Implementing a Lüscher Analysis with Multiple Partial Waves and Decay Channels

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Motivations and Overview

- use the Lüscher two-particle formalism for studying hadronic resonances
- develop implementation to be simple yet general
- computationaly simple fitting strategies
- provide software with all of these features
- more details (and software) NPB **924**, 477 (2017)

The Lüscher Quantization Condition

$$\det[1 + F^{(\boldsymbol{P})}(S - 1)] = 0$$

- allows access to infinite-volume physics (S-matrix) from finite-volume physics (F-matrix)
- F matrix elements are known functions

$$\langle J' m_{J'} L' S' a' | F^{(\boldsymbol{P})} | J m_J L S a \rangle = \delta_{a'a} \delta_{S'S} \ \frac{1}{2} \Big\{ \delta_{J'J} \delta_{m_{J'} m_J} \delta_{L'L} + \langle J' m_{J'} | L' m_{L'} S m_S \rangle \langle L m_L S m_S | J m_J \rangle W^{(\boldsymbol{P}a)}_{L' m_{L'};\ L m_L} \Big\}$$

- total momentum $\boldsymbol{P}$, total angular momentum J, J', orbital angular momentum L, L', spin S, S', channels a, a'
- W can be expressed as sums over the Lüscher zeta functions $\mathcal{Z}_{lm}$

The K-matrix

- quantization condition relates single energy to entire S-matrix
  - must parameterize S-matrix (except for single channel and single partial wave)
  - easier to parameterize a Hermitian matrix than a unitary matrix
- introduce the K-matrix

$$S = (1 + iK)(1 - iK)^{-1} = (1 - iK)^{-1}(1 + iK)$$

- then introduce $\widetilde{K}$ via

$$K^{-1}_{L'S'a';LSa}(E_{cm}) = u_{a'}^{-L'-\frac{1}{2}}\,\widetilde{K}^{-1}_{L'S'a';LSa}(E_{cm})\,u_a^{-L-\frac{1}{2}}.$$

- the u_a are defined by (here L is size of the box)

$$E_{cm} = \sqrt{\left(\frac{2\pi}{L}u_a\right)^2 + m_{1a}^2} + \sqrt{\left(\frac{2\pi}{L}u_a\right)^2 + m_{2a}^2}$$

- $\widetilde{K}^{-1}$ elements expected to be smooth function of E_{cm}

The "Box Matrix" and Block Diagonalization

- rewrite quantization condition in terms of $\widetilde{K}$

$$\det(1 - B^{(\boldsymbol{P})}\widetilde{K}) = \det(1 - \widetilde{K}B^{(\boldsymbol{P})}) = 0$$

- block diagonalize in the little group irreps

$$|\Lambda\lambda nJLSa\rangle = \sum_{m_J} c_{m_J}^{J(-1)^L;\,\Lambda\lambda n}\,|Jm_J LSa\rangle$$

- little group irrep Λ, irrep row λ, occurrence index n
- group theoretical projections with Gram-Schmidt used to obtain coefficients
- in block-diagonal basis, box matrix has form

$$\langle\Lambda'\lambda'n'J'L'S'a'|\,B^{(\boldsymbol{P})}\,|\Lambda\lambda nJLSa\rangle = \delta_{\Lambda'\Lambda}\delta_{\lambda'\lambda}\delta_{S'S}\delta_{a'a}\,B^{(\boldsymbol{P}\Lambda_B Sa)}_{J'L'n';\;JLn}(E)$$

- $\Lambda_B = \Lambda$ only if $\eta^P_{1a}\eta^P_{2a} = 1$

K-Matrix Parametrizations

- $\widetilde{K}$-matrix for $(-1)^{L+L'} = 1$ has form

$$\langle \Lambda' \lambda' n' J' L' S' a' | \, \widetilde{K} \, | \Lambda \lambda n J L S a \rangle = \delta_{\Lambda' \Lambda} \delta_{\lambda' \lambda} \delta_{n' n} \delta_{J' J} \, \mathcal{K}^{(J)}_{L' S' a'; \; L S a}(E_{\mathrm{cm}})$$

- common parametrization

$$\mathcal{K}^{(J)-1}_{\alpha\beta}(E_{\mathrm{cm}}) = \sum_{k=0}^{N_{\alpha\beta}} c^{(Jk)}_{\alpha\beta} E^k_{\mathrm{cm}}$$

- α, β compound indices for (L, S, a)

- another common parametrization

$$\mathcal{K}^{(J)}_{\alpha\beta}(E_{\mathrm{cm}}) = \sum_p \frac{g^{(Jp)}_\alpha g^{(Jp)}_\beta}{E^2_{\mathrm{cm}} - m^2_{Jp}} + \sum_k d^{(Jk)}_{\alpha\beta} E^k_{\mathrm{cm}},$$

Fitting Subtleties

- goal: obtain best-fit estimates for paramters of $\widetilde{K}$ or $\widetilde{K}^{-1}$
- $\chi^2 = \sum_{ij} \mathcal{E}(r_i)\sigma_{ij}^{-1}\mathcal{E}(r_j)$
- residuals $\boldsymbol{r} = \boldsymbol{R} - \boldsymbol{M}(\boldsymbol{\alpha}, \boldsymbol{R})$
- observables $\boldsymbol{R}$, model parameters $\boldsymbol{\alpha}$
- i-th component of $\boldsymbol{M}(\boldsymbol{\alpha}, \boldsymbol{R})$ gives model prediction for i-th component of $\boldsymbol{R}$
- if model depends on any observables, covariance matrix must be recomputed and inverted each time parameters $\boldsymbol{\alpha}$ adjusted during minimization!
- if model independent of all observables $\mathrm{cov}(r_i, r_j) = \mathrm{cov}(R_i, R_j)$ simplifying minimization

Fitting: Spectrum Method

- choose $E_{\mathrm{cm},k}$ as observables
- model predictions come from solving quantization condition for α
- problems:
  - root finding requires many computations of zeta functions
  - model predictions depend on observables $m_{1a},\ m_{2a},\ L,\ \xi$ so MUST recompute covariance during minimization
- "Lagrange multiplier" trick removes obs. dependence in model
  - include $m_{1a},\ m_{2a},\ L,\ \xi$ as both observables and model parameters
- observations

$$\text{Observations } R_i:\quad \{E_{\mathrm{cm},k}^{(\mathrm{obs})},\ m_j^{(\mathrm{obs})},\ L^{(\mathrm{obs})},\ \xi^{(\mathrm{obs})}\ \},$$

- model parameters

$$\text{Model fit parameters } \alpha_k:\quad \{\ \kappa_i,\ m_j^{(\mathrm{model})},\ L^{(\mathrm{model})},\ \xi^{(\mathrm{model})}\ \},$$

Fitting: Determinant Residual Method

- introduce quantization determinant as residual
- better to use function of matrix A with real parameter μ:

$$\Omega(\mu, A) \equiv \frac{\det(A)}{\det[(\mu^2 + AA^\dagger)^{1/2}]}$$

- residuals
$$r_k = \Omega\Big(\mu, 1 - B^{(\boldsymbol{P})}(E^{(\mathrm{obs})}_{\mathrm{cm},k})\ \widetilde{K}(E^{(\mathrm{obs})}_{\mathrm{cm},k})\Big),$$
- do not need to perform zeta computations during minimization
- must recompute covariance matrix during minimization
- covariance recomputation still simpler than root finding required in spectrum method

Conclusion

- introduced implementation of Lüscher two-particle formalism that is simple while still general
- new fitting strategy: determinant residual method
- software available made available to the public
- successfully applied to ρ, $K^*(892)$, and Δ