

Duration of classicality of a degenerate quantum scalar field

[Based on: Phys. Rev. D **97**, no. 4, 043531 (2018) [arXiv:1710.02195 [hep-ph]]]

Seishi Enomoto (Univ. of Florida, USA)

Collaborators : Sankha S. Chakrabarty, Yaqi Han,
Pierre Sikivie, Elisa M. Todarello

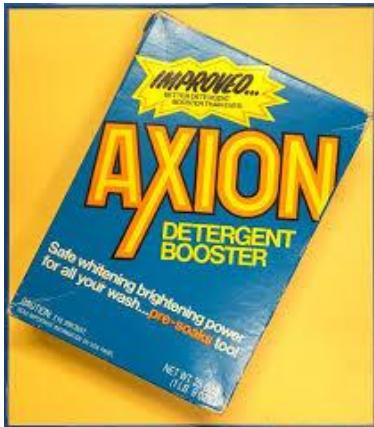


Outlook

1. Introduction
2. Classical description
3. Quantum description
4. Classical vs Quantum description
5. Gravitational self-interaction
6. Summary

1. Introduction

■ Axion is ...



- Solution to Strong CP problem
- Goldstone boson associated with global $U_{PQ}(1)$ symmetry breaking
- Possible candidate of cold dark matter (CDM)

■ Stable : $\Gamma_{a \rightarrow \gamma\gamma} = \frac{g_{a\gamma\gamma}^2 m_a^3}{64\pi} \sim [10^{49} \text{ sec}]^{-1} \left(\frac{m_a}{10^{-5} \text{ eV}} \right)^5 \ll [14 \text{ billion years}]^{-1}$

■ Bose-Einstein condensation (BEC) of dark matter axion

[P. Sikivie and Q. Yang, Phys. Rev. Lett. 103, 111301 (2009)]

1. There are a large number of identical bosons

- $n \sim 4 \times 10^{47} \text{ cm}^{-3} \left(\frac{f}{10^{12} \text{ GeV}} \right)^{\frac{5}{3}} \left(\frac{a(t_1)}{a(t)} \right)^3$ $\left[t_1 \sim 2 \times 10^{-7} \text{ s} \left(\frac{f}{10^{12} \text{ GeV}} \right)^{1/3} \right]$

2. The particle number is conserved
3. The particles are sufficiently degenerate
 - Velocity dispersion : $\delta v \sim 10^{-17}$
 - $\mathcal{N} = n \frac{(2\pi)^3}{\frac{4\pi}{3}(m\delta v)^3} \sim 10^{61} \left(\frac{f}{10^{12} \text{ GeV}} \right)^{8/3}$
4. The particles are in thermal equilibrium by gravity ($T_\gamma < 1 \text{ keV}$)

[O. Erken, P. Sikivie, H. Tam, and Q. Yang, Phys. Rev. D 85, 063520 (2012)]

[K. Saikawa and M. Yamaguchi, Phys. Rev. D 87, 085010 (2013)]

[J. Berges and J. Jaeckel, Phys. Rev. D 91, 025020 (2015)]



Axion can satisfy the BEC conditions!

Bose-Einstein condensation (BEC) of dark matter axion

[P. Sikivie and Q. Yang, Phys. Rev. Lett. 103, 111301 (2009)]

1. There are a large number of identical bosons

$$\blacksquare n \sim 4 \times 10^{47} \text{ cm}^{-3} \left(\frac{f}{10^{12} \text{ GeV}} \right)^{\frac{5}{3}} \left(\frac{a(t_1)}{a(t)} \right)^3 \quad \left(t_1 \sim 2 \times 10^{-7} \text{ s} \left(\frac{f}{10^{12} \text{ GeV}} \right)^{1/3} \right)$$

2. The particle number is conserved

3. The particles are sufficient

- Velocity dispersion :

$$\blacksquare \mathcal{N} = n \frac{(2\pi)^3}{\frac{4\pi}{3}(m\delta v)^3} \sim 10$$

- Is the classical
description still valid?
- If not, when is it break?

