{37}$$
$$= \frac{\alpha_s C_F}{2\pi}(4\zeta)\begin{cases} \left[\frac{1+x^2}{1-x}\ln\frac{x}{x-1} - \frac{2}{\sqrt{r_R-1}}\left[\frac{1+x^2}{1-x} - \frac{r_R}{2(1-x)}\right]\arctan\frac{\sqrt{r_R-1}}{2x-1} + \frac{r_R}{4x(x-1)+r_R}\right]_\oplus & x>1 \\ \left[\frac{1+x^2}{1-x}\ln\frac{4(p^z)^2}{-p^2} - \frac{2}{\sqrt{r_R-1}}\left[\frac{1+x^2}{1-x} - \frac{r_R}{2(1-x)}\right]\arctan\sqrt{r_R-1}\right]_+ & 0<x<1 \\ \left[\frac{1+x^2}{1-x}\ln\frac{x-1}{x} + \frac{2}{\sqrt{r_R-1}}\left[\frac{1+x^2}{1-x} - \frac{r_R}{2(1-x)}\right]\arctan\frac{\sqrt{r_R-1}}{2x-1} - \frac{r_R}{4x(x-1)+r_R}\right]_\ominus & x<0 \end{cases}$$
$$+ \frac{\alpha_s C_F}{2\pi}(4\zeta)\Big\{h(x,r_R) - |\eta|\,h(1+\eta(x-1),r_R)\Big\}.$$

- Unregulated divergence in the $\delta(1-x)$ part? **No!**

$$\lim_{|x|\to\infty}\tilde{q}^{(1)}_{\rm OM}(x,p^z,p^z_R,-p^2,\mu_R)\sim\frac{1}{x^2},$$ integrable at infinity, no need to regularize!

- MSbar PDF:

$$q^{(1)}(x,\mu) = \frac{\alpha_s C_F}{2\pi}(4\zeta)\begin{cases} 0 & x>1 \\ \left[\frac{1+x^2}{1-x}\ln\frac{\mu^2}{-p^2} - \frac{1+x^2}{1-x}\ln\left[x(1-x)\right] - (2-x)\right]_+ & 0<x<1 \\ 0 & x<0 \end{cases}.$$

# Matching coefficient

- Matching coefficient for isovector quasi-PDF in **quark**:

$$\xi = \frac{x}{y}$$

$$C^{\rm OM}\left(\xi, \frac{\mu_R}{p^z_R}, \frac{\mu}{p^z}, \frac{p^z}{p^z_R}\right) - \delta(1-\xi) \tag{40}$$

$$= \frac{\alpha_s C_F}{2\pi} \begin{cases} \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1} - \frac{2(1+\xi^2)-r_R}{(1-\xi)\sqrt{r_R-1}}\arctan\frac{\sqrt{r_R-1}}{2\xi-1} + \frac{r_R}{4\xi(\xi-1)+r_R}\right]_\oplus & \xi>1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{4(p^z)^2}{\mu^2} + \frac{1+\xi^2}{1-\xi}\ln\left[\xi(1-\xi)\right] + (2-\xi) - \frac{2\arctan\sqrt{r_R-1}}{\sqrt{r_R-1}}\left\{\frac{1+\xi^2}{1-\xi} - \frac{r_R}{2(1-\xi)}\right\}\right]_+ & 0<\xi<1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi-1}{\xi} + \frac{2}{\sqrt{r_R-1}}\left[\frac{1+\xi^2}{1-\xi} - \frac{r_R}{2(1-\xi)}\right]\arctan\frac{\sqrt{r_R-1}}{2\xi-1} - \frac{r_R}{4\xi(\xi-1)+r_R}\right]_\ominus & \xi<0 \end{cases}$$
$$+ \frac{\alpha_s C_F}{2\pi}\left\{h(\xi, r_R) - |\eta|\, h\big(1+\eta(\xi-1), r_R\big)\right\},$$

