S-matrix Bootstrap revisited

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S-matrix Bootstrap I: QFT in AdS [arXiv:1607.06109]
S-matrix Bootstrap II: two-dimensional amplitudes [arXiv:1607.06110]
S-matrix Bootstrap III: higher dimensional amplitudes [arXiv:1708.06765]
S-matrix Bootstrap IV: multiple amplitudes [work in progress]

with A. Homrich, M. Paulos, J. Toledo, B. Van Rees, P. Vieira

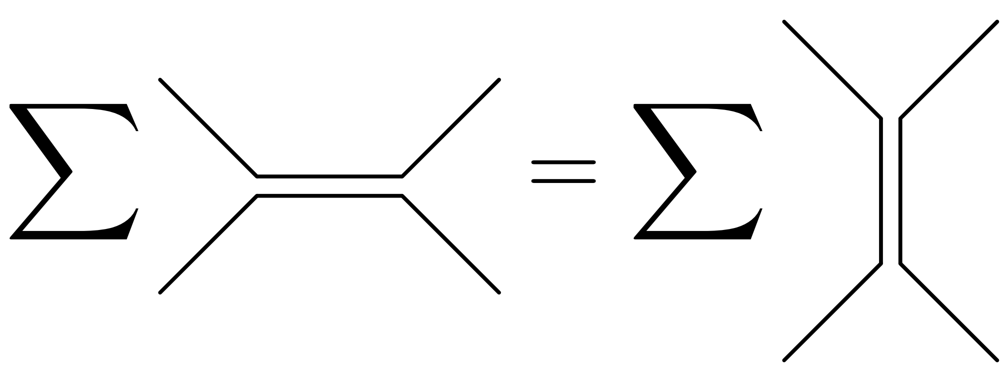
SUSY 2018, Barcelona, 27th of July, 2018

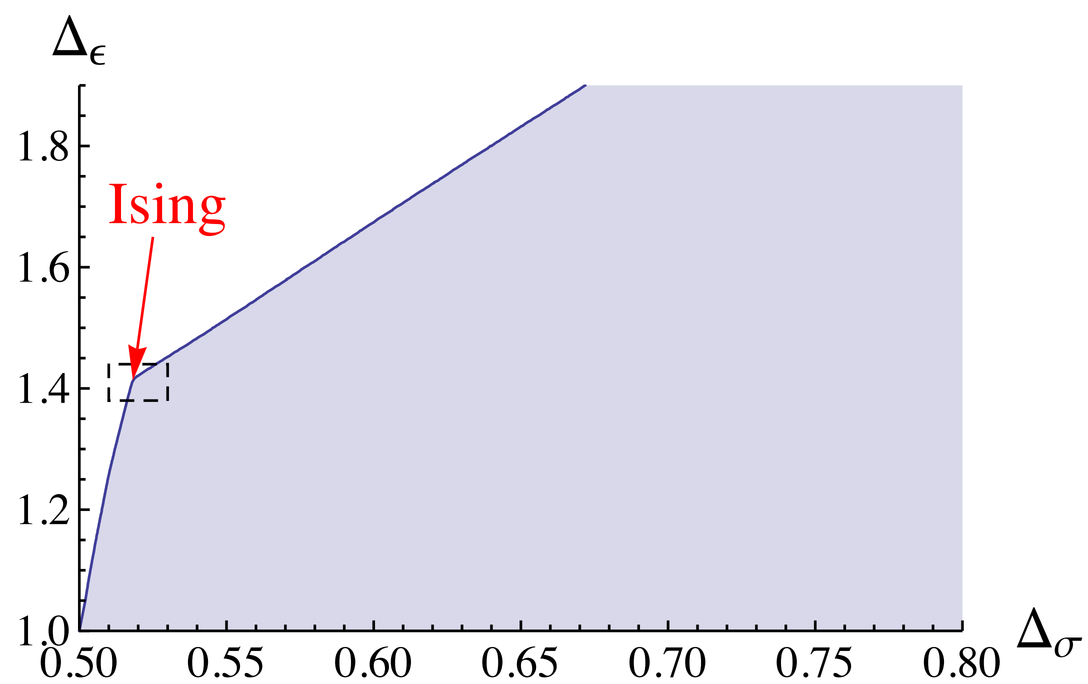
Motivation

Bootstrap Philosophy: bound the space of theories by imposing consistency conditions on physical observables.

Goal: extend recent success in CFT to massive QFT.

[Rattazzi, Rychkov, Tonni, Vichi '08] + many others





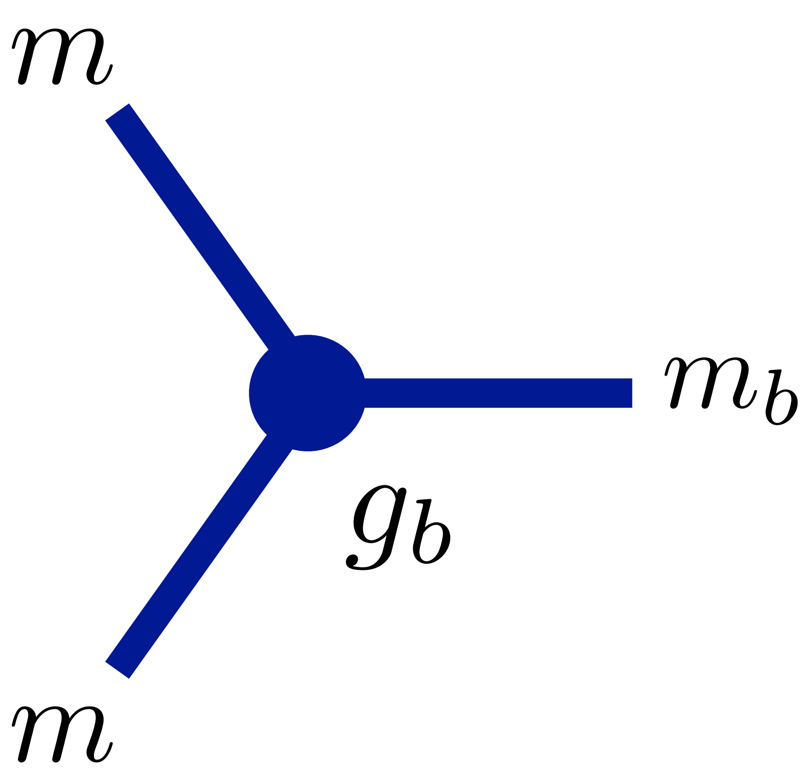
from [El-Showk et al '12]

Revisit the S-matrix Bootstrap program of the 60's and 70's.

Question

What is the **maximal coupling** compatible with the fundamental properties of QFT?

Lorentz invariance
Causality (analyticity)
Unitarity



$$\max\ g_b^2 = ?$$

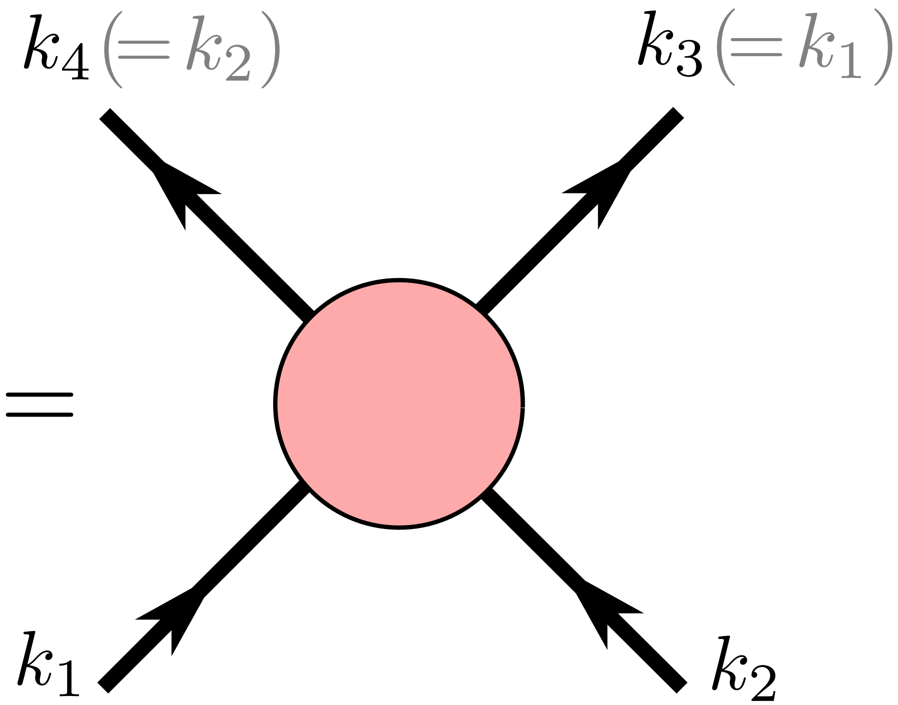
Outline

- S-matrix Bootstrap in D=2
- S-matrix Bootstrap in D>2
- Multiple Amplitudes Bootstrap in D=2
- Open questions

S-matrix Bootstrap
in 2D QFT

2 to 2 Scattering Amplitude

$S(s) =$



$$k_i^2 = m^2$$

$$s \equiv (k_1 + k_2)^2$$

$$t \equiv (k_2 - k_3)^2 \;\; = 4m^2 - s$$

$$u \equiv (k_3 - k_1)^2 = 0$$

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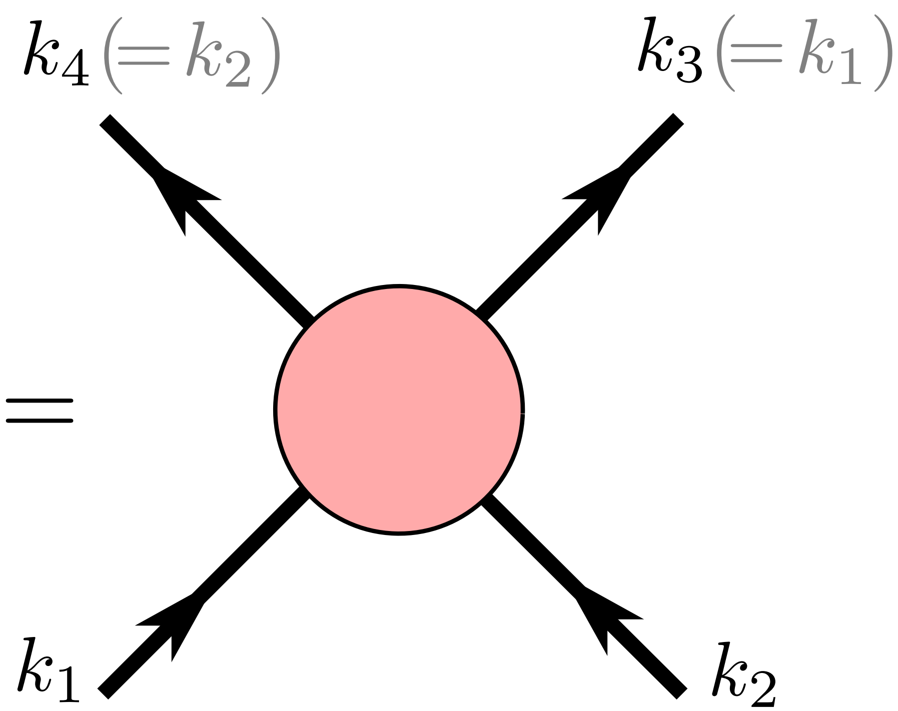