4. The particles are in thermal equilibrium by gravity ($T_\gamma < 1 \text{ keV}$)

[O. Erken, P. Sikivie, H. Tam, and Q. Yang, Phys. Rev. D 85, 063520 (2012)]

[K. Saikawa and M. Yamaguchi, Phys. Rev. D 87, 085010 (2013)]

[J. Berges and J. Jaeckel, Phys. Rev. D 91, 025020 (2015)]



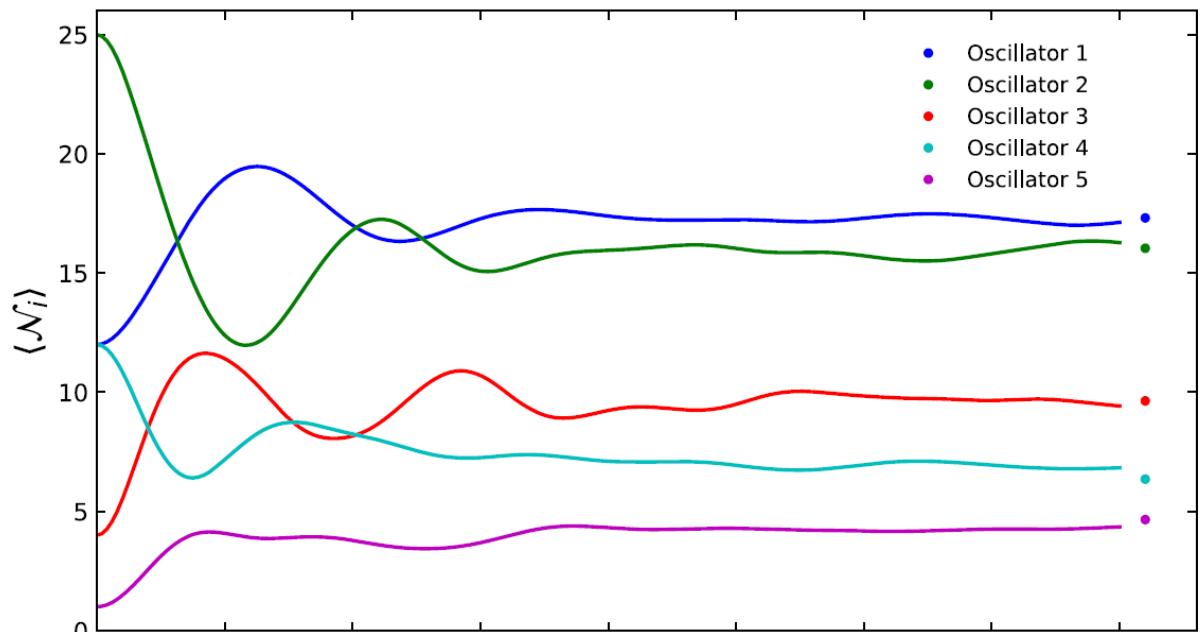
Axion can satisfy the BEC conditions!

