A possible alternative mechanism to SUSY: conservative extensions of the Poincaré group

*J.Phys.***A50**(2017)115401, *Int.J.Mod.Phys.***A32**(2016)1645041, `arXiv:1801.03463`

András LÁSZLÓ
`laszlo.andras@wigner.mta.hu`
Wigner RCP, Budapest, Hungary

SUSY2018

Barcelona, 26^{th} July 2018

Introduction

- **More symmetries simplify a theory.** Larger symmetry requirement reduces the number of variants of a field theoretical Lagrangian, and relates its coupling constants.
- **Grand unification (GUT) strategy.** Models with large, direct-indecomposable (unified) symmetry group is looked for.
- **Unification no-go theorems.** Spacetime symmetries (Poincaré group) and compact internal symmetries (compact gauge group) cannot be unified in an easy way (McGlinn1964, Coleman-Mandula1967).
- **Supersymmetry (SUSY).** The no-go theorems are circuimventable if some amount of "exotic" symmetries are allowed (Haag-Lopuszanski-Sohnius1975).
- **SUSY is not seen experimentally.** At present status (2018).
- **Mathematical alternatives can actually exist.** We found a group theoretical mechanism to possibly substitute SUSY for symmetry unification. There can be more.

(*We will talk only in global symmetries limit, for brevity.*)

General structure of Lie groups and the SUSY

- **Levi decomposition theorem**:

$$\underbrace{E}_{\text{finite dim real Lie group}} = \underbrace{R}_{\substack{\text{degenerate directions of Killing form} \\ (\textit{radical}\text{, or } \textit{solvable part})}} \rtimes \underbrace{\overbrace{L_1}^{(\textit{simple})} \times \ldots \times \overbrace{L_n}^{(\textit{simple})}}_{\substack{\text{non-degenerate directions of Killing form} \\ (\textit{Levi factor}\text{, or } \textit{semisimple part})}}$$

E.g.: the symmetries of flat plane (translations $\rtimes$ rotations) is a typical example.

- **Traditional gauge theory folklore:** only (semi)simple groups are important. $\mathrm{SU}(N)$ etc.

- **Poincaré group**:

$$\underbrace{\mathcal{P}}_{\text{Poincaré group}} = \underbrace{\mathcal{T}}_{\text{translation group (radical)}} \rtimes \underbrace{\overbrace{\mathcal{L}}^{(\textit{simple})}}_{\text{homogeneous Lorentz group (Levi factor)}}$$

is a typical demonstration of Levi's decomposition theorem.

super-Poincaré group (SUSY):

$$\underbrace{\mathcal{P}_s}_{\text{super-Poincaré group}} = \underbrace{\mathcal{S}}_{\text{supertranslation group (radical)}} \rtimes \underbrace{\mathcal{L}}_{\text{homogeneous Lorentz group (Levi factor)}}$$

is a similar example, with a bit larger radical (two-step nilpotent).

Supertranslations: a transformation group on the vector bundle of superfields. Action:

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \xmapsto{\begin{pmatrix} \epsilon^A \\ d^a \end{pmatrix}} \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma^a_{AA'}\,\mathrm{i}\big(\theta^A \bar{\epsilon}^{A'} - \epsilon^A \bar{\theta}^{A'}\big) \end{pmatrix}$$

on the "supercoordinates" and the affine spacetime coordinates.

All possible extensions of the Poincaré group

O'Raifeartaigh theorem (1965) — all possible finite dim extensions of the Poincaré group:

- **Either:**

$$
\begin{array}{ccccc}
E & = & R & \rtimes & L_1 \times \ldots \times L_n \\
\mathcal{P} & = & \mathcal{T} & \rtimes & \mathcal{L}
\end{array}
$$

(A) Trivial extension: $\sim$Coleman-Mandula.

(*Other symmetries are independent from Poincaré.*)

(B) Extended radical: not (A), radical bigger than $\mathcal{T}$.

(SUSY, extended SUSY, and our new example.)

- **Or:**

$$
\begin{array}{ccccc}
E & = & R & \rtimes & L_1 \times \ldots \times L_n \\
\mathcal{P} & = & \mathcal{T} & \rtimes & \mathcal{L}
\end{array}
$$

(C) Poincaré embedded into simple Lie group.

(*Conform*, E_8, $\mathrm{SO}(3{,}11)$, $\mathrm{SO}(1{,}13)$ *theories*.

Heavy symmetry breaking needed

for a gauge-theory-like limit.)

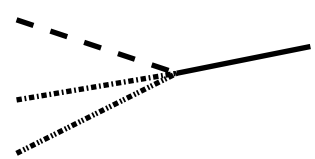
How (extended) SUSY works Lie group theoretically?

- **Unification via *extended super-Poincaré group***:

(nonvanishing adjoint subgroup action)

$$\underbrace{\underbrace{\Big(\big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\mathcal{Z}}_{\text{central charges}}\big) . \underbrace{\mathcal{Q}_{\mathrm{ext}}}_{\text{pure supertranslations}}\Big)}_{\text{group of supertranslations (radical)}} \rtimes \Big(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\text{compact internal symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz group}}\Big)}_{\text{extended super-Poincaré group}}$$