- Matching coefficient for isovector **nucleon** quasi-PDF

$$p^z \to yP^z,\ \eta = yP^z / p^z_R$$

**RI/MOM matching also preserves particle number conservation of the nucleon PDF!**

# Comparison to two-step matching procedure

RI/MOM renormalization in coordinate space

Converting RI/MOM to MSbar scheme

Fourier Transform to obtain *x*-distribution of quasi-PDF in the MSbar scheme

Matching MSbar quasi-PDF to MSbar PDF

M. Constantinou and H. Panagopoulos, 2017;
J. Green, K. Jansen, and F. Steffens, 2017;
C. Alexandrou et al., 2017, 2018.

RI/MOM renormalization in coordinate space

Fourier Transform to obtain *x*-distribution of quasi-PDF in the RI/MOM scheme

Matching RI/MOM quasi-PDF to MSbar PDF

Stewart and Zhao, 2017;
J.W. Chen et al. (LP3), 2017, 2018.

# Other schemes

Transverse momentum cut-off scheme (Xiong, Ji, Zhang, and Y.Z., 2014):

Linear divergence

$$C^{\Lambda_T}\left(\xi, \frac{\mu}{p^z}, \frac{\Lambda}{P^z}\right) = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\begin{cases} \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_\oplus & \xi > 1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{4(p^z)^2}{\mu^2} + \frac{1+\xi^2}{1-\xi}\ln\xi(1-\xi) + 1 - \frac{2\xi}{1-\xi} + \frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_+ & 0<\xi<1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi-1}{\xi} - 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_\ominus & \xi < 0 \end{cases}$$

MSbar scheme: gives convergent matching integrals (Izubuchi, Ji, Jin, Stewart and Y.Z., 2018)

$$C^{\overline{\rm MS}}\left(\xi, \frac{\mu}{|y|P^z}\right) = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1} + 1 + \frac{3}{2\xi}\right)_{+(1)}^{[1,\infty]} - \frac{3}{2\xi} & \xi > 1 \\ \left(\frac{1+\xi^2}{1-\xi}\left[-\ln\frac{\mu^2}{y^2P_z^2} + \ln\left(4\xi(1-\xi)\right)\right] - \frac{\xi(1+\xi)}{1-\xi}\right)_{+(1)}^{[0,1]} & 0<\xi<1 \\ \left(-\frac{1+\xi^2}{1-\xi}\ln\frac{-\xi}{1-\xi} - 1 + \frac{3}{2(1-\xi)}\right)_{+(1)}^{[-\infty,0]} - \frac{3}{2(1-\xi)} & \xi < 0 \end{cases}$$

$$+ \frac{\alpha_s C_F}{2\pi}\delta(1-\xi)\left(\frac{3}{2}\ln\frac{\mu^2}{4y^2P_z^2} + \frac{5}{2}\right).$$

Plus functions with $\delta$-function at $x$=1

# Other schemes

Transverse momentum cut-off scheme (Xiong, Ji, Zhang, and Y.Z., 2014):

Unregulated UV divergence in the plus function

$\sim -\frac{3}{2|\xi|},\ \xi\to\infty$

$$C^{\Lambda_T}\left(\xi, \frac{\mu}{p^z}, \frac{\Lambda}{P^z}\right) = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\begin{cases} \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1+\frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_{\oplus} & \xi>1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{4(p^z)^2}{\mu^2}+\frac{1+\xi^2}{1-\xi}\ln\xi(1-\xi)+1-\frac{2\xi}{1-\xi}+\frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_{+} & 0<\xi<1 \\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi-1}{\xi}-1+\frac{1}{(1-\xi)^2}\frac{\Lambda_T}{P^z}\right]_{\ominus} & \xi<0 \end{cases}$$

MSbar scheme: gives convergent matching integrals (Izubuchi, Ji, Jin, Stewart and Y.Z., 2018)