An example

■ Quantum

■ Initial state

$$|12, 25, 4, 12, 1\rangle$$



■ Classical

■ Initial state

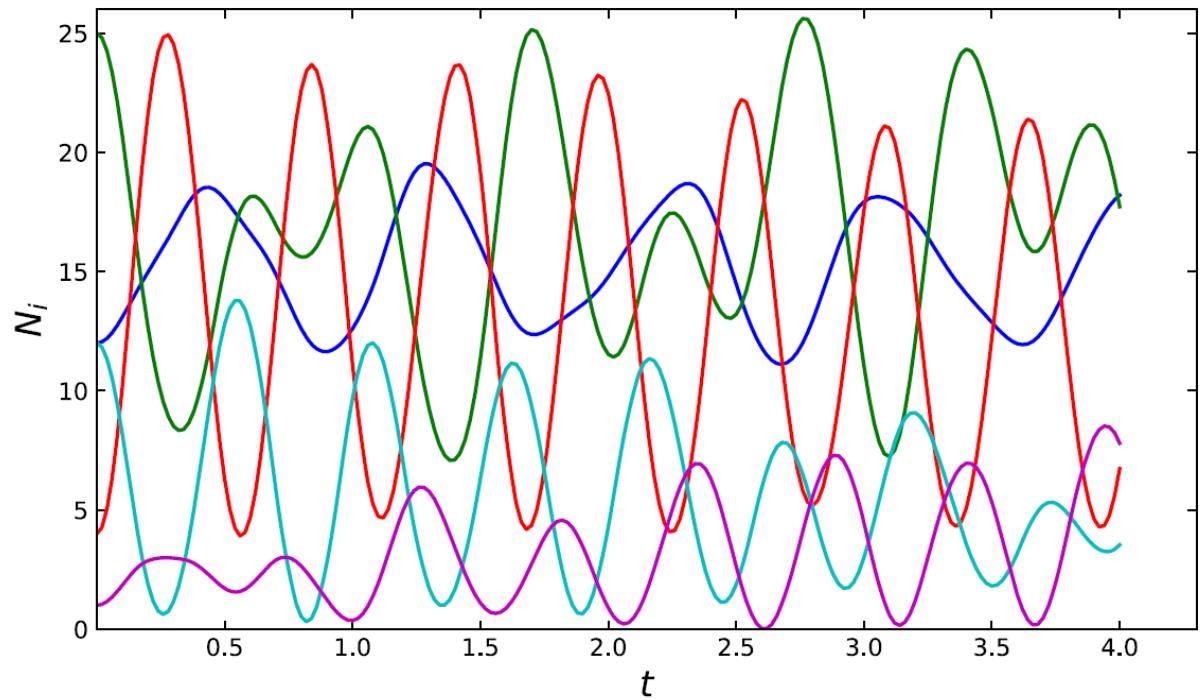
$$N_1 = 12$$

$$N_2 = 25$$

$$N_3 = 4$$

$$N_4 = 12$$

$$N_5 = 1,$$



2. Classical description

■ Ex. 1) Attractive $\lambda\phi^4$ ($\lambda < 0$)

■ Lagrangian $\left[\Lambda, f : \text{constants} \right]$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \Lambda^4 \left(1 - \cos \frac{\phi}{f} \right) \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \dots \quad \left(m \equiv \frac{\Lambda^2}{f}, \quad \lambda \equiv -\frac{\Lambda^4}{f^4} \right)\end{aligned}$$

■ EOM

$$\partial^2\phi - m^2\phi - \frac{1}{3!}\lambda\phi^3 = 0$$

■ Non-relativistic limit

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2m}} (\psi(t, \vec{x}) e^{-imt} + \psi(t, \vec{x})^\dagger e^{+imt})$$


$$\left\{ \begin{array}{l} [\psi(t, \vec{x}), \psi(t, \vec{y})] = 0, \quad [\psi(t, \vec{x}), \psi(t, \vec{y})^\dagger] = \delta^3(\vec{x} - \vec{y}) \\ i\partial_t \psi = -\frac{1}{2m} \vec{\nabla}^2 \psi + \frac{\lambda}{8m^2} \psi^\dagger \psi^2 \end{array} \right.$$

■ Classical description

■ ψ (operator) \rightarrow Ψ (c-number) : Schrödinger-Gross-Pitaevskii equation

$$i\partial_t \Psi = -\frac{1}{2m} \vec{\nabla}^2 \Psi + \frac{\lambda}{8m^2} |\Psi|^2 \Psi$$

■ One can regard

■ number density as: $n_c = |\Psi|^2$

■ velocity as : $\vec{v} = \frac{1}{n} \cdot \frac{1}{2im} (\Psi^* (\vec{\nabla} \Psi) - (\vec{\nabla} \Psi^*) \Psi)$

which satisfy

■ continuity equation: $\partial_t n_c + \vec{\nabla} \cdot (n_c \vec{v}) = 0$

■ Euler-like equation : $\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{m} \vec{\nabla} U - \vec{\nabla} q$

$$\left. \begin{array}{l} U \equiv \frac{\lambda}{8m^2} n_c, \\ q \equiv -\frac{1}{2m^2} \frac{\vec{\nabla}^2 \sqrt{n}}{\sqrt{n}} \end{array} \right)$$

■ Homogeneous solution : ($\Psi = \Psi_0(t)$) “potential” “quantum pressure”