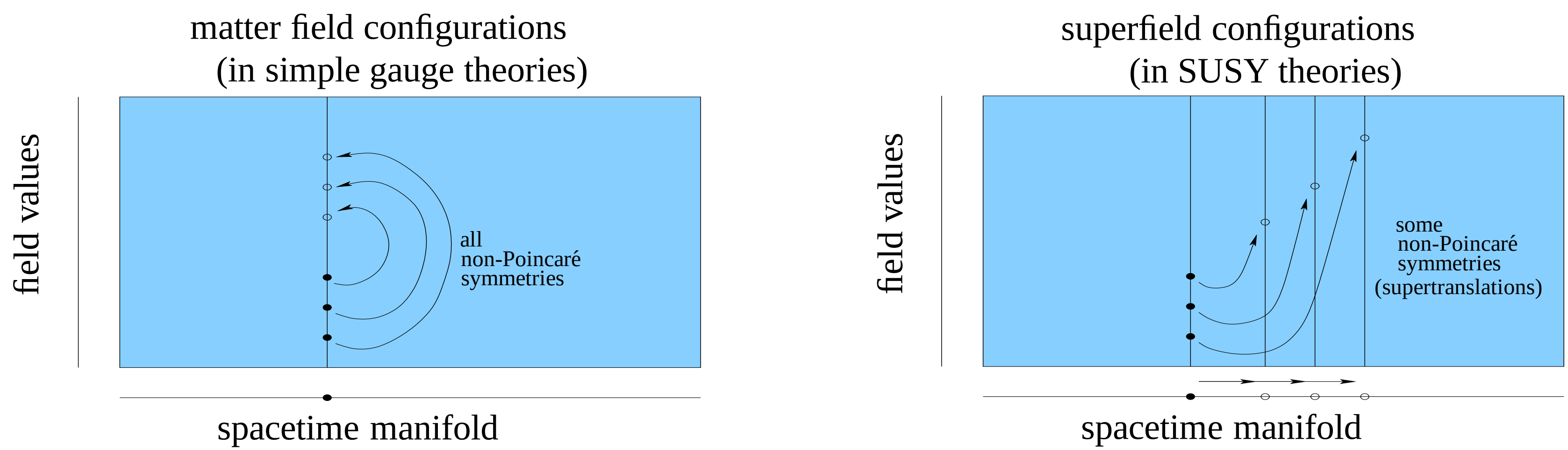
- **The extended super-Poincaré group is direct-indecomposable (unified).**
  $\Rightarrow$ Connects spacetime symmetries with compact internal (gauge) symmetries.
  $\Rightarrow$ Connects potentially independent compact internal symmetries with each-other.
  $\Rightarrow$ Running of coupling factors do unify.



Running of gauge couplings

- **Operated by O'Raifeartaigh theorem case B.** Via the extension of the radical.

- **Symmetry breaking still needed.** Because in (extended) SUSY, the non-Poincaré symmetries couple too strongly to spacetime.



(Not a vector bundle automorphism group over spacetime.)

Experimental hint not seen for this, so symmetry breaking needed for gauge-theory-like limit. (bug? feature?)

A possible alternative mechanism to SUSY

- ***Conservative* extensions of the Poincaré group.**
  $\Leftrightarrow$ The non-Poincaré symmetries are all *internal*, i.e. do not act on spacetime.
  $\Leftrightarrow$ There exists $\mathcal{P} \stackrel{i}{\longrightarrow} E \stackrel{o}{\longrightarrow} \mathcal{P}$ homomorphisms, such that $o \circ i =$ identity.
  $\Leftrightarrow$ Symm. breaking not needed for gauge-theory-like limit (vector bundle automorphism).
- **The (extended) super-Poincaré is non-conservative extension of Poincaré group.**
- **All possible conservative extensions of the Poincaré group:**

(nonvanishing adjoint subgroup action)

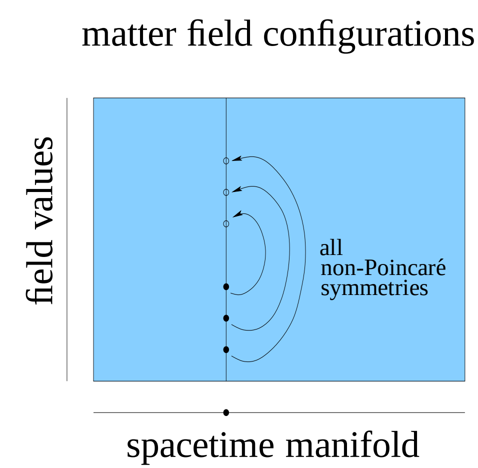
$$\underbrace{\Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\underbrace{\mathcal{N}}_{\text{nilpotent internal symmetries}} \Big) \rtimes \Big(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\text{compact internal symmetries}}}_{\text{all internal (gauge) symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz group}} \Big)}_{\text{unified global symmetries of matter fields}}$$

O'Raifeartaigh theorem + energy non-negativity $\Rightarrow$ these are only possible ones.
Similar gauge – spacetime symmetry unification to extended SUSY, via extended radical.

- **In a conservative Poincaré extension, non-Poincaré symmetries are all internal.**

any conservative extension of Poincaré group $=$

$\{$internal symmetries$\}$ $\rtimes$ $\{$Poincaré$\}$ $\Longleftrightarrow$



(SUSY does not admit this property.)

- **Constructed an example:**
See *J.Phys.***A50**(2017)115401 and *Int.J.Mod.Phys.***A32**(2016)1645041, with $\mathcal{G} = \mathrm{U}(1)$.
It is the symmetry group of a QFT-inspired algebra valued fields.

Price to pay: the full internal symmetry group is not purely $\{$compact$\}$ but

$\{$nilpotent$\}$ $\rtimes$ $\{$compact$\}$

(Some non-propagating, zero-energy gauge field modes present.)