$\sim -\frac{1}{\xi^2},\ \xi\to\infty$

$$C^{\overline{\rm MS}}\left(\xi, \frac{\mu}{|y|P^z}\right) = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1+\frac{3}{2\xi}\right)_{+(1)}^{[1,\infty]} - \frac{3}{2\xi} & \xi>1 \\ \left(\frac{1+\xi^2}{1-\xi}\left[-\ln\frac{\mu^2}{y^2P_z^2}+\ln\left(4\xi(1-\xi)\right)\right]-\frac{\xi(1+\xi)}{1-\xi}\right)_{+(1)}^{[0,1]} & 0<\xi<1 \\ \left(-\frac{1+\xi^2}{1-\xi}\ln\frac{-\xi}{1-\xi}-1+\frac{3}{2(1-\xi)}\right)_{+(1)}^{[-\infty,0]} - \frac{3}{2(1-\xi)} & \xi<0 \end{cases}$$
$$+\frac{\alpha_s C_F}{2\pi}\delta(1-\xi)\left(\frac{3}{2}\ln\frac{\mu^2}{4y^2P_z^2}+\frac{5}{2}\right).$$

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# Numerical results

- Take the iso-vector parton distribution $f_{u\text{-}d}$ as example:

$$f_{u-d}(x,\mu) = f_u(x,\mu) - f_d(x,\mu) - f_{\bar{u}}(-x,\mu) + f_{\bar{d}}(-x,\mu) \ ,$$

$$f_{\bar{u}}(-x,\mu) = -f_{\bar{u}}(x,\mu) \ , \qquad f_{\bar{d}}(-x,\mu) = -f_{\bar{d}}(x,\mu) \ .$$