■ $\Psi_0(t) = \sqrt{n_0} e^{i\delta\omega \cdot t}$ $\left(\delta\omega \equiv \frac{|\lambda|}{8m^2} n_0 \right)$

→ $n(t) = n_0 = (\text{const.}), \quad \vec{v}(t) = 0$

No evolution for number density

■ Perturbation : $\Psi(t) = \Psi_0(t) + \Psi_1(t, \vec{x}) + \dots$ $(|\Psi_0| \ll |\Psi_1|)$

$$i\partial_t \Psi_1 = -\frac{1}{2m} \vec{\nabla}^2 \Psi_1 - \delta\omega(2\Psi_1 + \Psi_1^*) + \dots$$

■ Density perturbation

■ $\delta(t, \vec{k}) \equiv \int_V d^3x e^{-i\vec{k}\cdot\vec{x}} \frac{\Psi_0\Psi_1^* + \Psi_0^*\Psi_1}{|\Psi_0|^2}$

for $k > k_J \equiv \sqrt{4m\delta\omega}$

stable

$$\delta(t, \vec{k}) = \cos \omega(k)t \delta(0, \vec{k}) + \frac{1}{\omega(k)} \sin \omega(k)t \partial_t \delta(0, \vec{k})$$

$\left(\quad \omega(k) \equiv \frac{|\vec{k}|}{2m} \sqrt{\vec{k}^2 - \vec{k}_J^2} \quad \right)$

■ for $0 < k < k_J$

unstable

$$\delta(t, \vec{k}) = \underbrace{\cosh \gamma(k) t}_{\text{Red}} \delta(0, \vec{k}) + \frac{i}{\omega(k)} \underbrace{\sinh \gamma(k) t}_{\text{Red}} \partial_t \delta(0, \vec{k})$$

$\left(\gamma(k) \equiv \frac{|\vec{k}|}{2m} \sqrt{\vec{k}_J^2 - \vec{k}^2} \right)$

**Growing base is
dependent on Initial
perturbation**

3. Quantum description

■ Quantum description

$$i\partial_t \psi = -\frac{1}{2m} \vec{\nabla}^2 \psi + \frac{\lambda}{8m^2} \psi^\dagger \psi^2$$

- Number density : $n_q = \langle * | \psi^\dagger \psi | * \rangle \equiv \langle \psi^\dagger \psi \rangle$ [c.f. $n_c = |\Psi|^2$]
- Expansion by annihilation operator

$$\psi(t, \vec{x}) = \sum_{\vec{k}} u_{\vec{k}}(t, \vec{x}) \color{orange} a_{\vec{k}}(t)$$

c-number wave function
(orthogonal & complete set)

Annihilation (creation) operator

$$[a_{\vec{k}}(t), a_{\vec{k}'}^\dagger(t)] = \delta_{\vec{k}\vec{k}'} \quad (\text{others}) = 0$$

$$\sum_k u_{\vec{k}}(t, \vec{x})^* u_{\vec{k}}(t, \vec{y}) = \delta^3(\vec{x} - \vec{y})$$

$$\frac{1}{V} \int_V d^3x u_{\vec{k}}(t, \vec{x}) u_{\vec{k}'}(t, \vec{x})^* = \delta_{\vec{k}\vec{k}'}$$

- We choose to be

$$u_{\vec{k}}(t, \vec{x}) = \frac{1}{\sqrt{N}} e^{i\vec{k} \cdot \vec{\chi}(t, \vec{x})} \Psi(t, \vec{x})$$

$$\left. \begin{array}{l} N = \int_V d^3x n : \text{total number in the system,} \\ \vec{\chi}(t, \vec{x}) : \text{comoving coordinate } (n_c d^3 \vec{\chi} = (\text{const.})) \end{array} \right]$$