In a *conservative* Poincaré group extension: there exists a homomorphism of

$$\left(\underbrace{\mathcal{N}}_{\text{nilpotent internal symmetries}} \rtimes \left(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\text{compact internal symmetries}} \times \underbrace{\mathcal{P}}_{\text{Poincaré symmetries}}\right)\right)_{\text{unified group, acting on fundamental field degrees of freedom}} \longrightarrow \left(\underbrace{\mathcal{G}_1 \times \ldots \times \mathcal{G}_m}_{\text{compact internal symmetries}} \times \underbrace{\mathcal{P}}_{\text{Poincaré symmetries}}\right)_{\text{observed symmetries, acting on some derived field quantities which are function of fundamental degrees of freedom}}$$

No immediate contradiction with experimental situation.

(Nilpotent internal symmetries can be "hidden" in fundamental d.o.f.)

Distant analogy:

$$\underbrace{\Psi}_{\mathrm{U}(1)\times\text{Poincaré acts on it}} \longmapsto \underbrace{\text{Fierz bilinears of } \Psi}_{\text{only Poincaré acts on it}}$$

The Fierz bilinears forget the fundamental $\mathrm{U}(1)$ symmetry of Dirac bispinor fields.

(But such "forgetting function" mechanism works also for semi-direct product.)

Perspectives

Least exotic solution to gauge–spacetime and gauge–gauge symmetry unification: *conservative* unification pattern.

(nonvanishing adjoint subgroup action)

$$\Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\mathcal{N}}_{\substack{\text{nilpotent internal symmetries}\\ \text{(all the exoticity goes here)}}} \Big) \rtimes \Big(\underbrace{\mathrm{U}(1)\times \mathrm{SU}(2)\times \mathrm{SU}(3)}_{\text{Standard Model internal symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz group}} \Big)$$

Unification happens not because of a heavy symmetry breaking.

But because of common adjoint subgroup action on "hidden" nilpotent internal symmetries.

Minimal exoticity.

Unification achieved not by symmetry *breaking* but by symmetry *hiding*.

Bonus: L.Šnobl, *J.Phys.* **A43**(2010)505202 says: max 1 copy of $\mathrm{U}(1)$ can be present.

Summary

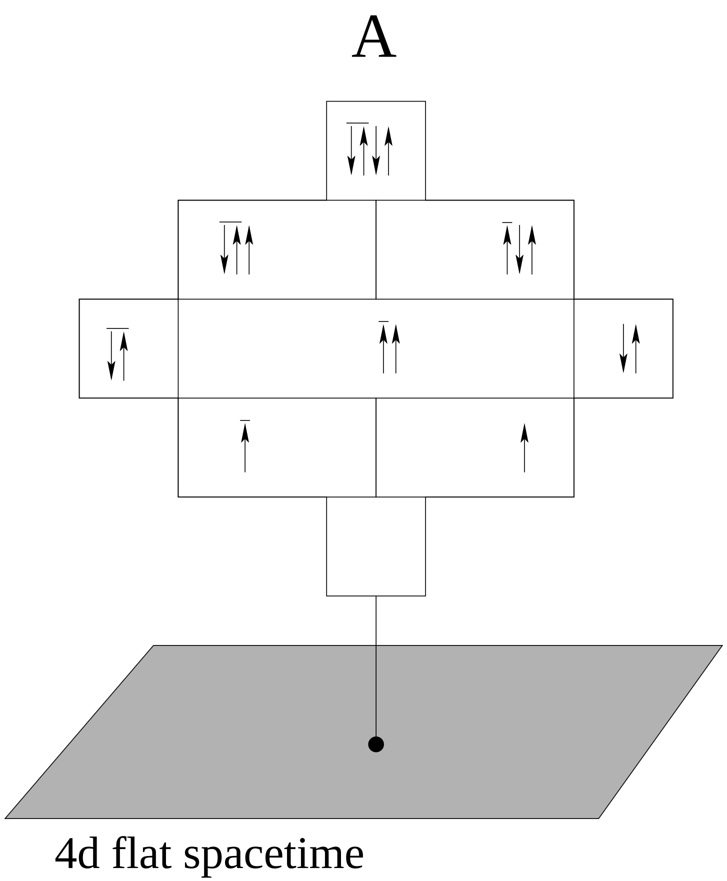
- **SUSY experimentally not visible at present.** As of 2018 status.
- **Mathematical alternatives to SUSY exist.** Are also O'Raifeartaigh B type, as SUSY.
- **The alternative: "conservative" extensions of the spacetime symmetries.** The complementing symmetries to spacetime symmetries are all strictly internal.
- **Example constructed.** At present, merely with $\mathrm{U}(1)$ as compact gauge group.
- **It connects gauge and spacetime symmetries.** Similarly to extended SUSY.
- **Harmonizes with present experimental situation.** Extra symmetries are "hidden".

*J.Phys.***A50**(2017)115401, *Int.J.Mod.Phys.***A32**(2016)1645041, `arXiv:1801.03463`.

Backup

The example group

Automorphism group of an algebra valued classical fields over flat spacetime.
The algebra: $A \cong \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, QFT-inspired. ($S^*$: lower index 2-spinor space.)