- Input:
  - "MSTW 2008" PDF
  - NLO $\alpha_s(\mu)$

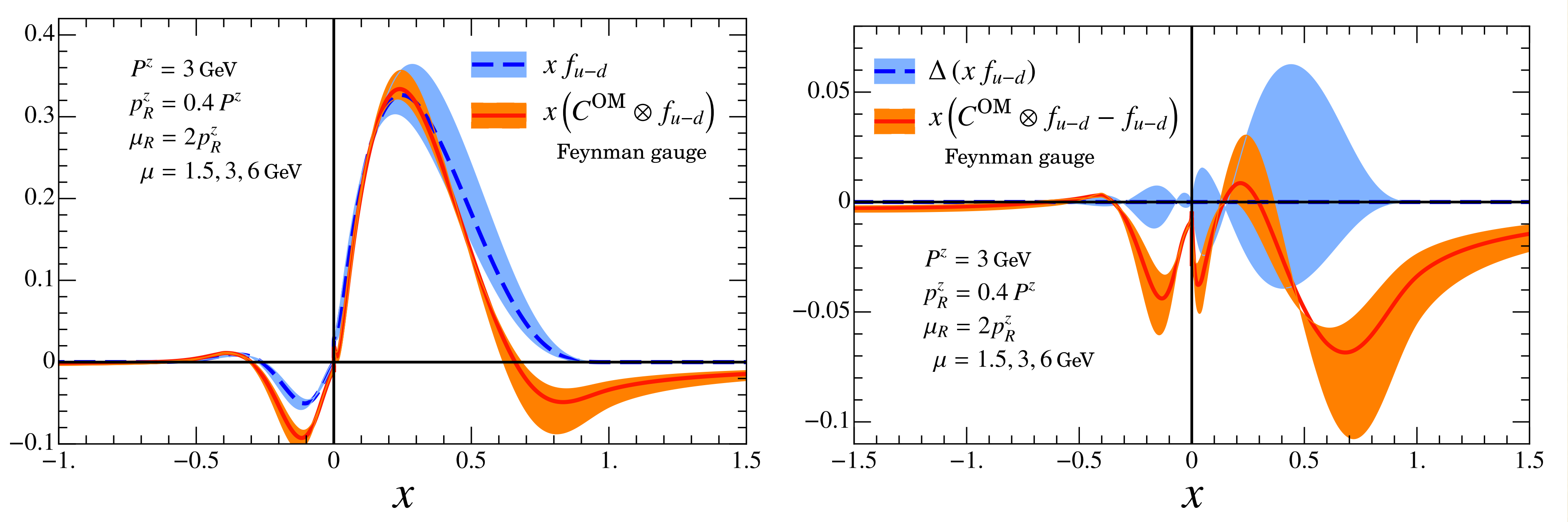
# Matching integral

$$\tilde{q}^{(1)}_{\mathrm{OM}}(x, P^z, p^z_R, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} C^{\mathrm{OM}}\left(\frac{x}{y}, \frac{\mu_R}{p^z_R}, \frac{\mu}{yP^z}, \frac{yP^z}{p^z_R}\right) f_{u-d}(y, \mu)$$

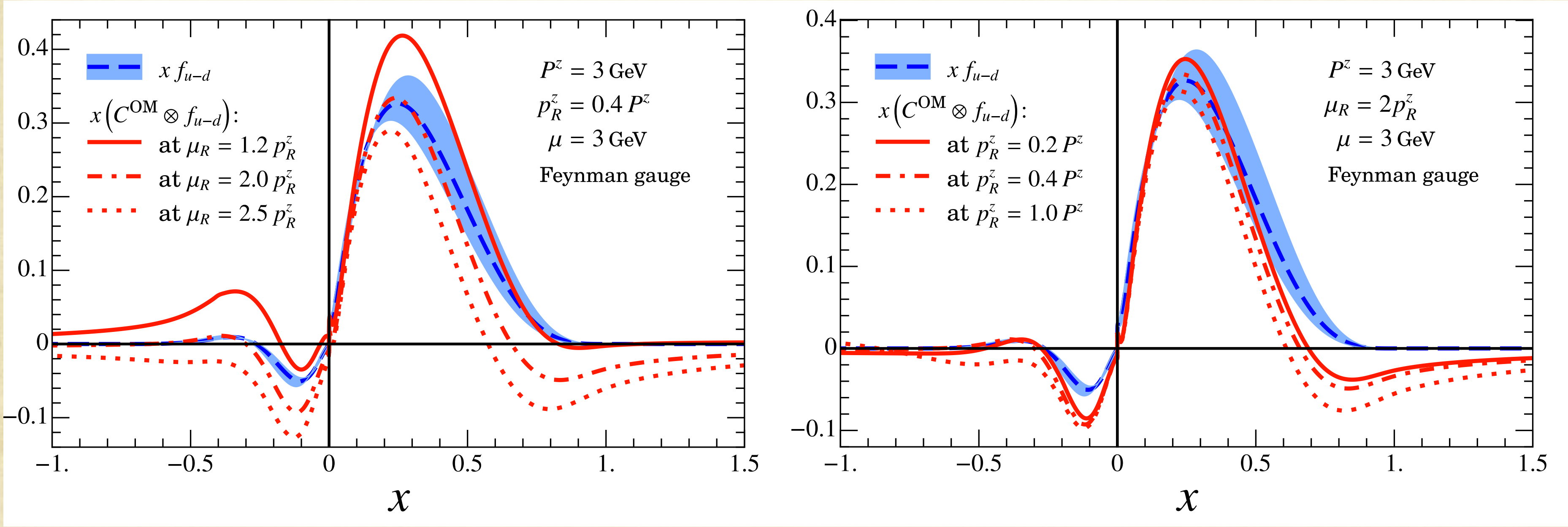
Four UV scales involved. Dependence on these scales introduces systematic uncertainty in the lattice calculation of PDF;

No singularities or divergences in MSbar or RI/MOM matching.