■ EOM for $\vec{a}_\vec{k}$

$$i\partial_t \vec{a}_\vec{k} = \sum_{\vec{k}'} M_{\vec{k}}^{\vec{k}'} \vec{a}_{\vec{k}'} + \frac{1}{2} \sum_{\vec{k}_2, \vec{k}_3, \vec{k}_4} \Lambda_{\vec{k}\vec{k}_2}^{\vec{k}_3\vec{k}_4} \vec{a}_{\vec{k}_2}^\dagger \vec{a}_{\vec{k}_3} \vec{a}_{\vec{k}_4}$$

$$\left(\begin{array}{l} M_{\vec{k}}^{\vec{k}'} = -\frac{N}{2} \Lambda_{\vec{k}\vec{0}}^{\vec{k}'\vec{0}} + \frac{1}{2mN} \int_V d^3x n \vec{V}(\vec{k} \cdot \vec{\chi}) \cdot \vec{V}(\vec{k}' \cdot \vec{\chi}) e^{-i(\vec{k}-\vec{k}') \cdot \vec{\chi}} \\ \Lambda_{\vec{k}_1\vec{k}_2}^{\vec{k}_3\vec{k}_4} = \frac{\lambda}{4m^2 N^2} \int_V d^3x n^2 e^{-i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4) \cdot \vec{\chi}} \end{array} \right)$$

■ Linearization

$$\vec{a}_\vec{k}(t) = \vec{A}_\vec{k} + \vec{b}_\vec{k}(t), \quad (\vec{A}_\vec{k}: c\text{-\#}, \quad " \vec{A}_\vec{k} \gg \vec{b}_\vec{k}")$$

→
$$\begin{cases} \vec{A}_\vec{k} = \sqrt{N} \delta_{\vec{k}\vec{0}} & (\leftrightarrow \text{corresponding to classical description}) \\ i\partial_t \vec{b}_\vec{k} = \sum_{\vec{k}'} \left[\left(M_{\vec{k}}^{\vec{k}'} + N \Lambda_{\vec{k}\vec{0}}^{\vec{k}'\vec{0}} \right) \vec{b}_{\vec{k}'} + \frac{N}{2} \Lambda_{\vec{k}\vec{k}'}^{\vec{0}\vec{0}} \vec{b}_{\vec{k}'}^\dagger \right] + O(\vec{b}^2) \end{cases}$$

■ Total number

$$N = \sum_{\vec{k}} \langle \vec{a}_\vec{k}^\dagger \vec{a}_\vec{k} \rangle \sim \underbrace{|\vec{A}_\vec{0}|^2}_{\text{classical description}} + \underbrace{\sum_{\vec{k} \neq \vec{0}} \langle \vec{b}_\vec{k}^\dagger \vec{b}_\vec{k} \rangle}_{\text{deviation by quantum effect}}$$

■ Homogeneous background ($\Psi = \Psi_0(t)$)

■ $\Psi = \sqrt{n_0} e^{i\delta\omega \cdot t}$

$$\left(n_0: \text{const.}, \quad \delta\omega \equiv \frac{|\lambda|n_0}{8m^2} \right)$$

■ $i\partial_t \begin{pmatrix} \mathbf{b}_{\vec{k}} \\ \mathbf{b}_{-\vec{k}}^\dagger \end{pmatrix} = \begin{pmatrix} \frac{\vec{k}^2}{2m} - \delta\omega & -\delta\omega \\ +\delta\omega & -\frac{\vec{k}^2}{2m} + \delta\omega \end{pmatrix} \begin{pmatrix} \mathbf{b}_{\vec{k}} \\ \mathbf{b}_{-\vec{k}}^\dagger \end{pmatrix} + O(\mathbf{b}^2)$

■ for $k > k_J \equiv \sqrt{4m\delta\omega}$

$$\mathbf{b}_{\vec{k}}(t) = \left(\underbrace{\cos \omega(k)t}_{} + i \underbrace{\frac{\delta\omega - \vec{k}^2/2m}{\omega(k)} \sin \omega(k)t}_{} \right) \mathbf{b}_{\vec{k}}(0) - i \frac{\delta\omega}{\omega(k)} \sin \omega(k)t \mathbf{b}_{-\vec{k}}^\dagger(0)$$

stable

$$\left(\omega(k) \equiv \frac{|\vec{k}|}{2m} \sqrt{\vec{k}^2 - \vec{k}_J^2} \right)$$