"spin algebra" valued fields:

$$\Big(\varphi,\; \xi_{(+)\,A'},\; \xi_{(-)\,A'},\; \varepsilon_{(+)\,[B'C']},\; v_{DD'},\; \varepsilon_{(-)\,[BC]},\; \chi_{(+)\,[B'C']A},\; \chi_{(-)\,A'[BC]},\; \omega_{[A'B'][CD]}\Big)$$

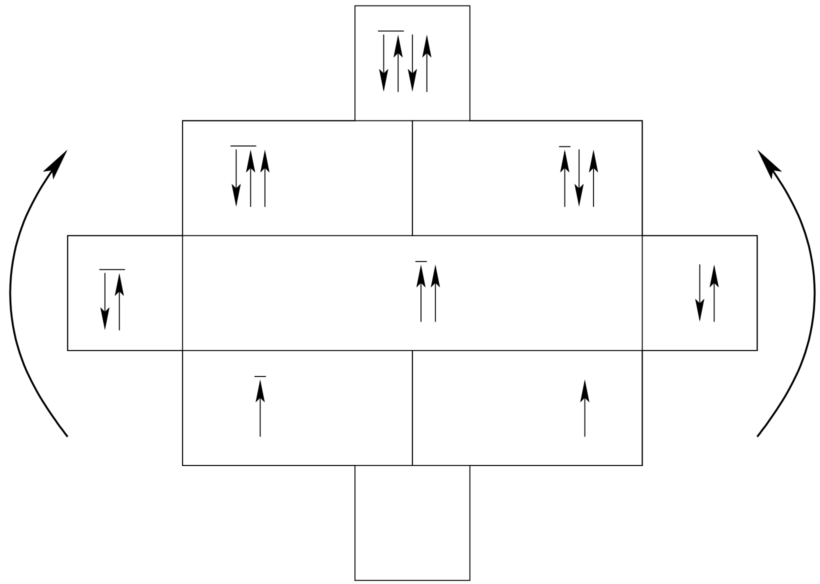
similar to superfields.

Encodes:
creation op. algebra for 2 fundamental d.o.f.,
Pauli principle,
charge conjugation.

The example group has favorable properties:
unified, has Poincaré, has $\mathrm{U}(1)$, minimal exoticity, has unitary reps, implies causal structure.

The nilpotent algebra automorphisms act as “dressing transformations”.

$\mathcal{N}$:



(generators $\longmapsto$ generators $+$ some higher polynomial of generators)

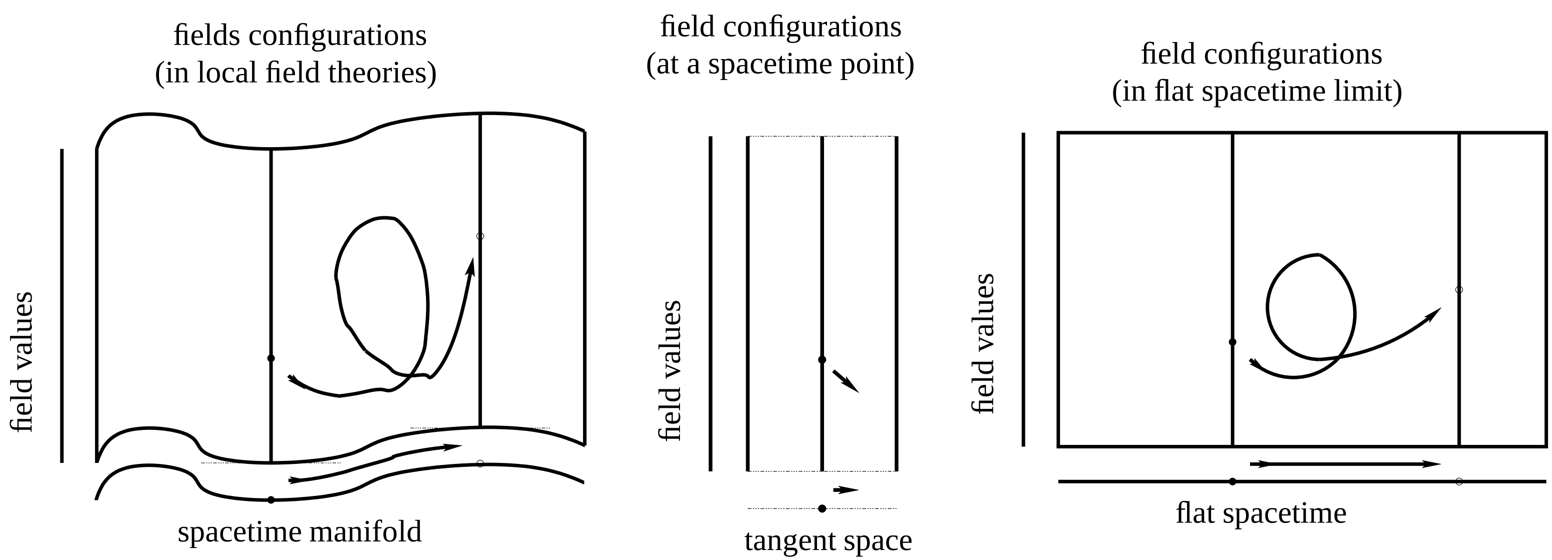
The full automorphism group of spin algebra is a unified gauge–Lorentz group:

$$\underbrace{\mathcal{N}}_{\text{"exotic" nilpotent gauge symmetries}} \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{usual compact gauge symmetries}} \times \underbrace{\mathcal{L}}_{\text{usual Lorentz symmetries}} \Big)$$

(Early attempts: R.M.Wald, S.Anco: *Phys.Rev.*D39(1989)2297 with algebra valued fields.)

What is a field theory?

In a (classical) field theory we have finite degrees of freedom at points of 4d spacetime.