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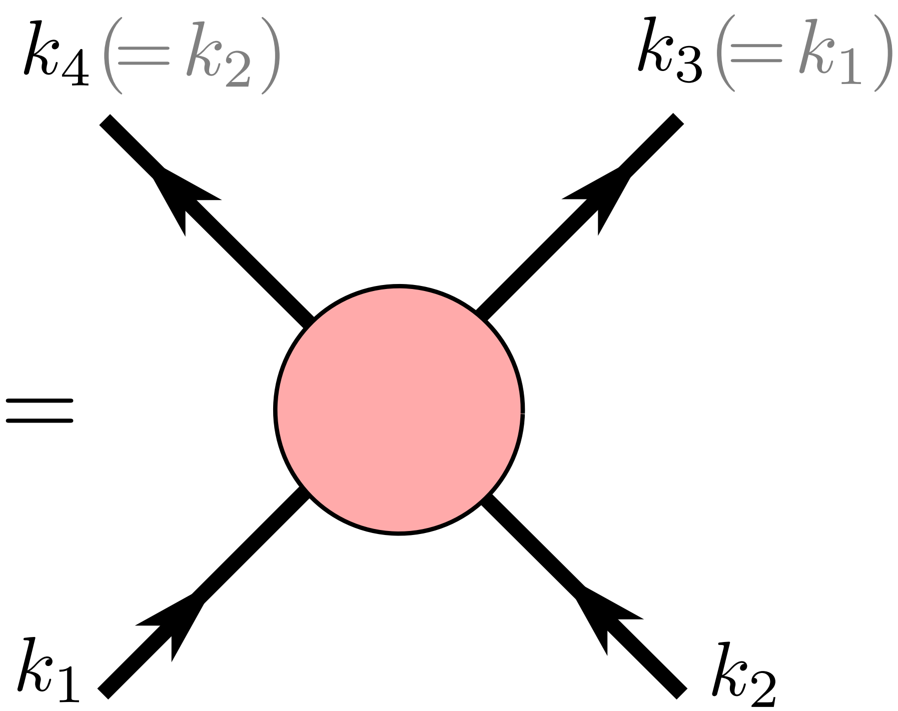

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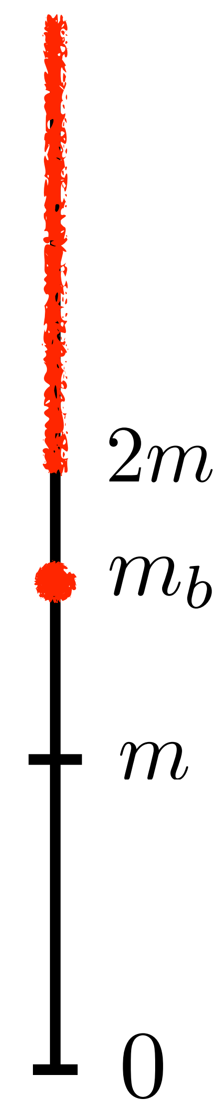

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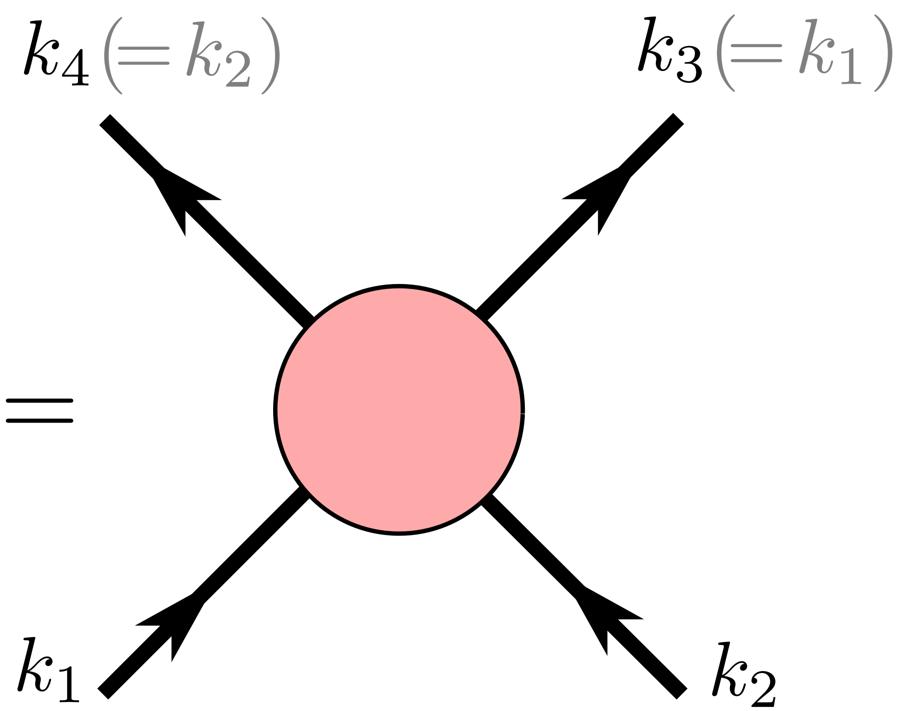

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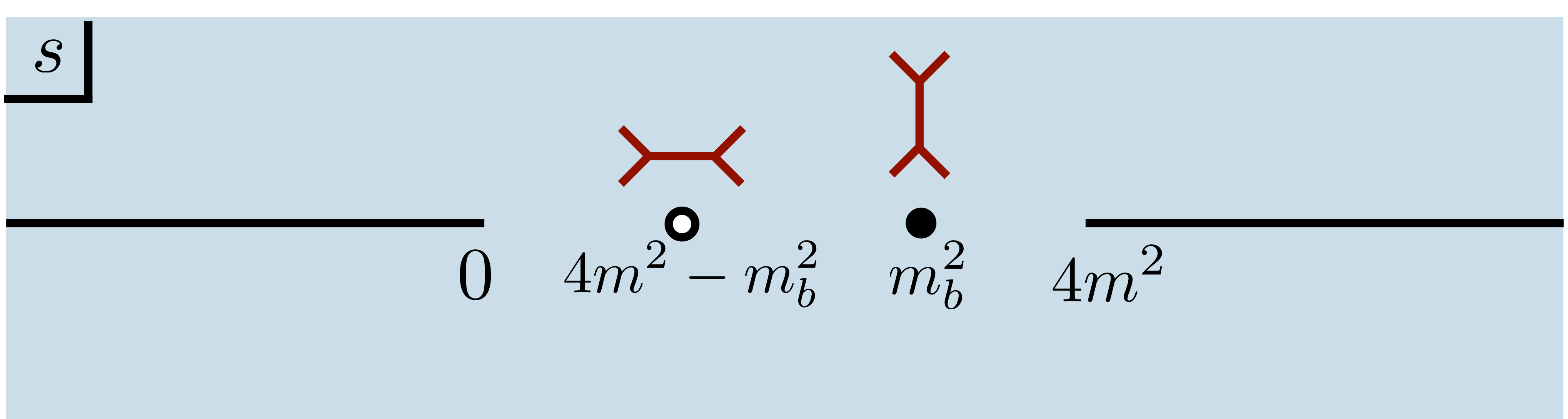

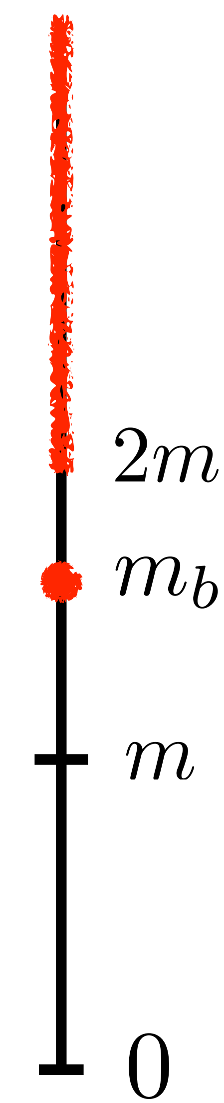

Constraints + question

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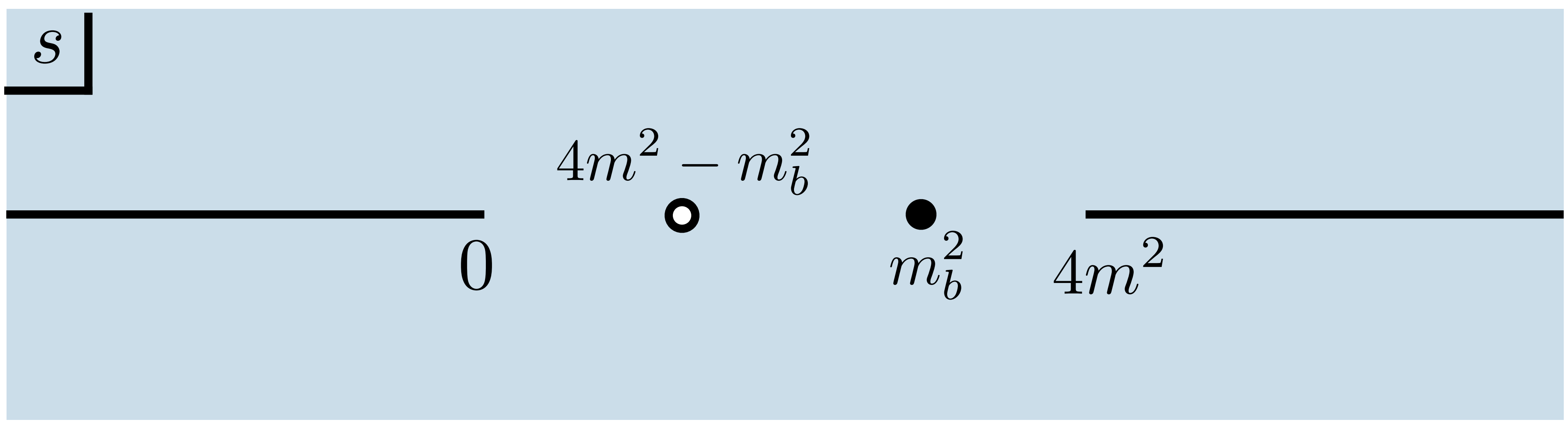