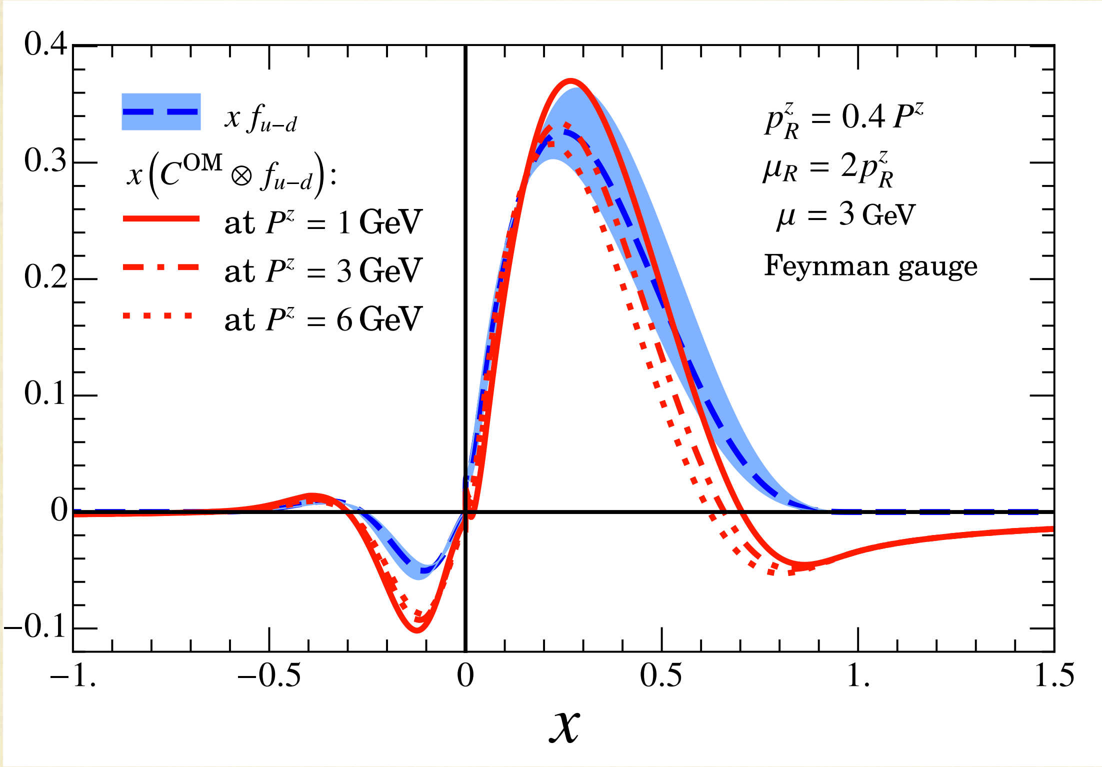
# Variation of factorization scale $\mu$