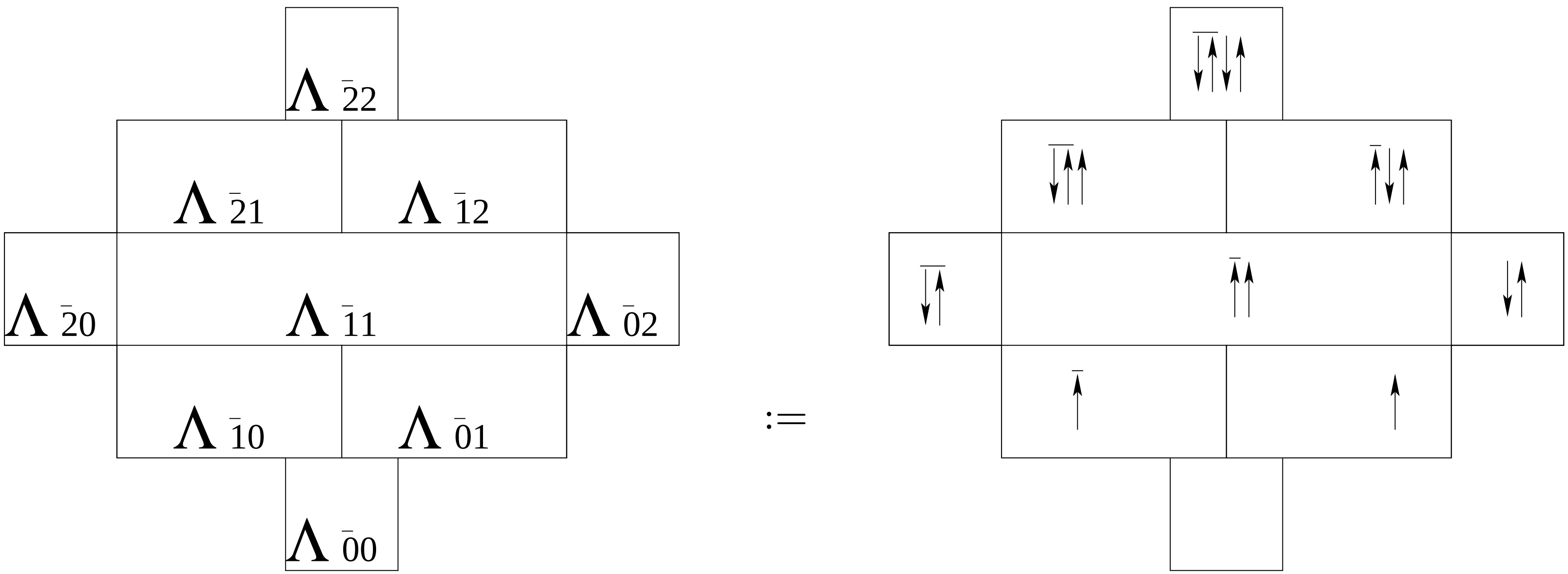
Structure of Lagrangian is determined by its *first order symmetries* at points of spacetime.

Such symmetry generators are finitely many, because they act on finite degrees of freedom. (They have the same structure as if we looked at global symmetries limit.)

Thus, such symmetries form always finite dimensional real Lie groups. (Can be shown.)

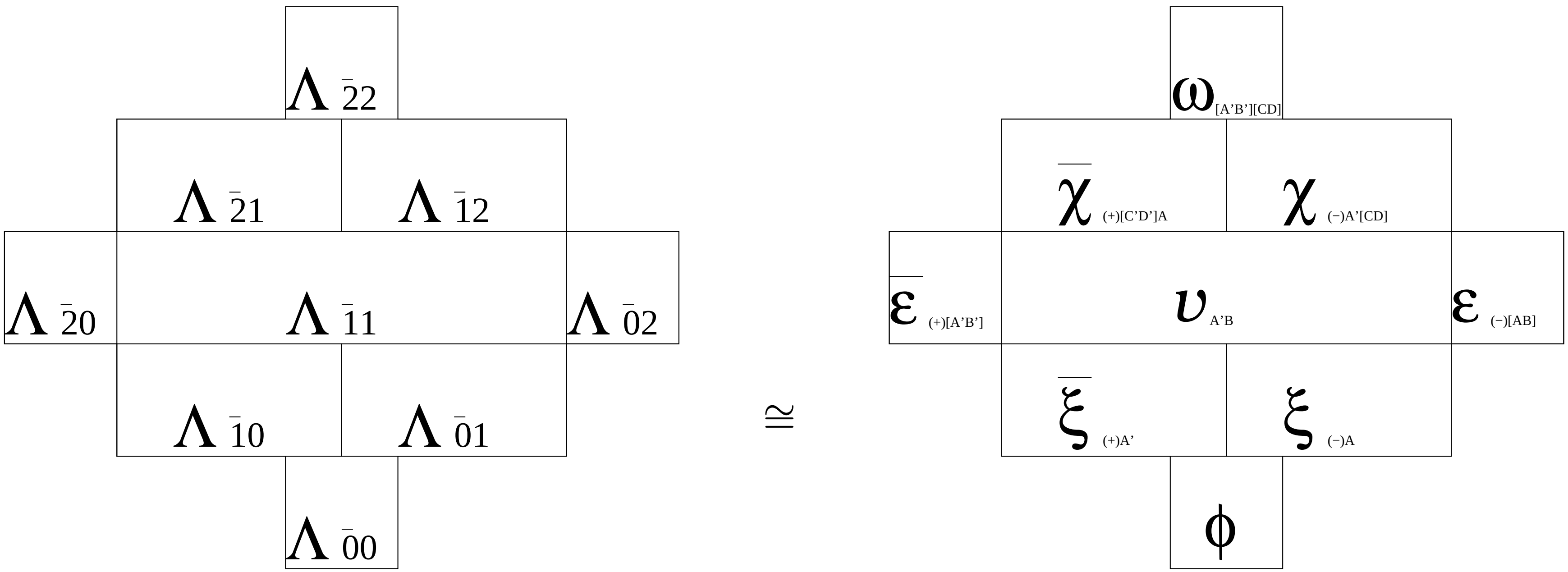
On the example group

Constructed example for $\mathcal{G}=\mathrm{U}(1)$ in JPhys**A50**(2017)115401 and IJMP**A32**(2016)1645041.
It is the automorphism group of a QFT-inspired algebra, named spin algebra:



$\equiv$ algebra of creation operators of fermion and antifermion in the limit if we only had spin.
(Encoding 2 fundamental degrees of freedom, Pauli principle, and charge conjugation.)