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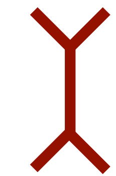
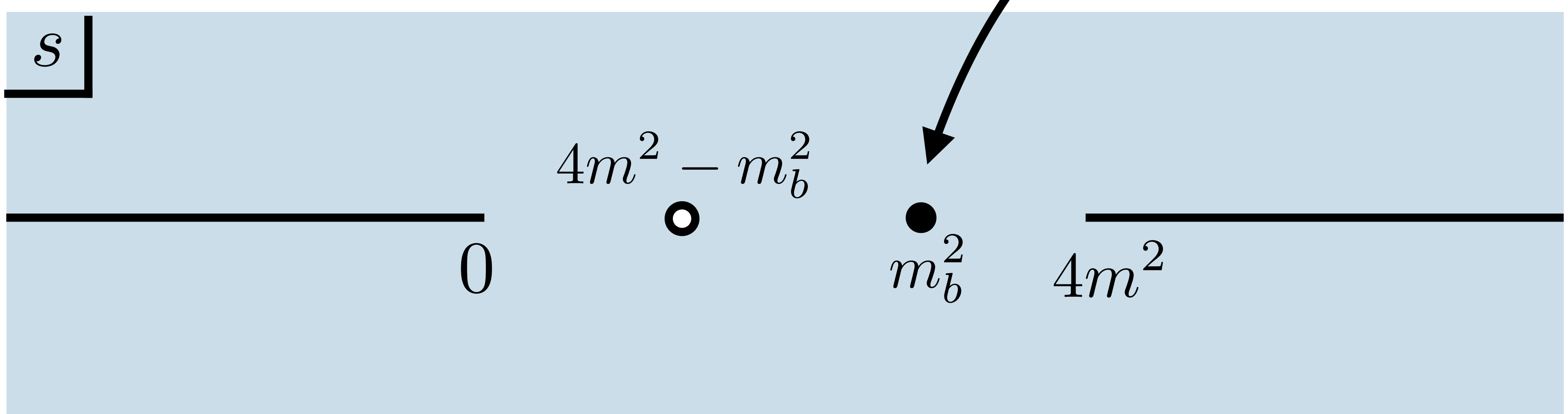
cubic coupling

$$S(s) \sim \frac{g_b^2}{s - m_b^2}$$





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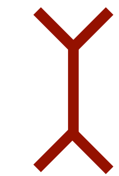

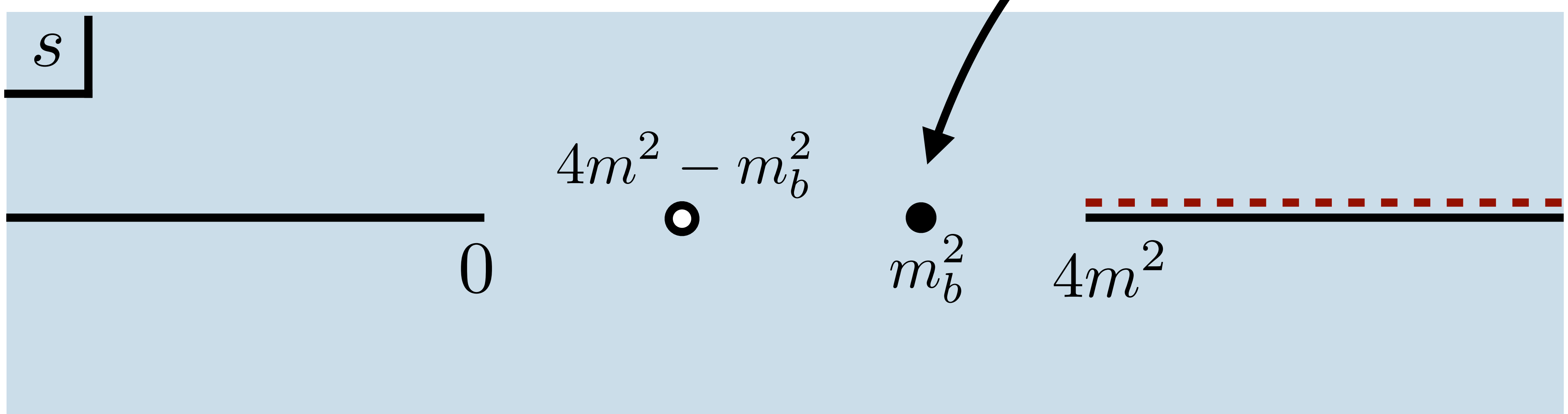

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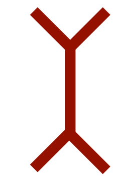

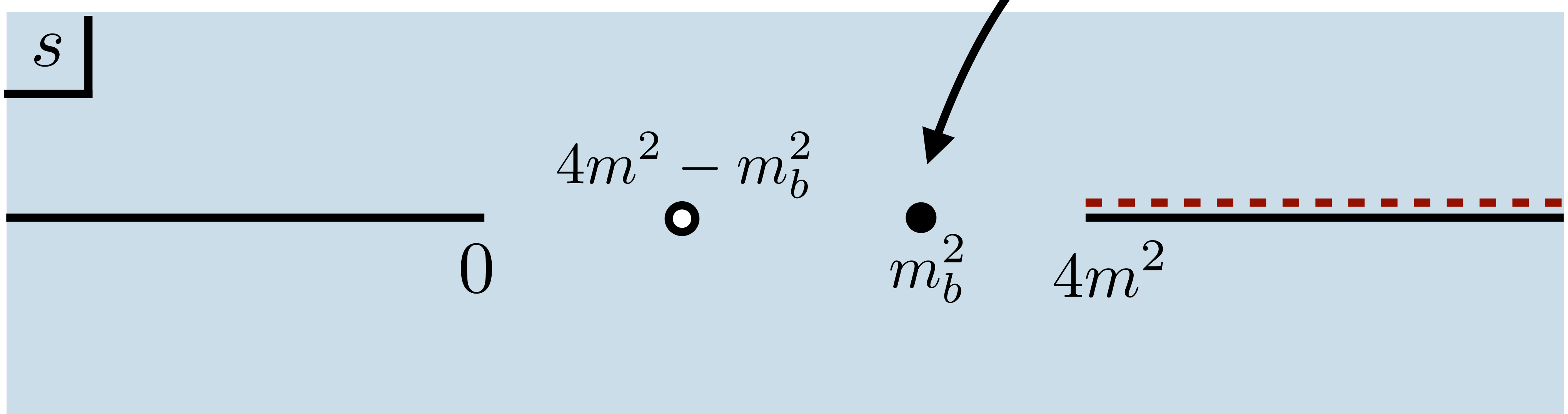

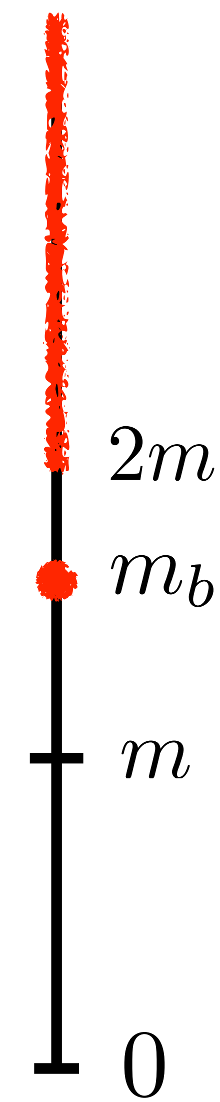

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Question: for given spectrum, $\boxed{\max\, g_b^2 = ?}$

Analytic solution

$$S_{opt}(s) = \frac{\sqrt{s(4m^2-s)} + \sqrt{m_b^2(4m^2-m_b^2)}}{\sqrt{s(4m^2-s)} - \sqrt{m_b^2(4m^2-m_b^2)}} \equiv [m_b](s)$$

[Symanzik '61]
[Creutz '72]

CDD factor
[Castillejo, Dalitz, Dyson]

Pole at $s = m_b^2 > 2$

No particle production $|S_{opt}(s)|^2 = 1\,, \qquad s > 4m^2\,.$

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Proof:

$$h(s) \equiv \frac{S(s)}{[m_b](s)}$$

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$h(s)$ analytic in the plane minus the cut

$|h(s)| \leq 1$ bounded at all boundaries

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$\Downarrow$ maximum modulus principle

$$|h(m_b^2)| = \left| \frac{g_b^2}{\mathrm{Res}_{s=m_b^2}[m_b](s)} \right| \leq 1$$

Maximum cubic coupling

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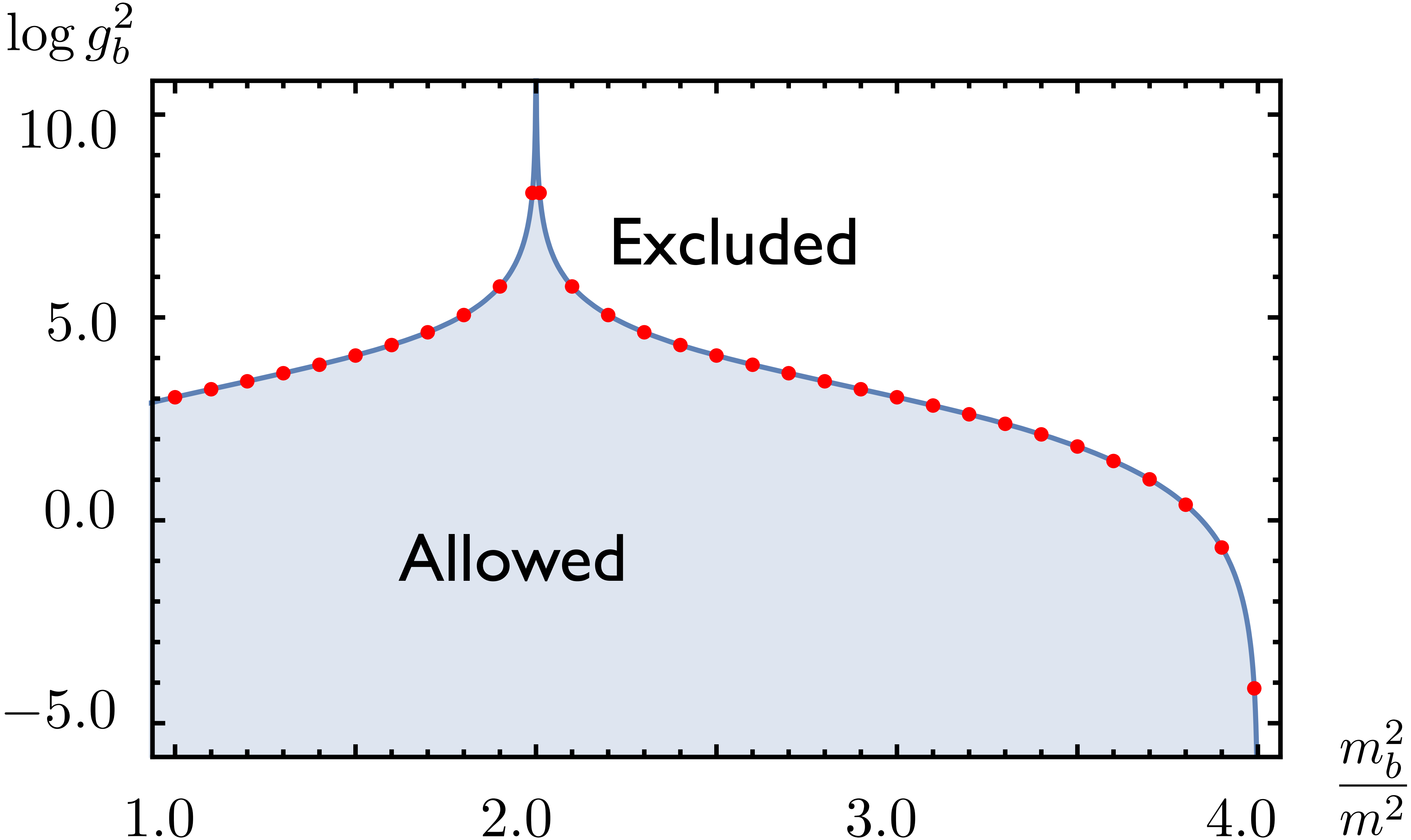