# Variation of RI/MOM scales $\mu_R$, $p_R{}^z$

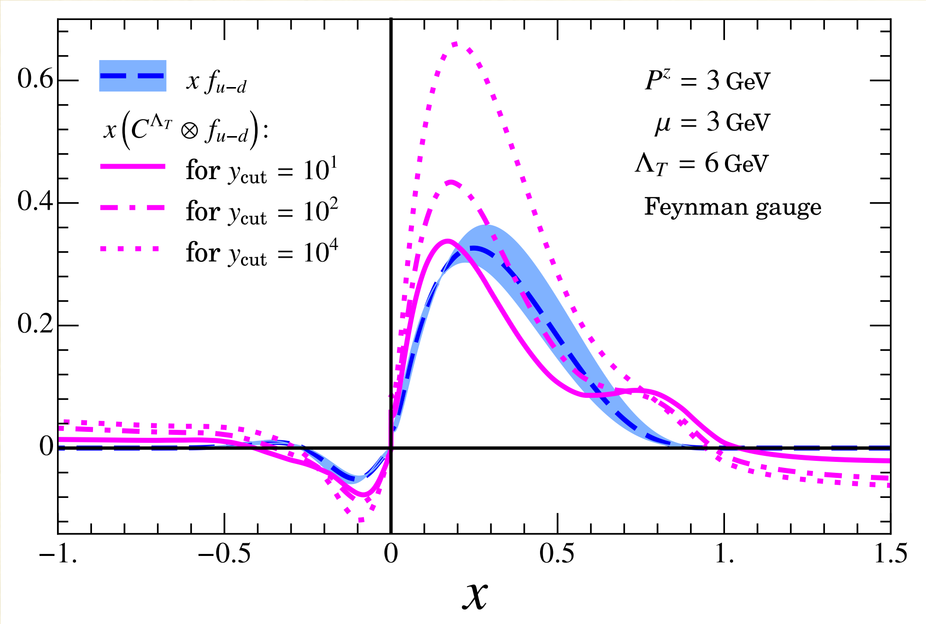


# Variation of nucleon momentum $P^z$



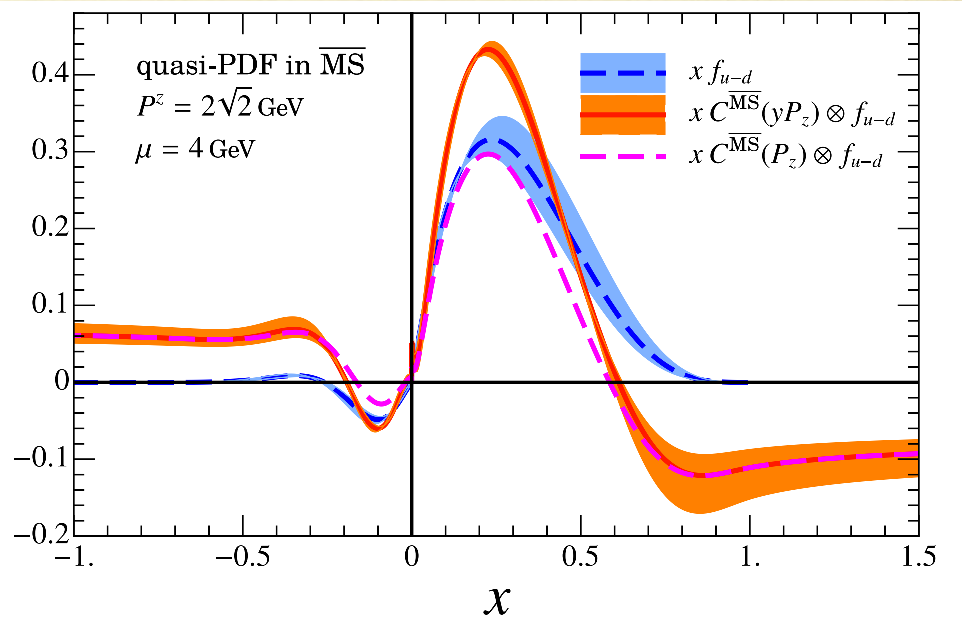
# Other schemes

Transverse momentum cut-off scheme




Xiong, Ji, Zhang, Zhao, 2014;

MSbar scheme




Izubuchi, Ji, Jin, Stewart and Y.Z., 2018

Recall unregulated UV divergence when $x/y \to \infty$, and $y/x \to \infty$, use a hard cut-off $y_{cut}=10^{\pm n}$.

# Scale dependence of the matching correction

- Dependence of $\mu$ and $P^z$ follow the Altarelli-Parisi equation, whose solution is known so we can resum the large logarithms of $\mu/P^z$;

- Dependence of $\mu_R$, $p_R^z$ is more complicated, and is scheme dependent. Large terms in one-loop correction could be resummed with a "renormalization group equation" (RGE),

$$\frac{d\tilde{q}(z, P^z, p^z_R, \mu_R)}{d\ln\mu_R} = \tilde{\gamma}(z, p^z_R, \mu_R)\, \tilde{q}(z, P^z, p^z_R, \mu_R)\,,$$

- It is simpler to make good choices of scales;

- The final result of the PDF from lattice calculation should be independent of the intermediate scales $P^z$, $\mu_R$, $p_R^z$ . Two-loop matching would be useful to test these perturbative uncertainties.