Spin algebra can also be represented via two-spinor formalism:



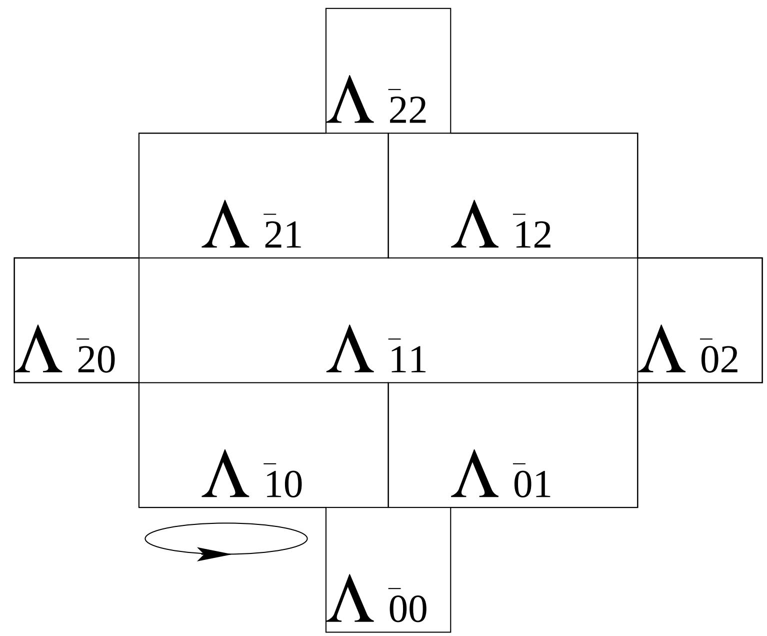
$\cong\ \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, where S^* is lower index two-spinor space (it is a 9-tuple of spinor fields).

Represents spin QFT degrees of freedom as classical fields at points of spacetime.
(Kind of semiclassical limit).

Regular automorphisms of spin algebra are $\mathrm{U}(1)$ gauge + Lorentz symmetries.

$$\underbrace{e_i}_{\text{generators}} \longmapsto \underbrace{\mathrm{e}^{\mathrm{i}\varphi} e_i \; + \; \Omega^{\frac{1}{2}} e_i \; + \; \sum_{j=1}^{2} \alpha_{ij} e_j}_{\text{new generators}}$$

$(i{=}1{,}2$ and $\varphi{\in}\mathbb{R},\ \Omega{\in}\mathbb{R}^{+},\ \alpha{\in}\mathrm{SL}(2{,}\mathbb{C}))$



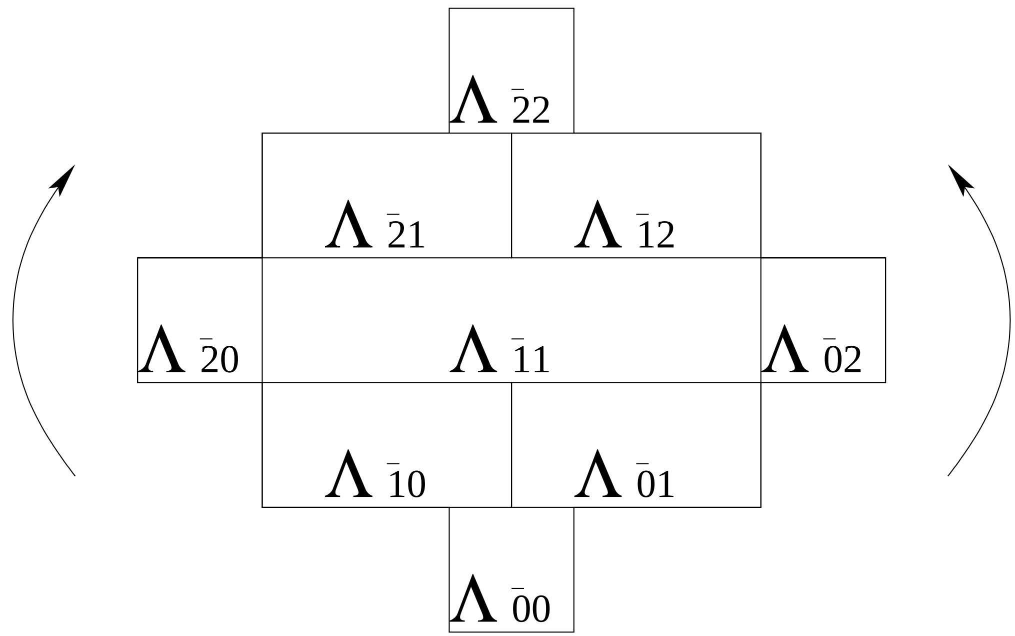
These preserve the space $\Lambda_{\bar{p}q}$ of pure p, q-forms.

Therefore, are equivalent to the usual transformations of two-spinor space ($\equiv \mathrm{U}(1) \times \mathcal{L}$).