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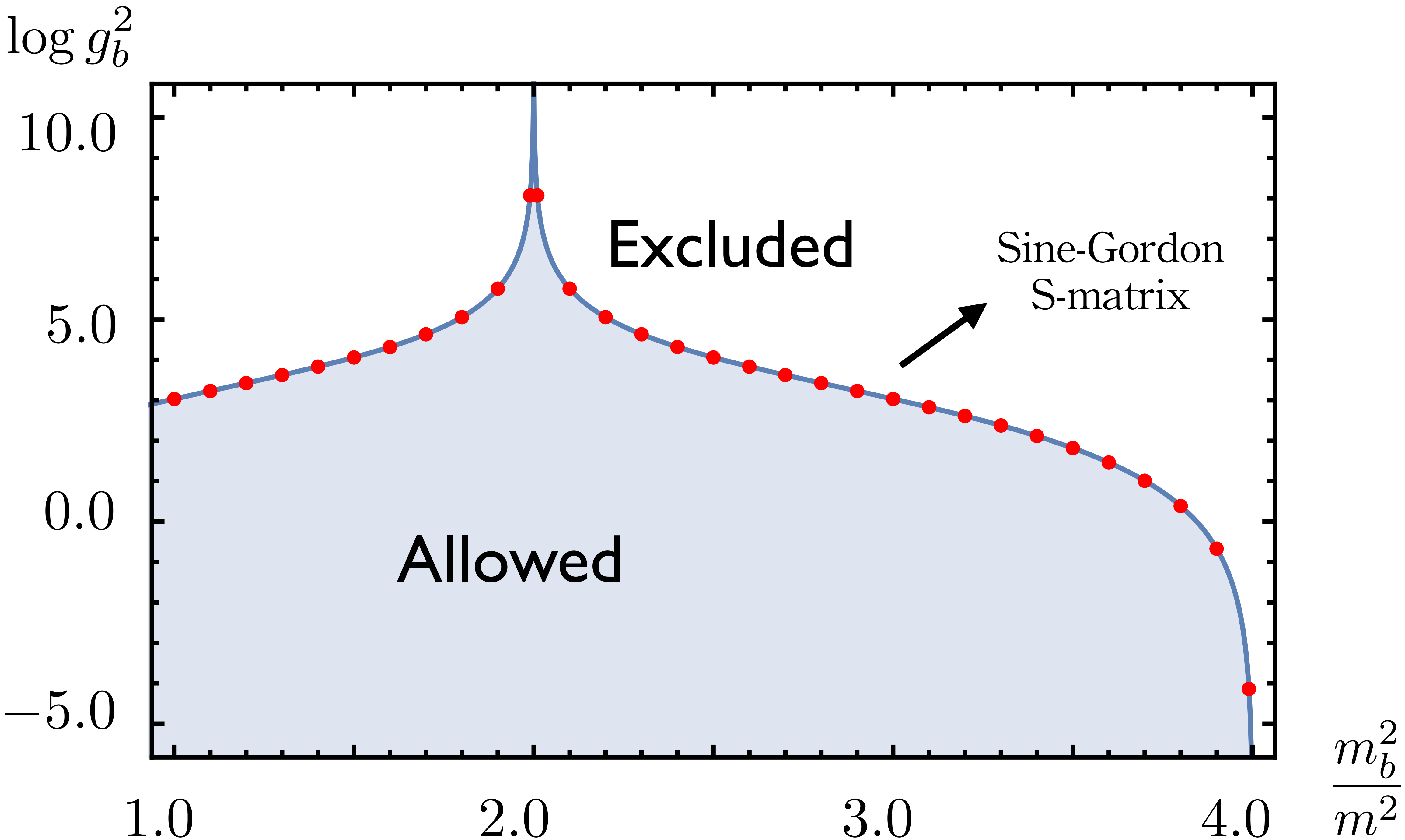
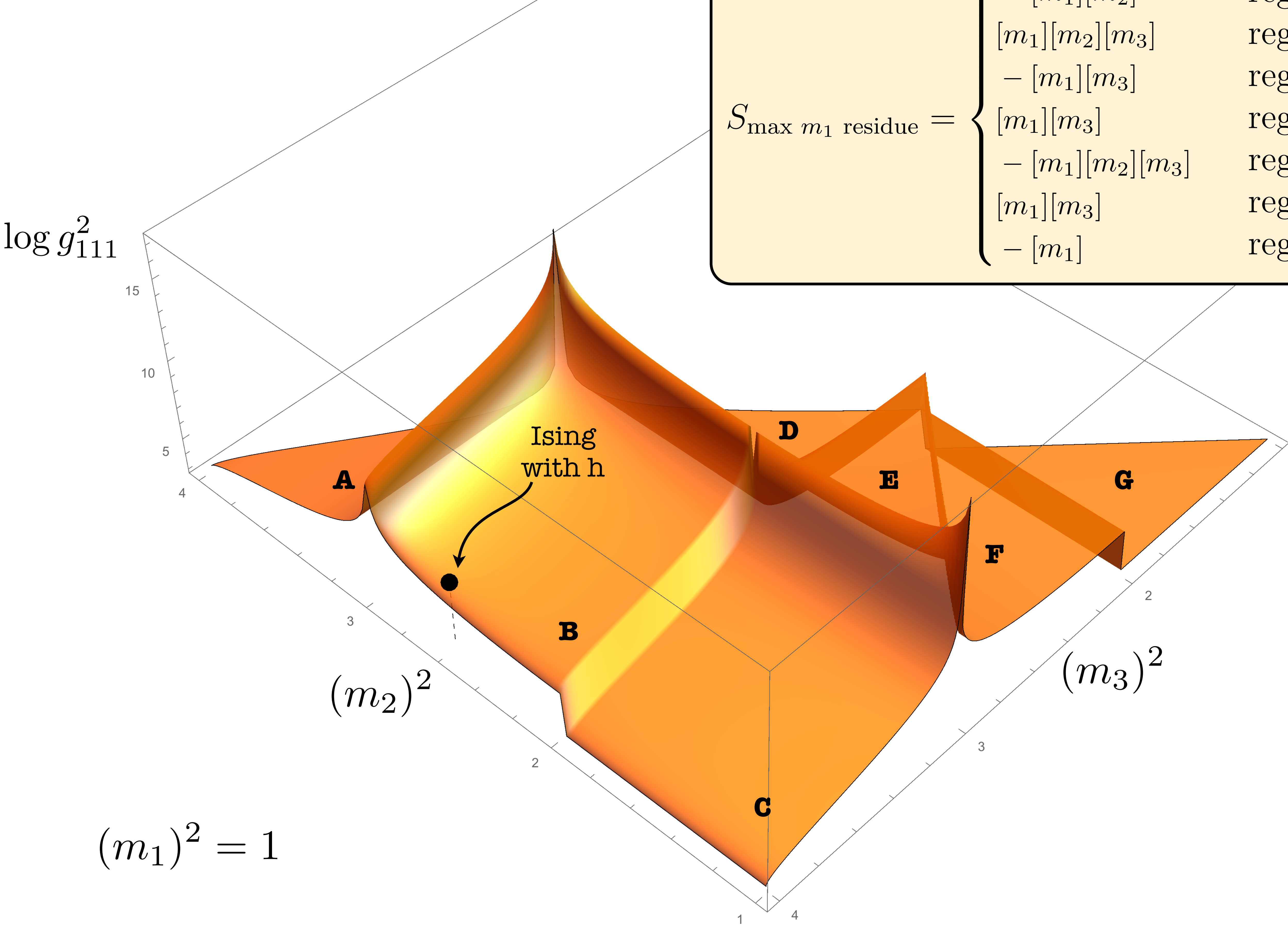


3 stable particles

$$
S_{\text{max } m_1 \text{ residue}} = \begin{cases} -[m_1][m_2] & \text{region A} \\ [m_1][m_2][m_3] & \text{region B} \\ -[m_1][m_3] & \text{region C} \\ [m_1][m_3] & \text{region D} \\ -[m_1][m_2][m_3] & \text{region E} \\ [m_1][m_3] & \text{region F} \\ -[m_1] & \text{region G} \end{cases}
$$