■ for $0 < k < k_J$

$$\mathbf{b}_{\vec{k}}(t) = \left(\underbrace{\cosh \gamma(k)t}_{} + i \underbrace{\frac{\delta\omega - \vec{k}^2/2m}{\omega(k)} \sinh \gamma(k)t}_{} \right) \mathbf{b}_{\vec{k}}(0) - i \frac{\delta\omega}{\omega(k)} \sinh \gamma(k)t \mathbf{b}_{-\vec{k}}^\dagger(0)$$

unstable

$$\left(\gamma(k) \equiv \frac{|\vec{k}|}{2m} \sqrt{\vec{k}_J^2 - \vec{k}^2} \right)$$

Quantum fluctuation is growing!

$$\sum_{\vec{k} \neq \vec{0}} \langle \mathbf{b}_{\vec{k}}^\dagger \mathbf{b}_{\vec{k}} \rangle \sim \sum_{0 < |\vec{k}| < k_J} \langle \mathbf{b}_{\vec{k}}^\dagger \mathbf{b}_{\vec{k}} \rangle \sim \frac{N|\lambda|}{32\pi\sqrt{2\pi m t}} e^{k_J^2 t / 2m} \sim N$$

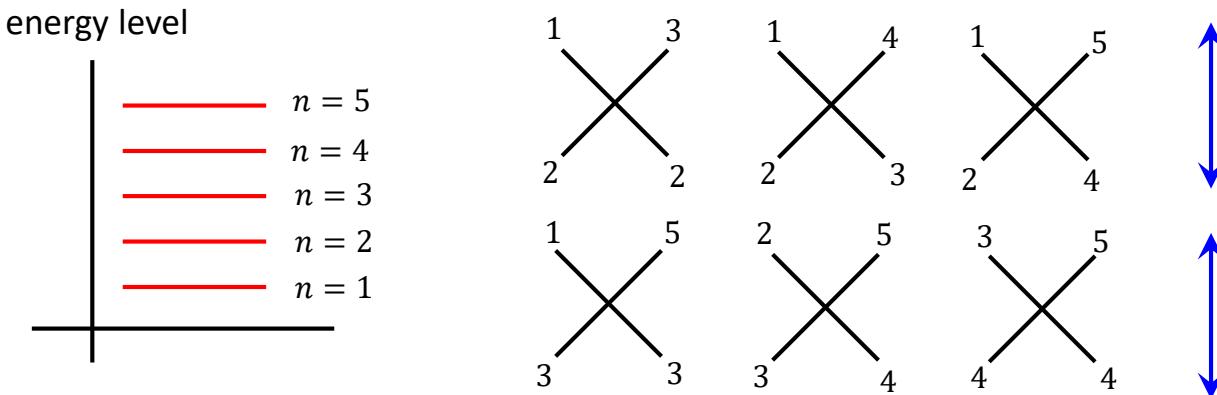
$\therefore t \sim \frac{2m}{k_J^2} \ln \frac{32\pi^{3/2} n_0}{k_J^3}$ (duration of classicality)

4. Classical vs Quantum description

■ Ex. 2) Numerical results by simplified model (five oscillators)

■ Hamiltonian

$$H = \sum_{n=1}^5 n \mathbf{a}_n^\dagger \mathbf{a}_n + \frac{1}{4} \sum_{i,j,k,l} \Lambda_{ij}^{kl} \mathbf{a}_k^\dagger \mathbf{a}_l^\dagger \mathbf{a}_i \mathbf{a}_j$$



■ EOM : quantum

$$i\partial_t \mathbf{a}_n = [\mathbf{a}_n, H]$$

$$= n \mathbf{a}_n + \frac{1}{2} \sum_{i,j,k,l} \Lambda_{ij}^{kn} \mathbf{a}_k^\dagger \mathbf{a}_l^\dagger \mathbf{a}_i \mathbf{a}_j$$



$$\langle \mathcal{N}_n \rangle \equiv \langle * | \mathbf{a}_n^\dagger \mathbf{a}_n | * \rangle$$