The exotic automorphisms ($\mathcal{N}$) act as “dressing” transformations.

$$\underbrace{e_i}_{\text{generators}} \quad \overset{\mathcal{N}}{\longmapsto} \quad \underbrace{e_i \; + \; \text{higher polynomials}}_{\text{new generators}}$$

$(i{=}1{,}2)$



In QFT analogy: 1-particle spaces not conserved. Are “dressed” by higher particle content.

These symmetries have nilpotent Lie algebra ($\Rightarrow$ degenerate directions of Killing form).

They are invisible when truncated to 1-particle theory.

The full automorphism group of spin algebra is a unified gauge–Lorentz group:

$$\underbrace{\mathcal{N}}_{\text{"exotic" nilpotent gauge symmetries}} \rtimes \Big(\underbrace{\mathcal{G}}_{\text{usual compact gauge symmetries}} \times \underbrace{\mathcal{L}}_{\text{usual Lorentz symmetries}} \Big)$$

which is direct-indecomposable. ($\mathcal{G} = \mathrm{U}(1)$ in the example.)

How it bypasses Coleman-Mandula?

Essence of Coleman-Mandula-like no-go theorems:

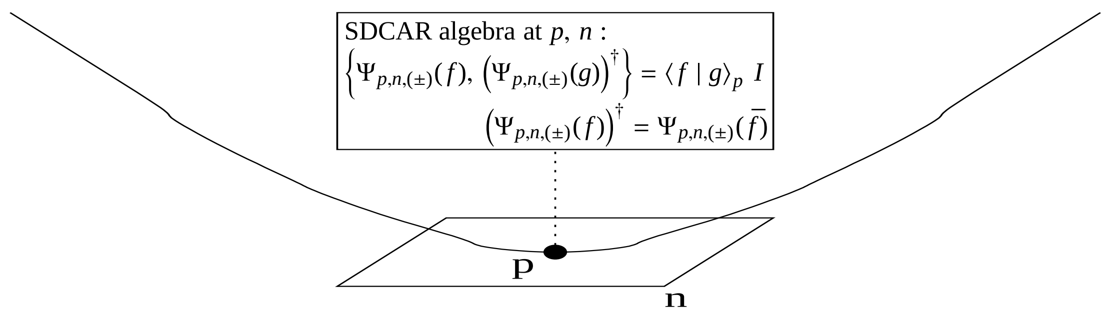
- Full symmetry group is a Poincaré group extension $\Rightarrow$ O'Raifeartaigh A or B or C.
- Complementing symmetries to Poincaré symmetries preserve 1-particle Fock subspace and have positive definite invariant "internal" scalar product $\Rightarrow$ no O'Raifeartaigh B.
- No symmetry breaking present $\Rightarrow$ no O'Raifeartaigh C.

Our mechanism: internal group is $\mathcal{N} \rtimes \mathcal{G} \Rightarrow$ internal scalar product degenerates on $\mathcal{N} \Rightarrow \checkmark$
SUSY: similar degeneration over the pure supertranslations $\mathcal{Q} \Rightarrow \checkmark$

Let A be spin algebra.

Then, one can show that there is a correspondence:

$$A \quad \longleftrightarrow \quad \mathrm{Aut}(A)\text{-covariant family of SDCAR algebras}$$
$$\text{(parametrized by mass shell and } \mathbb{Z} \times \mathbb{Z}\text{-splitting of } A\text{)}$$



(SDCAR algebra: algebra of quantum field operators.)

So, it is really a QFT semiclassical limit.

SUSY viewed as an ordinary Lie group

One can write:

$$\underbrace{\mathcal{P}_s}_{\text{super-Poincaré group}} = \underbrace{\underbrace{\Big(\underbrace{\mathcal{T}}_{\text{normal subgroup of translations}} \cdot \underbrace{\mathcal{Q}}_{\text{pure supertranslations}}\Big)}_{=\mathcal{S}\text{ (normal subgroup of all supertranslations)}} \rtimes \underbrace{\mathcal{L}}_{\text{Lorentz group}}}_{\text{super-Poinceré group}}$$

: indicates nonvanishing adjoint subgroup action.

$\cdot$: indicates "semi-semidirect product" (right extension of a normal subgroup).

Parts not connected by or $\cdot$ are independent.

Traditionally, SUSY is presented as "super-Lie algebra":

$$
\begin{array}{ll}
\checkmark & [\,P_a\,,\,P_b\,]=0,\\
\checkmark & [\,P_a\,,\,Q_A\,]=0,\\
\checkmark & [\,P_a\,,\,\bar{Q}_{A'}\,]=0,\\
!!!\rightarrow & \{\,Q_A\,,\,Q_B\,\}=0,\\
!!!\rightarrow & \{\,\bar{Q}_{A'}\,,\,\bar{Q}_{B'}\,\}=0,\\
!!!\rightarrow & \{\,Q_A\,,\,\bar{Q}_{A'}\,\}=2\,\sigma^a_{AA'}\,P_a.
\end{array}
$$

If not a Lie algebra, how can it be a collection of infinitesimal transformations? Answer:

[Nucl.Phys.**B76**(1974)477, Phys.Lett.**B51**(1974)239]:
Take $\epsilon^A_{(i)}$ ($i=1,2$) "supercoordinate" (Grassmann valued two-spinor) basis.
Introduce new generators $\delta_{(i)}=\epsilon^A_{(i)}\,Q_A$ instead of Q_A. (Infinitesimal change of superfields.)
Will make ordinary Lie algebra.

Exponentiating this Lie algebra: super-Poincaré Lie group is obtained. SUSY is not so exotic!
Ordinary Lie group / Lie algebra theory also applies!