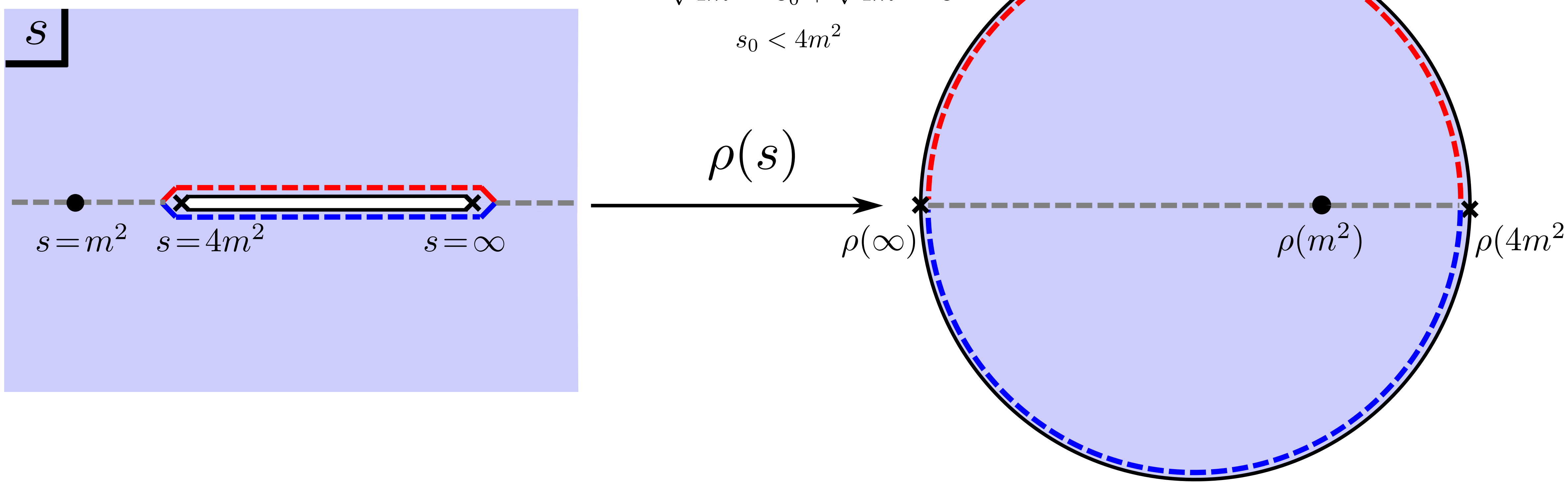


$(m_1)^2 = 1$

Numerical approach

$$\rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

$$s_0 < 4m^2$$



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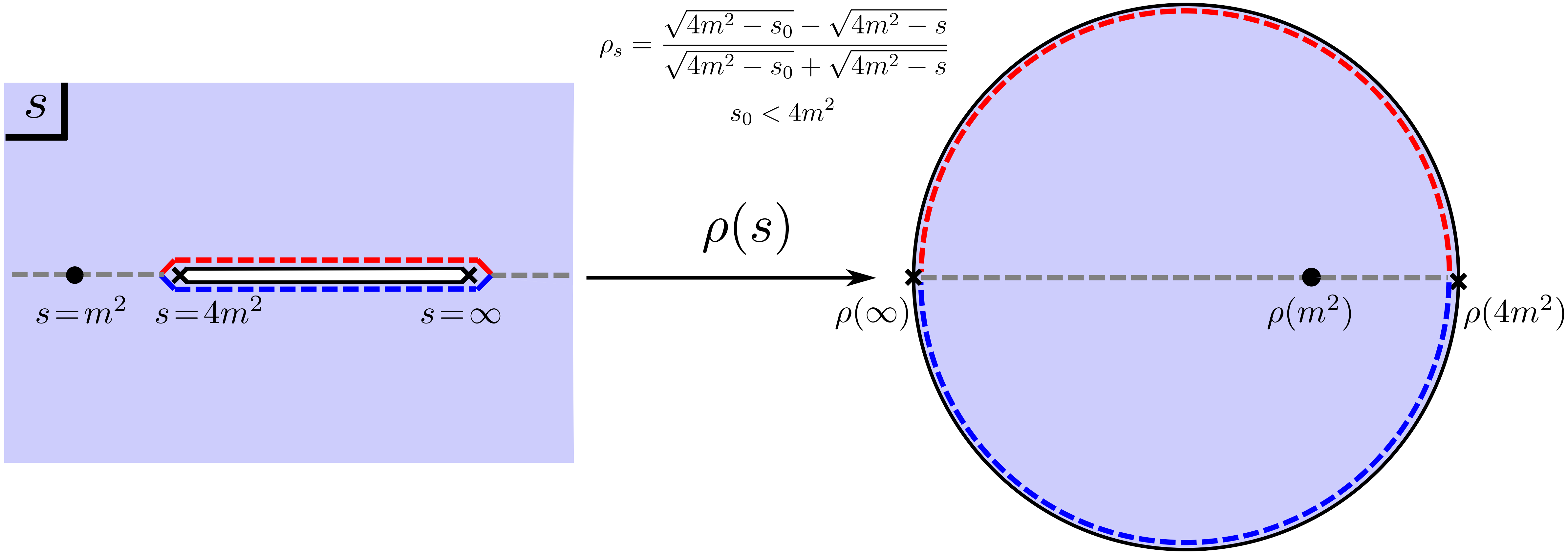

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Ansatz:

$$S_{ext}(s,t) = \frac{g_b^2}{s - m_b^2} + \frac{g_b^2}{t - m_b^2} + \sum_{a,b=0} c_{(ab)} \rho_s^a \rho_t^b$$

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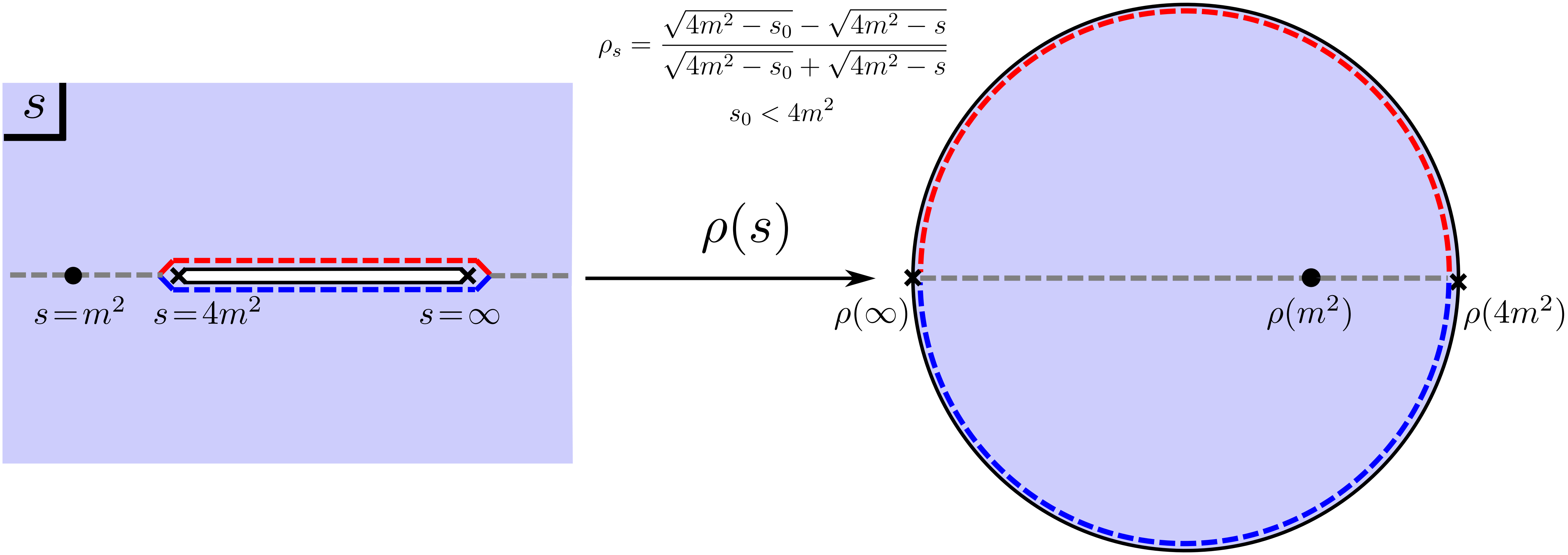

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Crossing symmetry and analyticity are automatic.

Unitarity gives quadratic constraints:

$$\left| S_{ext}(s, 4m^2 - s) \right|^2 \leq 1 \,, \qquad s > 4m^2$$

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Truncate to finite number of variables and **quadratic constraints**

$a + b \leq N_{\max}$ $\downarrow$ $\{g_b^2, c_{(ab)}\}$

$\downarrow$ at $s = s_1, s_2, \ldots, s_M$

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[Simmons-Duffin '15]

Use semidefinite programming (SDPB) to maximize g_b^2 subject to these constraints. This reproduces the analytic solution as $N_{\max} \to \infty$

S-matrix Bootstrap in d+1 QFT

2 to 2 Scattering Amplitude

$$\langle \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1, \mathbf{p}_2 \rangle = \mathbb{1} + i(2\pi)^{d+1} \delta^{(d+1)}(p_1 + p_2 - p_3 - p_4)\, T(s, t, u)$$

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Partial waves:

Gegenbauer polynomial
$x = \cos\theta$

$$S_\ell(s) = 1 + i\frac{(s-4m^2)^{\frac{d-2}{2}}}{\sqrt{s}} \int_{-1}^{1} dx (1-x^2)^{\frac{d-3}{2}} P_\ell^{(d)}(x)\, T(s,t,u)\Big|_{\substack{t \to -\frac{1-x}{2}(s-4m^2) \\ u \to -\frac{1+x}{2}(s-4m^2)}}$$

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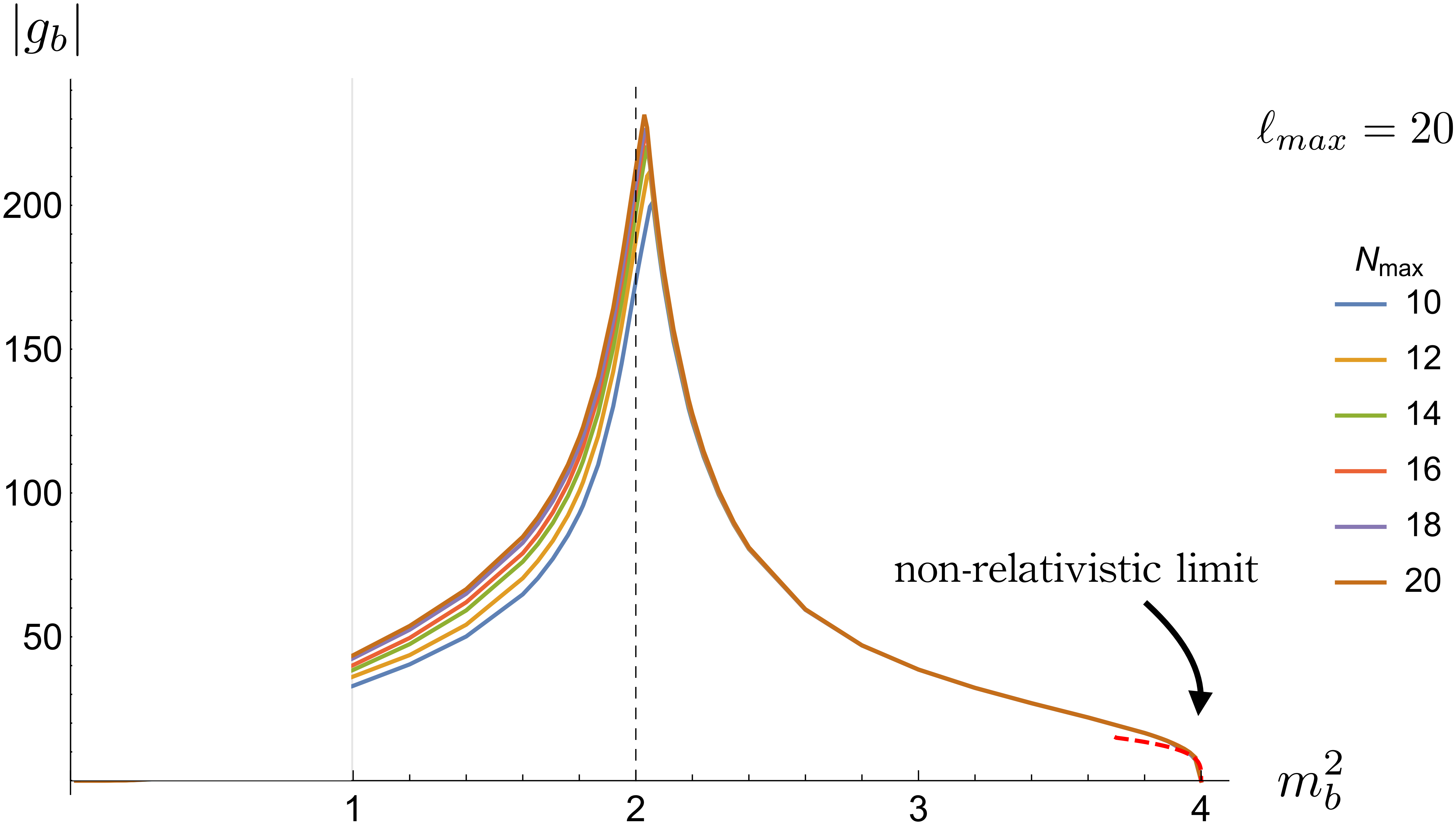
$\Rightarrow$ Quadratic constraints on the variables $\{g_b^2, \alpha_{(abc)}\}$

$a + b + c \leq N_{\max}$

Maximal cubic coupling in 3+1 QFT



Maximal quartic coupling

Ansatz with **no poles**. Maximize $\lambda = \dfrac{1}{32\pi} T(s = t = u = \dfrac{4}{3} m^2)$
$(\text{e.g.} \quad \pi^0 \pi^0 \to \pi^0 \pi^0)$