# Summary

- The implementation of the RI/MOM scheme on the nonperturbative renormalization of the quasi PDF in lattice QCD is discussed;
- The one step matching for RI/MOM quasi-PDF preserves vector current conservation, and leads to convergent matching integrals.
- Scale dependence of the matching correction introduces systematic uncertainty. RGE and NNLO calculation can be useful for high precision calculations.

# MSbar treatment

- Bare quasi-PDF:

$$\tilde{q}^{(1)}(x,p^z,\epsilon) = \frac{\alpha_s C_F}{2\pi}\left\{\frac{3}{2}\left(\frac{1}{\epsilon_{\rm UV}}-\frac{1}{\epsilon_{\rm IR}}\right)\delta(1-x)+\frac{\Gamma(\epsilon+\frac{1}{2})e^{\epsilon\gamma_E}}{\sqrt{\pi}}\frac{\mu^{2\epsilon}}{p_z^{2\epsilon}}\frac{1-\epsilon}{\epsilon_{\rm IR}(1-2\epsilon)}\right. \\ \left.\times\left[|x|^{-1-2\epsilon}\left(1+x+\frac{x}{2}(x-1+2\epsilon)\right)-|1-x|^{-1-2\epsilon}\left(x+\frac{1}{2}(1-x)^2\right)+I_3(x)\right]\right\}$$

$$I_3(x)=\theta(x-1)\left(\frac{x^{-1-2\epsilon}}{x-1}\right)^{[1,\infty]}_{+(1)}-\theta(x)\theta(1-x)\left(\frac{x^{-1-2\epsilon}}{1-x}\right)^{[0,1]}_{+(1)}-\delta(1-x)\pi\csc(2\pi\epsilon)+\theta(-x)\frac{|x|^{-1-2\epsilon}}{x-1}$$

- $\varepsilon$ expansion:

$$\int_0^\infty \frac{dx}{x^{1+\epsilon}}=\frac{1}{\epsilon_{\rm UV}}-\frac{1}{\epsilon_{\rm IR}}.$$

$$\frac{\theta(x)}{x^{1+\epsilon}}=\left[-\frac{1}{\epsilon_{\rm IR}}\delta(x)+\frac{1}{\epsilon_{\rm UV}}\frac{1}{x^2}\delta^+\left(\frac{1}{x}\right)\right] \\ +\left(\frac{1}{x}\right)^{[0,1]}_{+(0)}+\left(\frac{1}{x}\right)^{[1,\infty]}_{+(\infty)} \\ -\epsilon\left[\left(\frac{\ln x}{x}\right)^{[0,1]}_{+(0)}+\left(\frac{\ln x}{x}\right)^{[1,\infty]}_{+(\infty)}\right]+O(\epsilon^2)$$

Plus functions with $\delta$-function at $x=\pm\infty$, consistent with DimReg.

# MSbar treatment

- Renormalized quasi-PDF:

$$\delta\tilde{q}'^{(1)}(x,\mu/|p^z|,\epsilon_{\rm UV}) = \frac{\alpha_s C_F}{2\pi}\frac{3}{2\epsilon_{\rm UV}}\delta(1-x)\,,$$

$$\tilde{q}'^{(1)}(x,\mu/|p^z|,\epsilon_{\rm IR}) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\frac{1+x^2}{1-x}\ln\frac{x}{x-1}+1+\frac{3}{2x}\right)_{+(1)}^{[1,\infty]} - \left(\frac{3}{2x}\right)_{+(\infty)}^{[1,\infty]} & x>1 \\ \left(\frac{1+x^2}{1-x}\left[-\frac{1}{\epsilon_{\rm IR}}-\ln\frac{\mu^2}{4p_z^2}+\ln\big(x(1-x)\big)\right]-\frac{x(1+x)}{1-x}\right)_{+(1)}^{[0,1]} & 0<x<1 \\ \left(-\frac{1+x^2}{1-x}\ln\frac{-x}{1-x}-1+\frac{3}{2(1-x)}\right)_{+(1)}^{[-\infty,0]} - \left(\frac{3}{2(1-x)}\right)_{+(-\infty)}^{[-\infty,0]} & x<0 \end{cases}$$

$$+\frac{\alpha_s C_F}{2\pi}\left[\delta(1-x)\left(\frac{3}{2}\ln\frac{\mu^2}{4p_z^2}+\frac{5}{2}\right)+\frac{3}{2}\gamma_E\left(\frac{1}{(x-1)^2}\delta^+(\frac{1}{x-1})+\frac{1}{(1-x)^2}\delta^+(\frac{1}{1-x})\right)\right]$$

- Plus functions with $\delta$-function at $x=\pm\infty$ needed for V.C.C..