With super-Lie algebra, one can exactly generate those ordinary Lie algebras (Lie groups) which have the structure

$$\underbrace{\left(\mathcal{T} \;\cdot\; \mathcal{Q}\right)}_{=\mathcal{S}} \;\rtimes\; \mathcal{L}$$

with

- $\mathcal{L}$ being a subgroup,
- $\mathcal{S}$ being a complementing normal subgroup,
- $\mathcal{T}$ being a normal subgroup within $\mathcal{S}$, and
- $\mathcal{T}$ commutes with all $\mathcal{S}$ and $\mathcal{Q} \cong \mathcal{S}/\mathcal{T}$ is abelian. $\leftarrow$!!!
  (translations commute with all supertranslations,
  supertranslations without considering contribution of translations are abelian.)

$\uparrow$

This makes it possible to switch the sign of “odd” part ($\mathcal{Q}$) with Grassmannian basis.

Group theoretical constraints for unification

- Collection of all symmetries:

$$\underbrace{E}_{\text{Lie group}} = \underbrace{R}_{\substack{\text{degenerate directions of Killing form}\\ (\textit{radical}\text{, or }\textit{solvable part})}} \rtimes \underbrace{L_1 \times \cdots \times L_n}_{\substack{\text{non-degenerate directions of Killing form}\\ (\textit{Levi factor}\text{, or }\textit{semisimple part})}}$$

- Poincaré group:

$$\underbrace{\mathcal{P}}_{\text{Poincaré group}} = \underbrace{\mathcal{T}}_{\text{translations (radical)}} \rtimes \underbrace{\mathcal{L}}_{\text{homogeneous Lorentz group (Levi factor)}}$$

- Compact gauge group:

$$\underbrace{\mathcal{G}}_{\text{compact Lie group}} = \underbrace{\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)}_{\text{compact abelian part (radical)}} \times \underbrace{G_1 \times \cdots \times G_m}_{\text{compact non-abelian part (Levi factor)}}$$

Simplest solution is *conservative* unification pattern:
we inject subgroups where they naturally belong.

(nonvanishing adjoint subgroup action)

$$\Bigg(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\mathcal{N}}_{\substack{\text{nilpotent internal symmetries} \\ \text{(all the exoticity goes here)}}} \Bigg) \rtimes \Bigg(\underbrace{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}_{\text{Standard Model internal symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz group}} \Bigg)$$

Unification happens not because of a heavy symmetry breaking.
But because of common adjoint subgroup action on "hidden" nilpotent internal symmetries.
Minimal exoticity.

Lie bracket structure of our example group

See for details: `arXiv:1801.03463`, in Proceedings of QTS10 (2018).

$$
\begin{array}{rrcl}
 & [\mathrm{ad}_a,\, \mathrm{ad}_{a'}] & = & \mathrm{ad}_{[a,a']}, \\
 & \left[\mathrm{ad}_a,\, \nu_{\beta'}\right] & = & -\,\mathrm{ad}_{\nu_{\beta'}(a)}, \\
 & \left[\mathrm{ad}_a,\, \zeta_{\mathrm{i}\varphi'}\right] & = & -\,\mathrm{ad}_{\zeta_{\mathrm{i}\varphi'}(a)}, \\
 & [\mathrm{ad}_a,\, J_{cd}] & = & -\,\mathrm{ad}_{J_{cd}(a)}, \\
 & [\mathrm{ad}_a,\, P_c] & = & 0, \\
 & \left[\nu_\beta,\, \nu_{\beta'}\right] & = & 0, \\
 & \left[\nu_\beta,\, \zeta_{\mathrm{i}\varphi'}\right] & = & -\nu_{[\zeta_{\mathrm{i}\varphi'},\beta]}, \\
 & \left[\nu_\beta,\, J_{cd}\right] & = & -\nu_{[J_{cd},\beta]}, \\
 & \left[\nu_\beta,\, P_c\right] & = & 0, \\
\mathrm{U}(1)\ \text{generator}\ \rightarrow & \left[\zeta_{\mathrm{i}\varphi},\, \zeta_{\mathrm{i}\varphi'}\right] & = & 0, \\
 & \left[\zeta_{\mathrm{i}\varphi},\, J_{cd}\right] & = & 0, \\
 & \left[\zeta_{\mathrm{i}\varphi},\, P_c\right] & = & 0, \\
\text{Lorentz generator}\ \rightarrow & \left[J_{cd},\, J_{ef}\right] & = & \mathrm{i}\,g_{de}\,J_{cf} - \mathrm{i}\,g_{ce}\,J_{df} + \mathrm{i}\,g_{cf}\,J_{de} - \mathrm{i}\,g_{df}\,J_{ce}, \\
 & [J_{cd},\, P_e] & = & \mathrm{i}\,g_{de}\,P_c - \mathrm{i}\,g_{ce}\,P_d, \\
\text{translation generator}\ \rightarrow & [P_c,\, P_d] & = & 0
\end{array}
$$

Here, $A \cong \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$, $\longleftarrow$ field algebra.

$\beta, \beta' \in \mathrm{Re}\left(\Lambda_{\bar{1}2} \otimes \Lambda^*_{\bar{1}0} \oplus \Lambda_{\bar{2}1} \otimes \Lambda^*_{\bar{0}1}\right) \subset \mathrm{Re}\,(\mathrm{Lin}(A))$, $\longleftarrow$ nilpotent symmetry generator

$a, a' \in \mathrm{Re}\,(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2}) \subset \mathrm{Re}\,(A)$, $\longleftarrow$ nilpotent symmetry generator

$\varphi, \varphi' \in \mathbb{R}$ $\longleftarrow$ $\mathrm{U}(1)$ symmetry generator.