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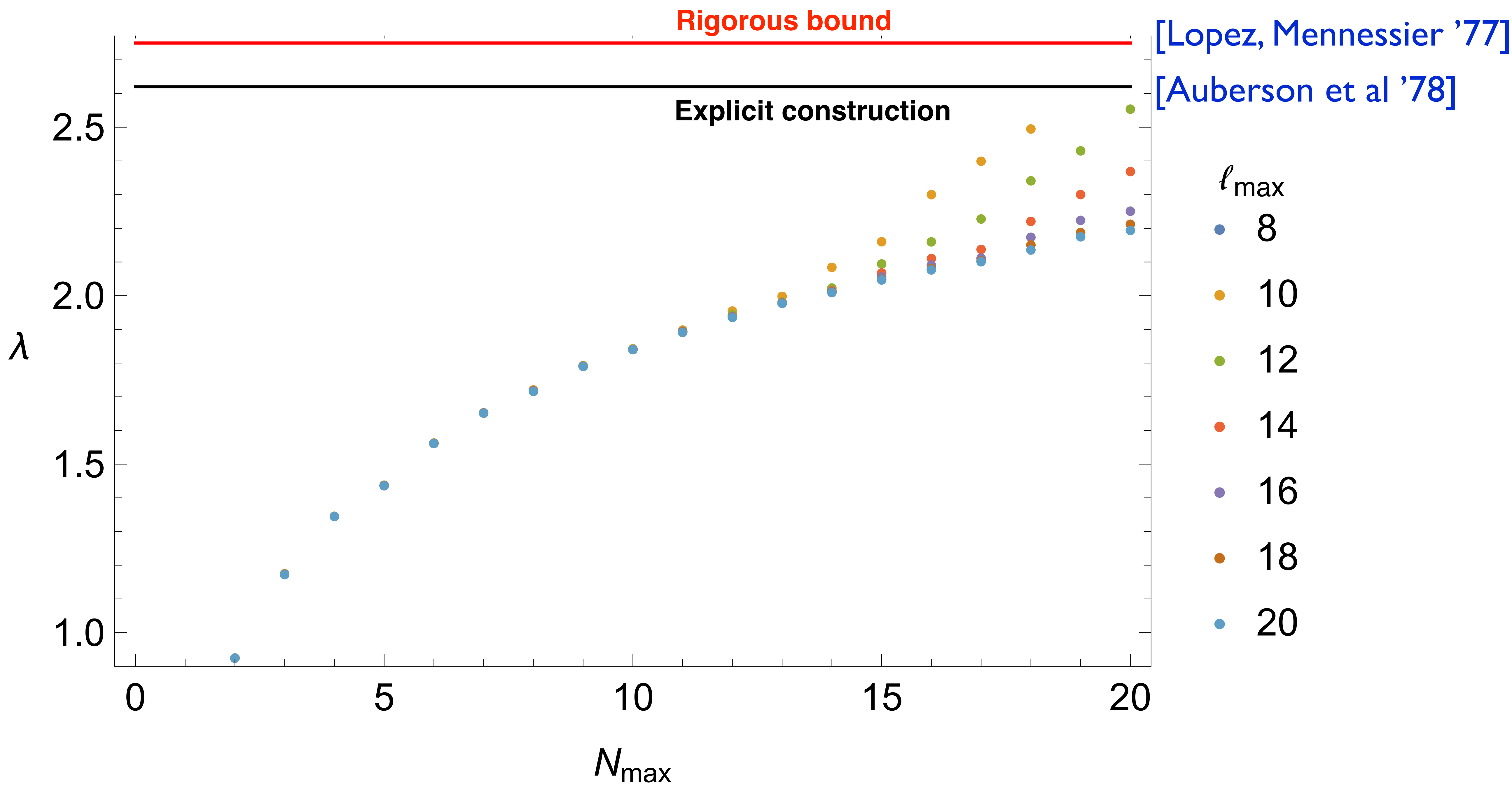

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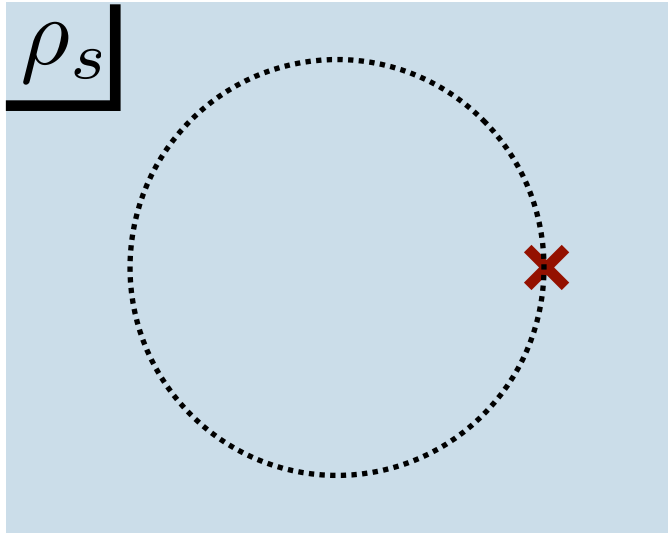
Improved ansatz with threshold bound state:

$$T(s,t,u) = \beta\left(\frac{1}{\rho_s - 1} + \frac{1}{\rho_t - 1} + \frac{1}{\rho_u - 1}\right) + \sum_{\substack{a,b,c=0\\ a+b+c\le N_{\max}}} \alpha_{(abc)}\rho_s^a\rho_t^b\rho_u^c$$



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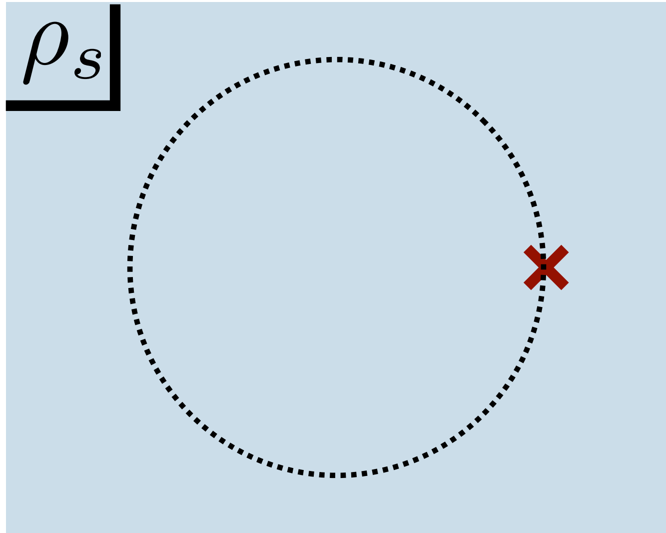