# ETMC's matching which has V.C.C.

- Corresponds to a modified MSbar quasi-PDF of quark

$$\tilde{q}^{(1)}(x,\mu/|p^z|,\epsilon_{\rm IR}) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \left(\frac{1+x^2}{1-x}\ln\frac{x}{x-1}+1+\frac{3}{2x}\right)_{+(1)}^{[1,\infty]} - \left(\frac{3}{2x}\right)_{+(\infty)}^{[1,\infty]} & x>1 \\ \left(\frac{1+x^2}{1-x}\left[-\frac{1}{\epsilon_{\rm IR}}-\ln\frac{\mu^2}{4p_z^2}+\ln\left(x(1-x)\right)\right]-\frac{x(1+x)}{1-x}\right)_{+(1)}^{[0,1]} & 0<x<1 \\ \left(-\frac{1+x^2}{1-x}\ln\frac{-x}{1-x}-1+\frac{3}{2(1-x)}\right)_{+(1)}^{[-\infty,0]} - \left(\frac{3}{2(1-x)}\right)_{+(-\infty)}^{[-\infty,0]} & x<0 \end{cases}$$

$$+\frac{\alpha_s C_F}{2\pi}\left[\delta(1-x)-\frac{1}{2}\frac{1}{x^2}\delta^+\left(\frac{1}{x}\right)-\frac{1}{2}\frac{1}{(1-x)^2}\delta^+\left(\frac{1}{1-x}\right)\right]\left(\frac{3}{2}\ln\frac{\mu^2}{4p_z^2}+\frac{5}{2}\right).$$

C. Alexandrou et al., 2018

# Ratios

A. Radyushkin, 2017;Zhang, Chen and Monahan, 2018; Izubuchi, Ji, Jin, Stewart and Y.Z., 2018

- Ratio of the quasi-PDF in quark in coordinate space:

$$\tilde{Q}^{(1)}(\zeta, z^2, \mu, \epsilon_{\rm IR}) = \frac{\alpha_s C_F}{2\pi}\left\{\frac{3}{2}\left(\ln\frac{\mu^2 z^2 e^{2\gamma_E}}{4}+1\right)e^{-i\zeta} + \left(-\frac{1}{\epsilon_{\rm IR}} - \ln\frac{z^2\mu^2 e^{2\gamma_E}}{4} - 1\right)h(\zeta) + \frac{2(1-i\zeta-e^{-i\zeta})}{\zeta^2} + 4i\zeta e^{-i\zeta}\,{}_3F_3(1,1,1,2,2,2,i\zeta)\right\}.$$

$$\zeta = zp^z$$

$$\tilde{Q}(0,z^2,\mu) = \frac{\alpha_S C_F}{2\pi}\cdot\left(\frac{3}{2}\ln\frac{\mu^2 z^2 e^{2\gamma_E}}{4}+1\right)$$

- V.C.C. is satisfied by the ratio:

$$\lim_{z\to 0}\frac{\tilde{Q}(zp^z, z^2, \mu, \varepsilon_{IR})}{\tilde{Q}(0, z^2, \mu, \varepsilon_{IR})} \sim z^n \ln(z^2) \to 0$$

- F.T. of the ratio should be similar to the ETMC matching coefficient.

- Can treat ~Q($0, z^2, \mu^2$) as additional renormalization constant for small |z|: modify both the RI/MOM to MSbar conversion factor for the quasi-PDF, and the matching.