■ EOM : classical

$$i\partial_t \mathbf{A}_n = n \mathbf{A}_n + \frac{1}{2} \sum_{i,j,k,l} \Lambda_{ij}^{kn} \mathbf{A}_k^\dagger \mathbf{A}_l^\dagger \mathbf{A}_i \mathbf{A}_j$$



$$N_n = |\mathbf{A}_n|^2$$

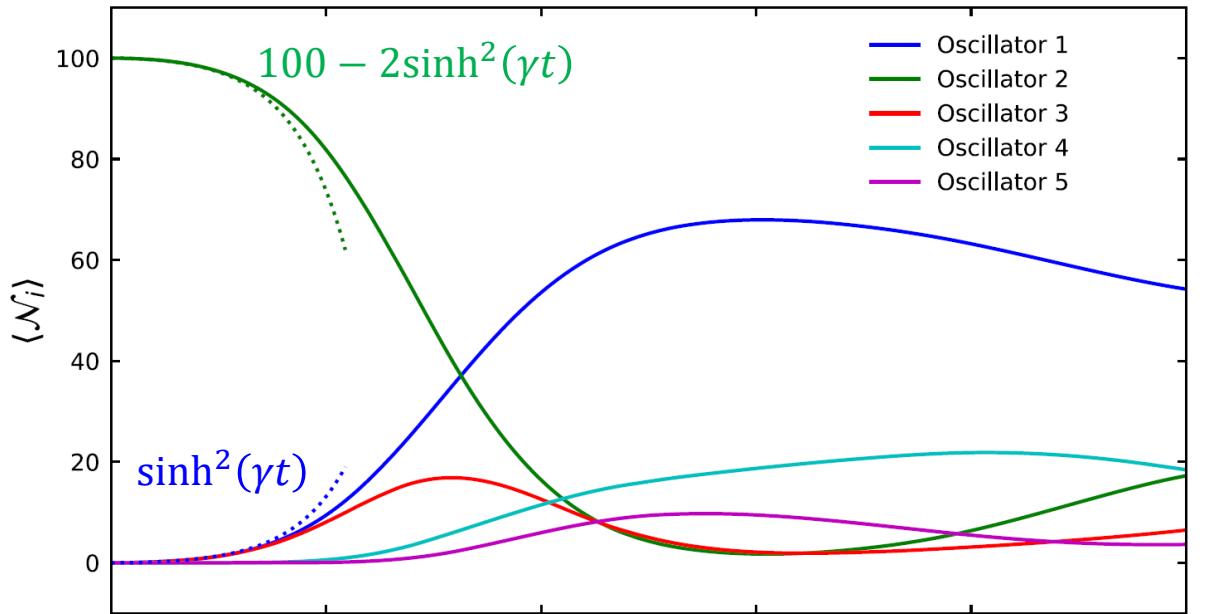
■ Results

■ Quantum

$$i\partial_t \mathbf{a}_n = n\mathbf{a}_n + \frac{1}{2} \sum_{i,j,k,l} \Lambda_{ij}^{kn} \mathbf{a}_k^\dagger \mathbf{a}_i \mathbf{a}_j$$