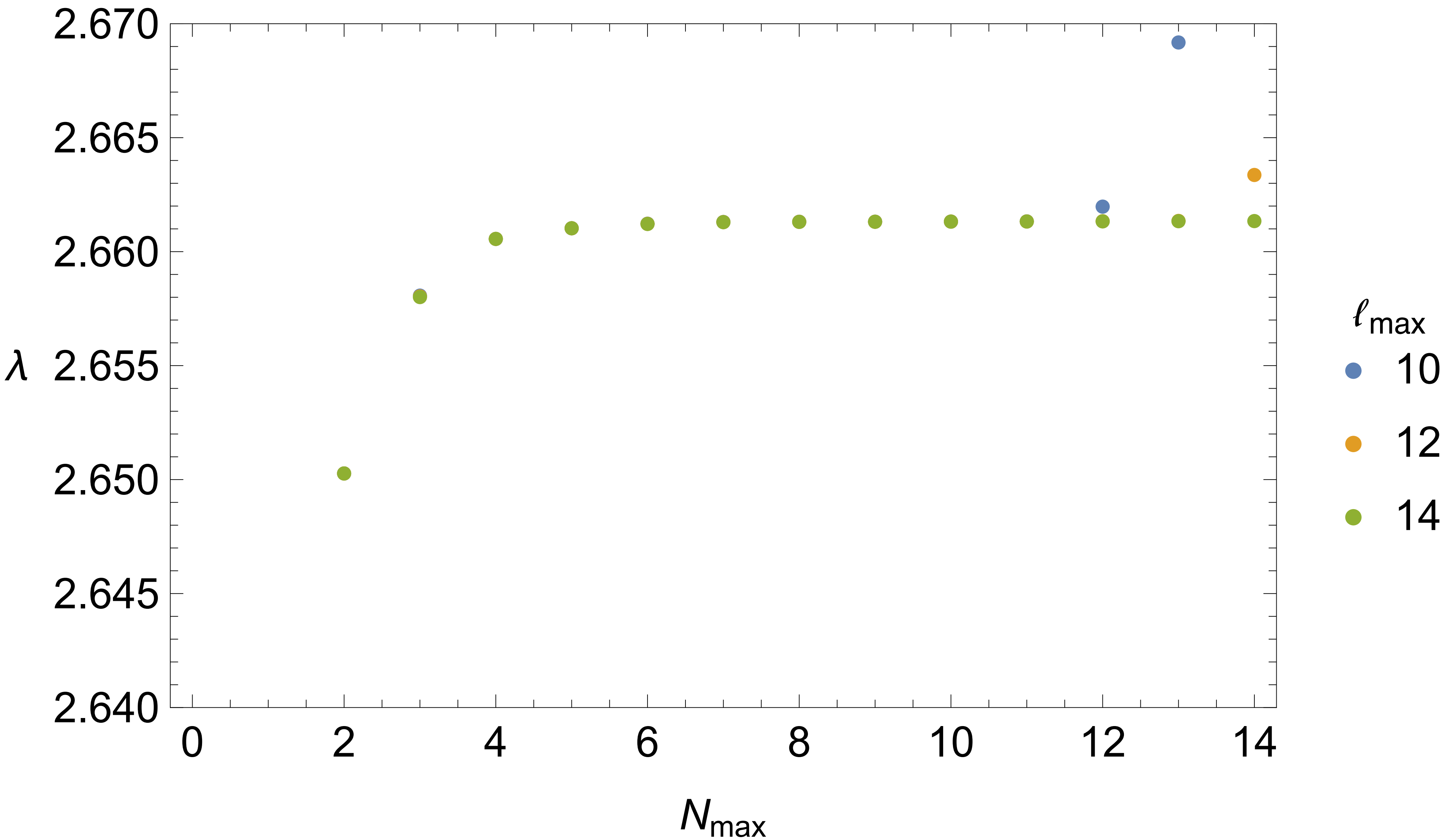

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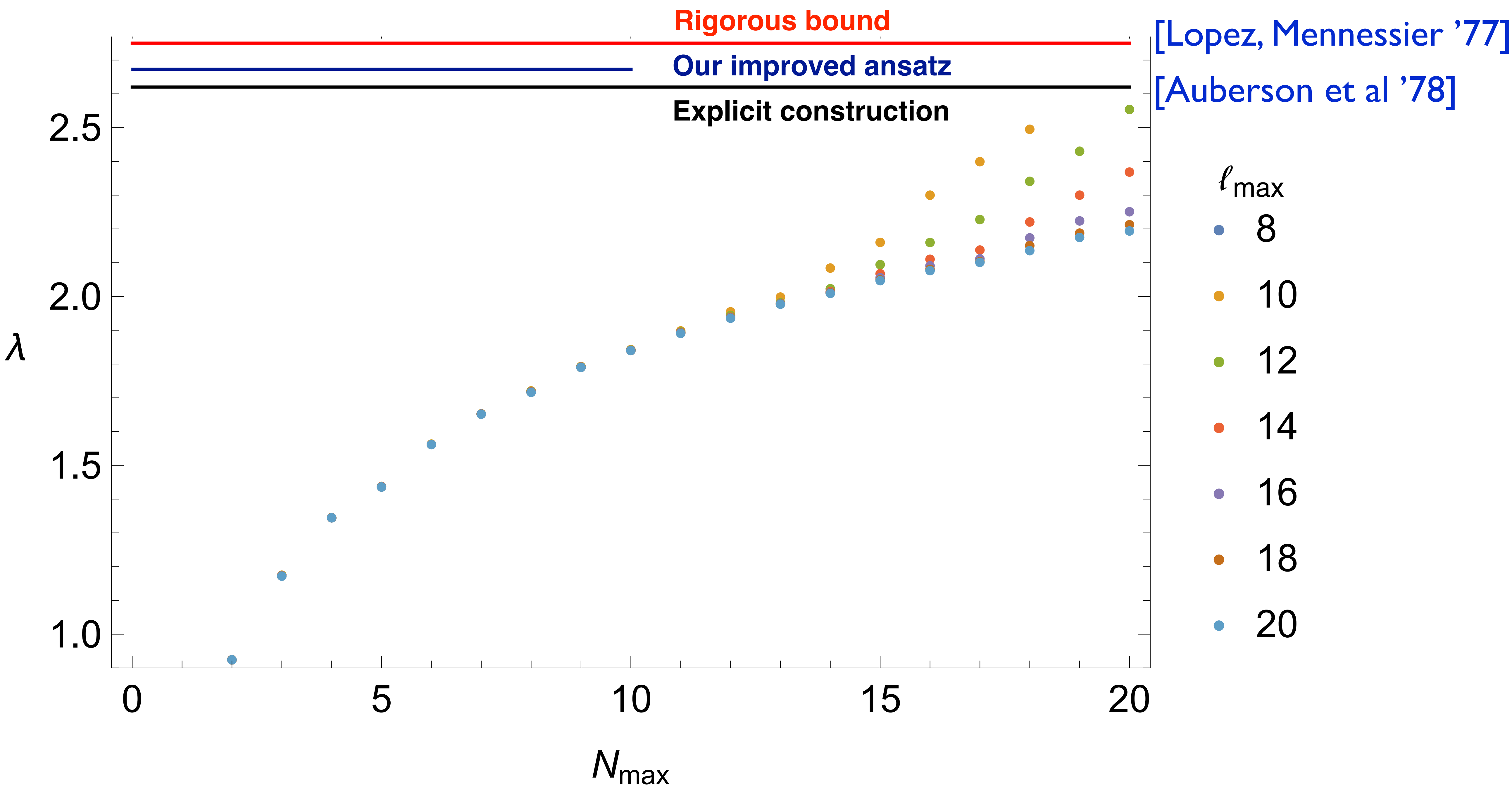
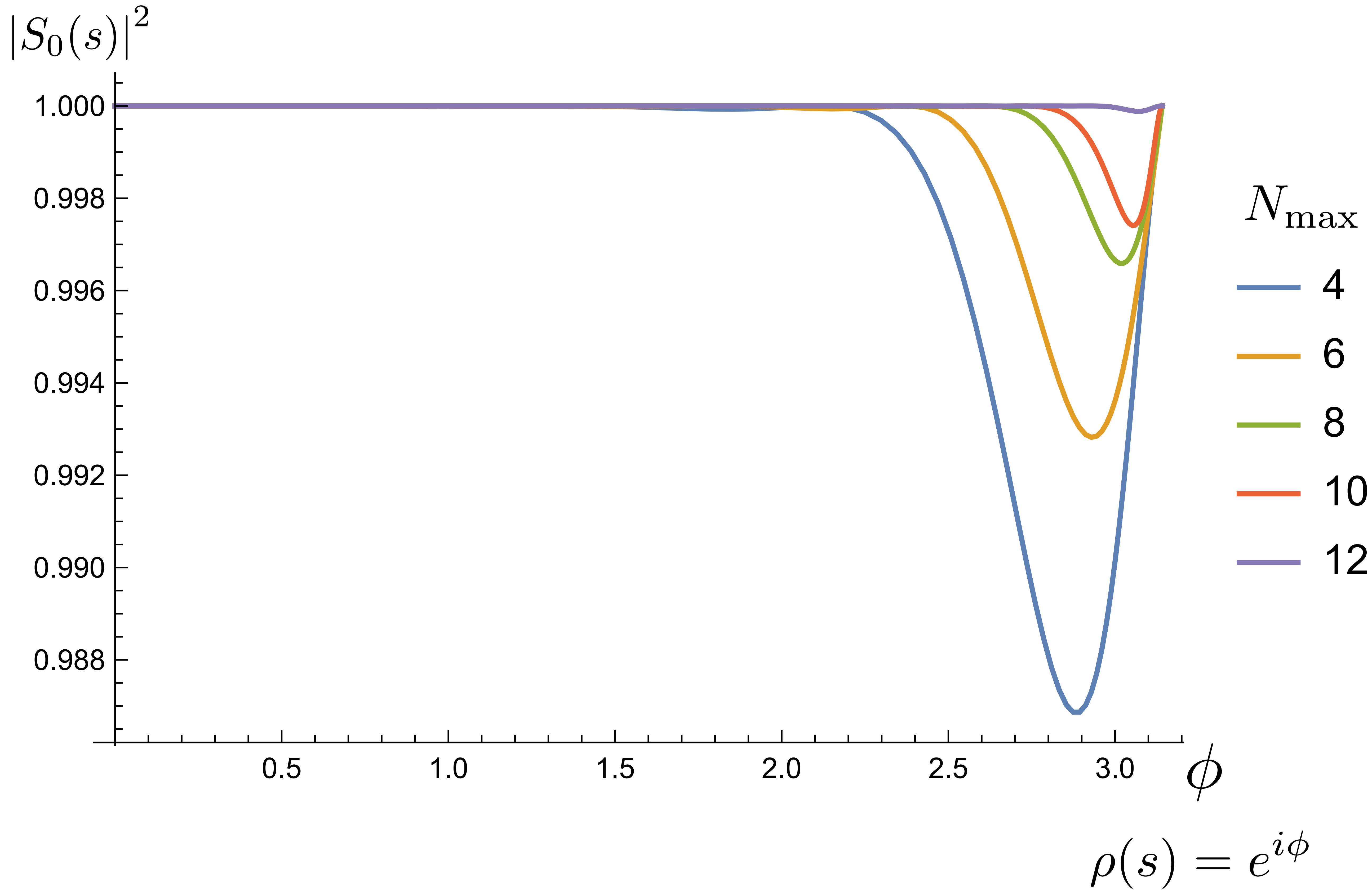


No particle production?

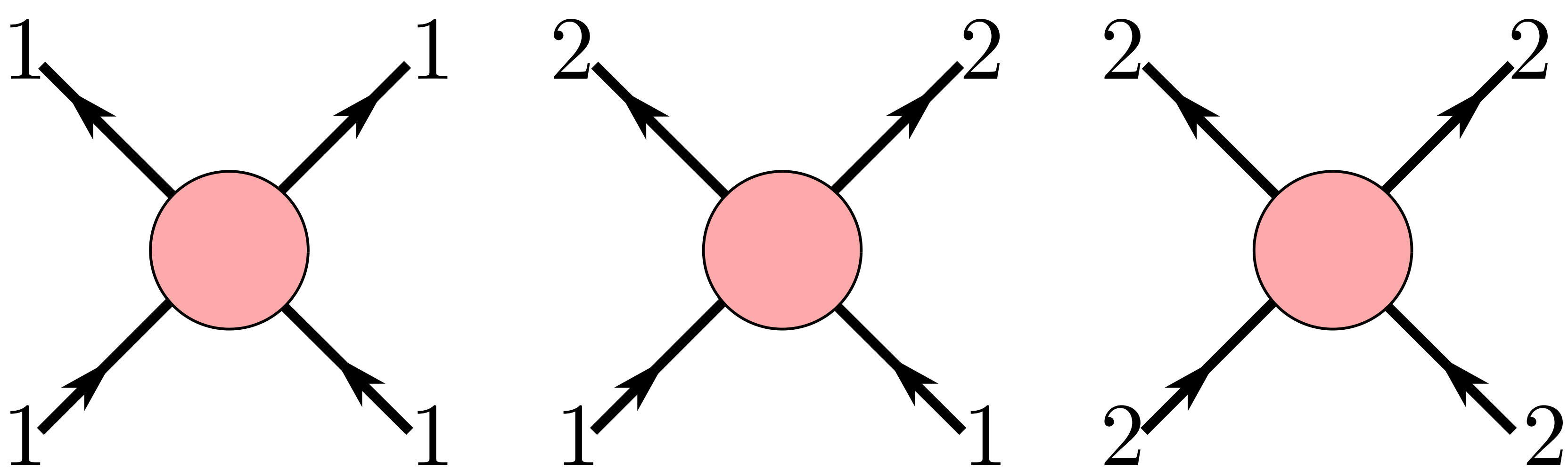


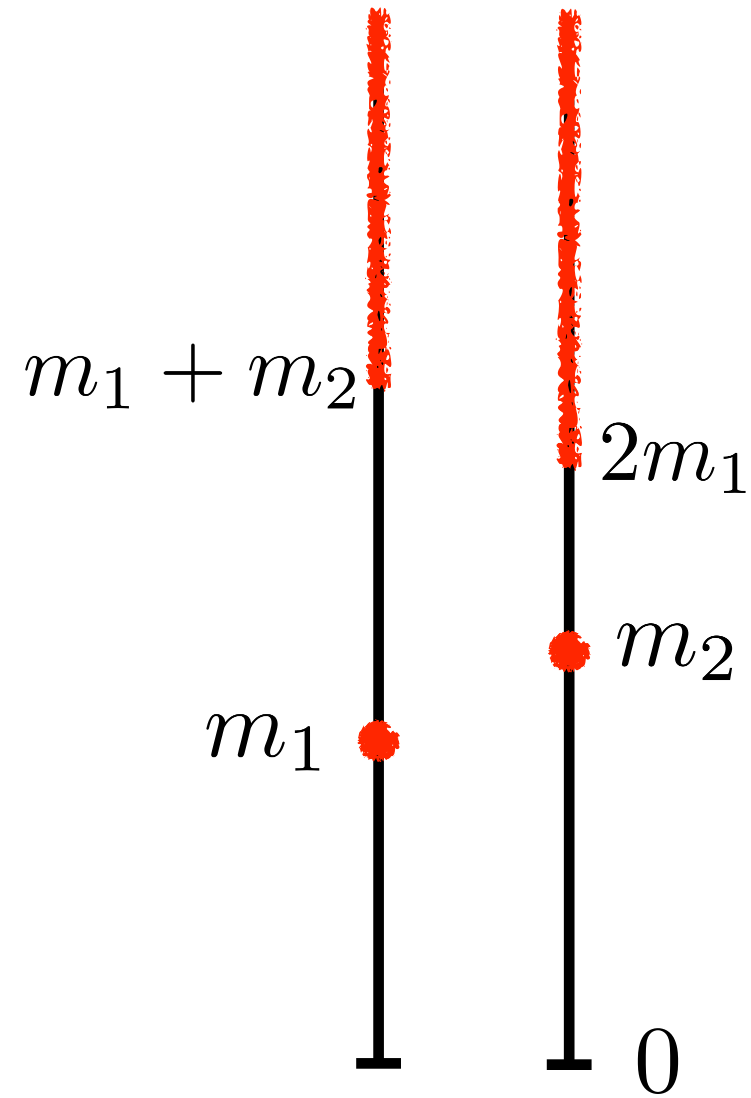
Multiple Amplitudes Bootstrap in 2D QFT

2 to 2 Scattering Amplitudes

Example: two stable particles

$\mathbb{Z}_2$ symmetry : $-$ $+$



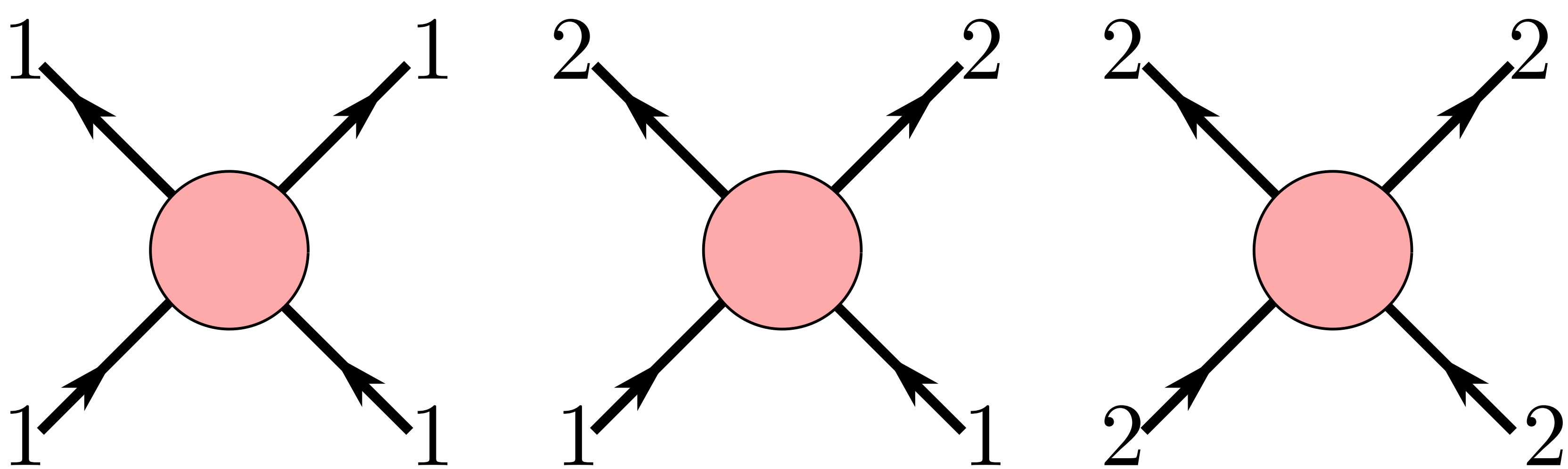


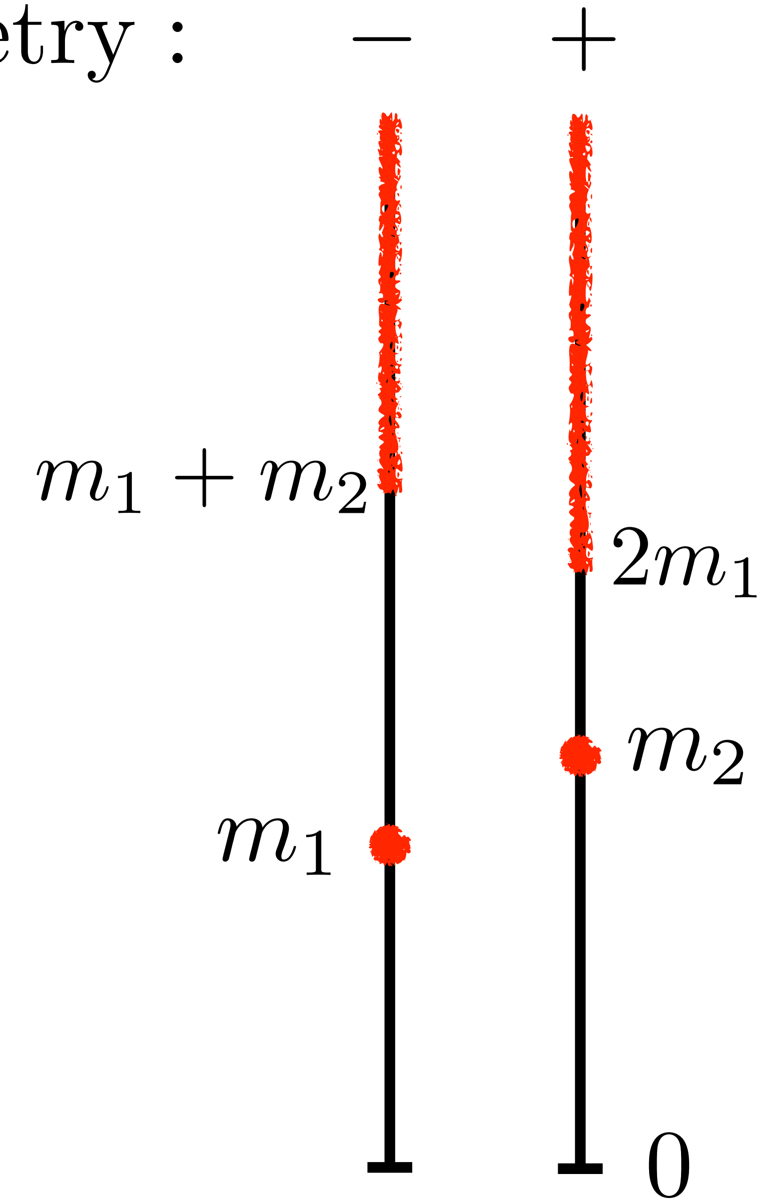
Maximize $\boxed{g_{112}^2}$

2 to 2 Scattering Amplitudes

Example: two stable particles

$\mathbb{Z}_2$ symmetry :





Maximize g_{112}^2

Unitarity: $|S_{11\to 11}|^2 + |S_{11\to 22}|^2 \leq 1$

Not zero in optimal solution

Unitarity

$$\mathrm{T} = \begin{bmatrix} T_{11\to 11} & T_{11\to 22} \\ T_{22\to 11} & T_{22\to 22} \end{bmatrix} \qquad \rho = \begin{bmatrix} \frac{\theta(s-4m_1^2)}{2\sqrt{s(s-4m_1^2)}} & 0 \\ 0 & \frac{\theta(s-4m_2^2)}{2\sqrt{s(s-4m_2^2)}} \end{bmatrix}$$

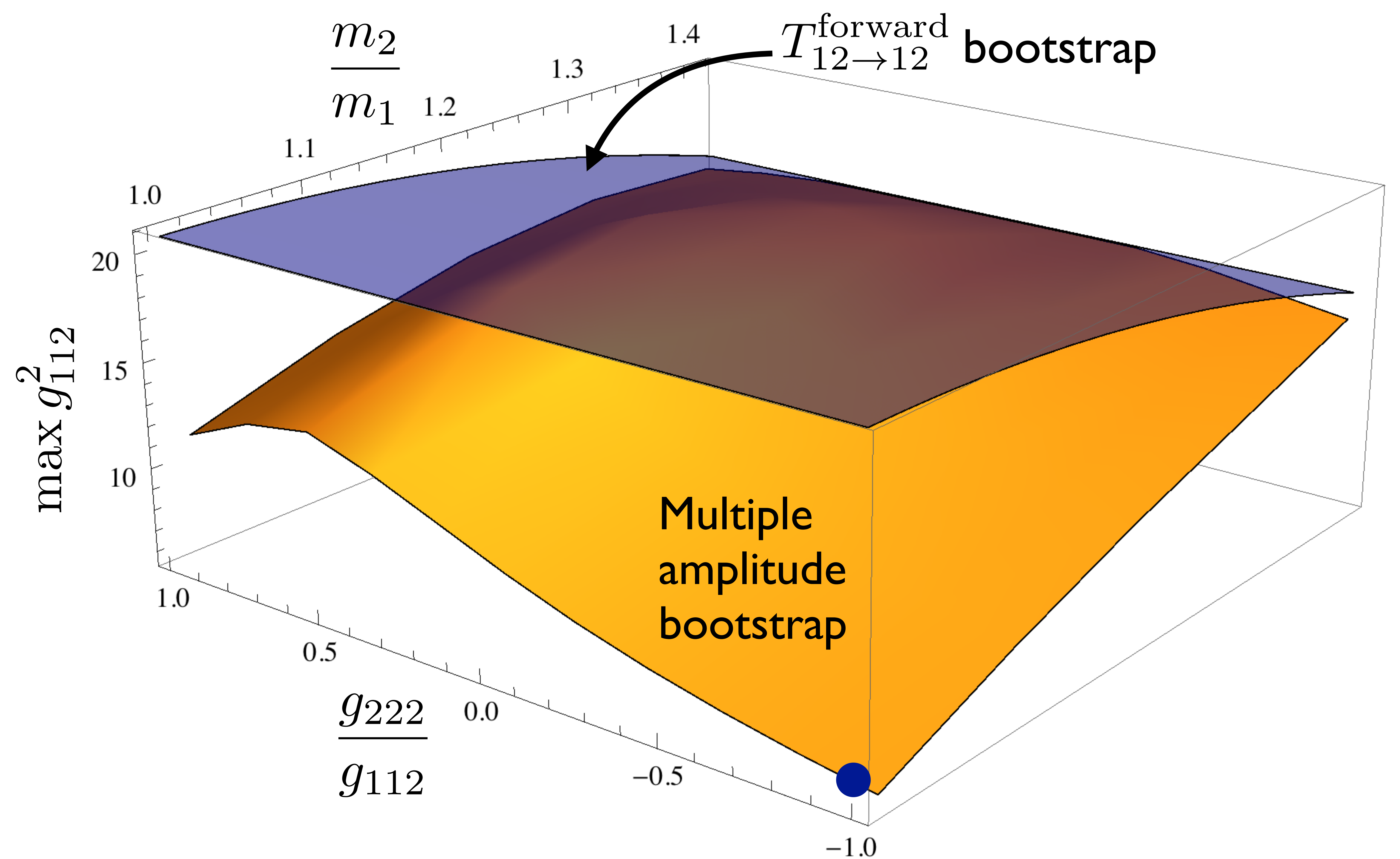
Unitarity is a matrix equation:

$$2\mathrm{Im}\ \mathrm{T} = \mathrm{T}^\dagger \rho \mathrm{T} + positive$$

This can be imposed in SDPB as a positive semi-definite matrix:

$$\begin{bmatrix} \mathbb{I} & \mathrm{T}^\dagger \sqrt{\rho} \\ \sqrt{\rho}\,\mathrm{T} & 2\mathrm{Im}\ \mathrm{T} \end{bmatrix} \succeq 0$$

Preliminary results



3-state Potts model saturates the bound for $m_2 = m_1$ and $\frac{g_{222}}{g_{112}} = -1$

Open questions

Outlook

- Anomalous thresholds (Landau diagrams)
- Particles with spin (internal and external)
- Particles with flavour (global symmetries)

  [He, Irrgang, Kruczenski '18]
  [Cordova, Vieira '18]
  [Paulos, Zheng '18]
  [in progress with Guerrieri, JP, Vieira]
- Massless particles?
- Connect with conformal bootstrap for D>2
- Other interesting questions? Maximize particle production? Resonances?

  [Doroud, Elias Miró '18]
- Can we input UV data about the QFT? Hard scattering? Form factors?

Thank you!