Initial state

$$|0, 100, 0, 0, 0\rangle$$



■ Classical

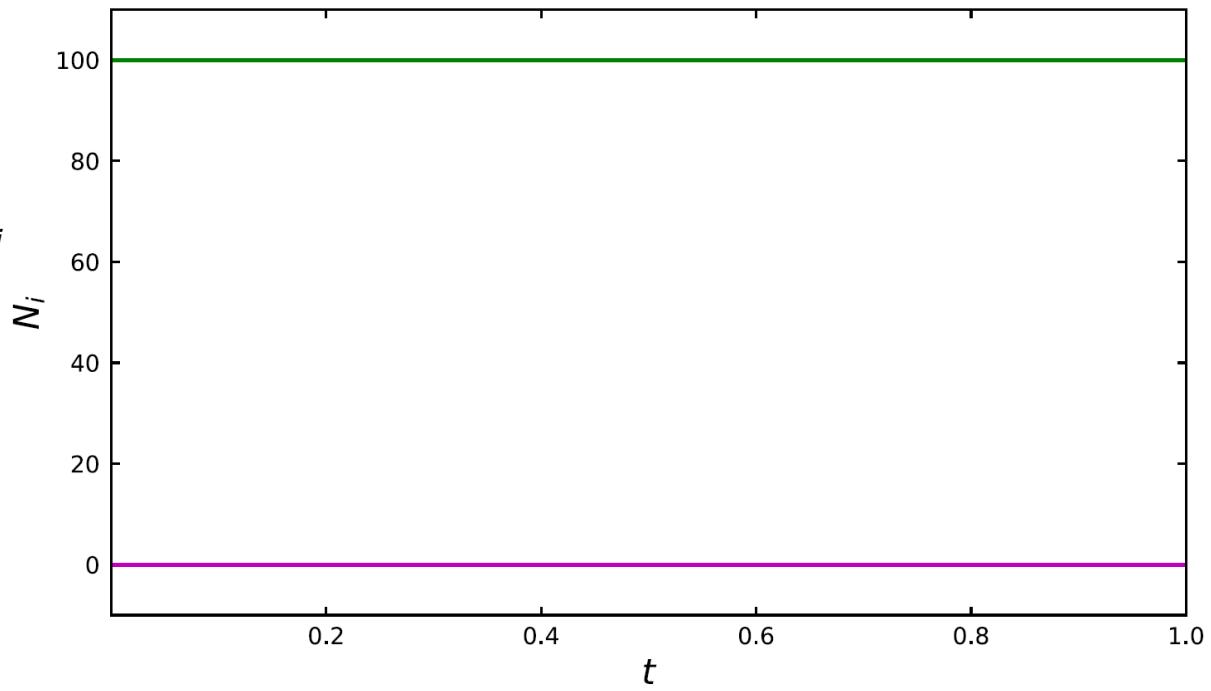
$$i\partial_t \mathbf{A}_n = n\mathbf{A}_n + \frac{1}{2} \sum_{i,j,k,l} \Lambda_{ij}^{kn} \mathbf{A}_k^\dagger \mathbf{A}_i \mathbf{A}_j$$

Initial state

$$N_1 = N_3 = N_4$$

$$= N_5 = 0,$$

$$N_2 = 100$$



5. Gravitational self-interaction

■ Ex.3) Gravitational Self-interaction

■ Hamiltonian [G : Newton constant]

$$H = \int_V d^3x \frac{1}{2m} \left(\vec{\nabla} \psi^\dagger(t, \vec{x}) \right) \cdot \left(\vec{\nabla} \psi(t, \vec{x}) \right) - \frac{Gm^2}{2} \int_V d^3x \int_V d^3y \psi^\dagger(t, \vec{x}) \psi(t, \vec{x}) \frac{1}{|\vec{x}-\vec{y}|} \psi^\dagger(t, \vec{y}) \psi(t, \vec{y})$$

■ EOM

$$i\partial_t \psi = \left[-\frac{1}{2m} \vec{\nabla}^2 + m\varphi \right] \psi \quad \left. \begin{array}{l} \varphi(t, \vec{x}) = -Gm \int_V d^3y \frac{\psi^\dagger(t, \vec{y}) \psi(t, \vec{y})}{|\vec{x}-\vec{y}|} \\ \text{: gravitational potential} \end{array} \right]$$

■ Classical description

■ Density perturbation:

$$\delta(t, \vec{k}) = C(\vec{k}) \left(\frac{t}{t_0} \right)^{2/3} + D(k) \left(\frac{t_0}{t} \right)$$

■ Quantum description

■ $N_{ex} = \sum_{0 < |\vec{k}| < k_J} \langle b_{\vec{k}}^\dagger b_{\vec{k}} \rangle \sim 0.26 N G m^2 \sqrt{mt_*} \left(\frac{t}{t_*} \right)^2$

■ Duration of classicality : $t_c \sim t_* \frac{1}{(Gm^2\sqrt{mt_*})^{1/2}}$

6. Summary

- When the axion thermalize, all conditions for their BEC are satisfied
- We estimated the duration of classicality with attractive contract interaction in case of the homogeneous background
 - Attractive $\lambda\phi^4$: $t_c \sim \frac{2m}{k_J^2} \ln \frac{32\pi^{3/2}n_0}{k_J^3}$,
 - Gravitational self interaction : $t_c \sim t_* \frac{1}{(Gm^2\sqrt{mt_*})^{1/2}}$
- The parametric resonance causes quanta to jump in pairs out of the condensate into all modes with wave vector less than some critical value

